On the Stability and Instability of Positive Steady States of a Semilinear Heat Equation in $\mathbb{R}^n$

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1. Introduction and Statements of Main Results

In this paper we shall consider the following Cauchy problem

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u + u^p & \text{in } \mathbb{R}^n \times (0, T), \\
\quad u(x, 0) &= \varphi(x) & \text{in } \mathbb{R}^n,
\end{aligned}
\]

(1.1)

where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, $p > 1$, $T > 0$ and $\varphi \neq 0$ is a given bounded continuous non-negative function in $\mathbb{R}^n$. It is known that there exists $T = T[\varphi] > 0$ such that (1.1) has a unique classical solution $u(x, t; \varphi)$ in $C^{2,1}(\mathbb{R}^n \times (0, T)) \cap C(\mathbb{R}^n \times [0, T])$ which is bounded in $\mathbb{R}^n \times [0, T']$ for all $T' < T[\varphi]$ with $\|u(\cdot, t; \varphi)\|_{L^\infty(\mathbb{R}^n)} \to \infty$ as $t \uparrow T[\varphi]$ if $T[\varphi] < \infty$. Due to the possible nonuniqueness of solutions of (1.1), in this paper we shall restrict ourselves to this particular solution $u(x, t; \varphi)$ of (1.1) and, in case $T[\varphi] = \infty$ we call $u(x, t; \varphi)$ the global solution, otherwise we say that $u(x, t; \varphi)$ blows up in finite time.

It seems that the stability question for the trivial steady state $u_0 \equiv 0$ of (1.1) was first considered by H. Fujita in [5] in 1966. He showed that for $1 < p < \frac{n+2}{n}$, the solution $u(x, t; \varphi)$ must blow up in finite time; and for $p > \frac{n+2}{n}$, the solution $u(x, t; \varphi)$ exists globally in time if $\varphi$ is sufficiently small, say, dominated by a small multiple of Gaussian (which decays exponentially at $x = \infty$). The borderline case $p = \frac{n+2}{n}$ belongs to global nonexistence and was settled later by Hayakawa in [9] and Kobayashi, Sirao, and Tanaka in [10]. Different proofs have been given by many authors including, for instance, Aronson and Weinberger in [1] and Weissler in [17]. Weissler also treated (1.1) in $L^p$-setting. For other related results we refer the readers to a recent survey by H. Levine; see [12].
Therefore we see that the trivial equilibrium $u_0 \equiv 0$ of (1.1) is unstable (under any reasonable topology) for $1 < p \leq \frac{n+2}{n}$, and it becomes "stable" in some sense when $p > \frac{n+2}{n}$. In the latter case the kind of "stability" and domains of attraction will have to be made more precise. First, the domain of attraction for $u_0 \equiv 0$ was improved from requiring $\varphi$ to have exponential decay at $x = \infty$ to polynomial decay at $x = \infty$ by Weissler; see [17]. Recently, the borderline decay at $x = \infty$ was found to be $|x|^{-\frac{2}{p-1}}$ at $\infty$ by Lee and Ni; see [11]. The most explicit domain of attraction for $u_0 \equiv 0$ in case $p > \frac{n}{n-2}$, $n \geq 3$, is due to X. Wang (see [16]) so far. In his Ph.D. thesis, among other things, Wang established the following result.

**Theorem A.** Suppose that $n \geq 3$ and $p > \frac{n}{n-2}$. If $\varphi \leq \lambda U$ for some constant $0 < \lambda < 1$, where

\begin{equation}
U(x) = \left[ \frac{2}{p-1} \left( n - 2 - \frac{2}{p-1} \right) \right]^{\frac{1}{p-1}} |x|^{-\frac{2}{p-1}},
\end{equation}

then the solution $u(x, t; \varphi)$ of (1.1) must satisfy that $0 \leq u \leq \lambda U$ for all $x \in \mathbb{R}^n$ and $t > 0$, and $\|u(\cdot, t; \varphi)\|_{L^\infty(\mathbb{R}^n)} \to 0$ as $t \to \infty$. Moreover, we have

\begin{equation}
\|u(\cdot, t; \varphi)\|_{L^\infty(\mathbb{R}^n)} \leq [(\lambda^{1-p} - 1)(p - 1)t]^{-\frac{1}{p-1}}
\end{equation}

for $t > 0$.

In fact, the decay rate (in time) (1.3) is sharp. We should also remark that the function $U$ defined by (1.2) is a singular equilibrium of (1.1) (in the distributional sense).

Theorem A also implies that in case $p > \frac{n}{n-2}$ the trivial steady state $u_0 \equiv 0$ is stable in the following sense. Setting

\begin{equation}
C^+_U(\mathbb{R}^n) = \left\{ \varphi \in C(\mathbb{R}^n) \mid \varphi \geq 0, \|\varphi U^{-1}\|_{L^\infty(\mathbb{R}^n)} < \infty \right\}
\end{equation}

and

\begin{equation}
B^{+}_{\lambda, U} = \left\{ \varphi \in C^+_U(\mathbb{R}^n) \mid \|\varphi\|_{C^+_U(\mathbb{R}^n)} \equiv \|\varphi U^{-1}\|_{L^\infty(\mathbb{R}^n)} < \lambda \right\},
\end{equation}

we see that Theorem A implies that if $\varphi \in B^{+}_{\lambda, U}$ for some $0 < \lambda < 1$, then the solution $u(x, t; \varphi)$ stays in $B^{+}_{\lambda, U}$ for all $t > 0$. Thus, for each $\lambda \in (0, 1)$, $B^{+}_{\lambda, U}$ is an invariant set and $u_0 \equiv 0$ is stable with respect to $C^+_U$ topology. (In fact, if we interpret $u^p$ as $|u|^{p-1}u$ in (1.1), by a simple comparison argument we see immediately that the above discussion holds for solutions without sign conditions.) Moreover, Theorem A indicates that when the exponent $p$ is fixed, $u_0 \equiv 0$ becomes "increasingly stable" as the dimension $n$ increases.
The purpose of this paper is to investigate the stability or instability of positive steady states of (1.1). Before we proceed further, we first collect a few known facts concerning such equilibria, i.e., the positive solutions of the elliptic equation

\[ \Delta u + u^p = 0 \]

in \( \mathbb{R}^n \), \( n \geq 3 \). (For \( n = 2 \), (1.6) does not have any positive solutions in \( \mathbb{R}^n \). See, e.g., [15].)

**Proposition B.**

(i) For \( 1 < p < \frac{n+2}{n-2} \), (1.6) possesses no positive solutions in \( \mathbb{R}^n \).

(ii) For \( p = \frac{n+2}{n-2} \), all positive solutions of (1.6) are given by (up to translation)

\[ u_\alpha(x) = \left( \frac{\lambda \sqrt{n(n-2)}}{\lambda^2 + |x|^2} \right)^{\frac{n-2}{2}}, \quad x \in \mathbb{R}^n, \]

where \( u_\alpha(0) = \alpha \) and \( \alpha = \left( \frac{\lambda^{-1} \sqrt{n(n-2)}}{\lambda^2 + |x|^2} \right)^{\frac{n-2}{2}} \).

(iii) For \( p > \frac{n+2}{n-2} \) the set of all positive radial solutions of (1.6) is a one-parameter family \( \{u_\alpha\}_{\alpha > 0} \) with \( u_\alpha(r), \quad r = |x| \), strictly decreasing in \( r \), \( u_\alpha(0) = \alpha \), \( u_\alpha(r) = \alpha u_1(\alpha^{\frac{p-1}{p-2}} r) \) and, as \( r \to \infty \)

\[ r^{\frac{2}{p-2}} u_\alpha(r) \to L \equiv \left[ \frac{2}{p-1} \left( n - 2 - \frac{2}{p-1} \right) \right]^{\frac{p-1}{p-2}} \]

for every \( \alpha > 0 \).

In fact, as we shall see later (Theorem 2.5), a much more precise asymptotic behavior of the \( u_\alpha \)'s than (1.8) is needed for our purposes. For (i) and (ii) above, the radial case is well known and easy to prove and the general case may be found in [2] or [3]. Part (iii) and much more may be found in [13], Theorem 1, and in [4], Theorem 5.26.

Using Proposition B, in [16] Wang also gave the following slightly different version of Theorem A which, however, turns out to be the starting point of something much more interesting.

**Theorem C.** Suppose that \( n \geq 3 \), \( p \geq \frac{n+2}{n-2} \). Let \( \{u_\alpha\}_{\alpha > 0} \) be the family of solutions of (1.6) given by Proposition B (ii) or (iii). Then the following conclusions hold.
(i) If $\varphi \leq \lambda u_\alpha$ for some $0 < \lambda < 1$ and some $\alpha > 0$, then the solution $u(x, t; \varphi)$ of (1.1) satisfies $0 < u < \lambda u_\alpha$ for all $t > 0$ and the decay rate (1.3) holds.

(ii) If $\varphi \geq \lambda u_\alpha$ for some $\lambda > 1$ and some $\alpha > 0$, then the solution $u(x, t; \varphi)$ of (1.1) blows up in finite time.

While it is natural to view Theorem C as yet another description for the domain of attraction for the trivial equilibrium $u_0 \equiv 0$ of (1.1), it may also be viewed as an instability result for all positive steady states of (1.1). It says that positive steady states of (1.1) are not only unstable with respect to the $L^\infty$ norm but, in fact, are unstable under a more refined weighted norm, namely, $\| (1 + |x|)^2 \varphi \|_{L^\infty(\mathbb{R}^n)}$ where $\lambda = \frac{2}{p-1}$ if $p > \frac{n+2}{n-2}$, and $\lambda = n - 2$ if $p = \frac{n+2}{n-2}$. Therefore it seems very interesting that the instability of $u_\alpha$, $\alpha > 0$, takes a dramatic turn when the exponent $p$ increases and exceeds another critical value $p_c$ defined in (1.11) below — the steady state $u_\alpha$ switches from being unstable in any reasonable sense to stable in some appropriate topologies and, moreover, $u_\alpha$ becomes “weakly asymptotically stable” in an even finer topology which comes from some higher order term of the asymptotic expansion of $u_\alpha$ near $\infty$. We would also like to remark that the usual asymptotic stability of $u_\alpha$ in any reasonable sense does not seem possible.

To state our results we first introduce some notation. Since we shall concentrate on the positive steady states of (1.1), we shall restrict ourselves to the case $n \geq 3$ and $p \geq \frac{n+2}{n-2}$ in the rest of this paper. Now, let $m = \frac{2}{p-1}$ and

\begin{equation}
\lambda_1 = \lambda_1(n, p) = \frac{(n - 2 - 2m) - \sqrt{(n - 2 - 2m)^2 - 8(n - 2 - m)}}{2},
\end{equation}

\begin{equation}
\lambda_2 = \lambda_2(n, p) = \frac{(n - 2 - 2m) + \sqrt{(n - 2 - 2m)^2 - 8(n - 2 - m)}}{2}.
\end{equation}

Observe that $n - 2 - 2m \geq 0$ and $\lambda_1$, $\lambda_2$ are the roots of the quadratic polynomial $P(z) \equiv z^2 + bz + c = 0$ where $b = -(n - 2 - 2m) \leq 0$ and $c = 2(n - 2 - m) > 0$. ($P(z)$ comes from equation (2.8) below.) Thus both $\lambda_1$, $\lambda_2$ are positive if they are real.

Then, direct computations by (1.9) and (1.10) show that $\lambda_1$, $\lambda_2 \in \mathbb{R}$ (i.e., $(n - 2 - 2m)^2 - 8(n - 2 - m) \geq 0$) if and only if $n > 10$ and $p \geq p_c$, where

\begin{equation}
p_c = \begin{cases} 
\frac{(n-2)^2-4n+8\sqrt{n-1}}{(n-2)(n-10)} & \text{if } n > 10, \\
\infty & \text{if } n \leq 10.
\end{cases}
\end{equation}
Or, alternatively, one may proceed as follows: \( \lambda_1, \lambda_2 \in \mathbb{R} \) if and only if the discriminant of \( P(z) \)

\[
 b^2 - 4c = (n - 2 - 2m)^2 - 4(n - 2 - 2m) - 4(n - 2)
\]

is non-negative, i.e., if and only if \( n - 2 - 2m \geq 2 + 2\sqrt{n - 1} \) where \( 2 + 2\sqrt{n - 1} \) is the positive root of

\[
 Q(z) = z^2 - 4z - 4(n - 2) = 0,
\]

which in turn is equivalent to that \( n > 10 \) and

\[
 p \geq 1 + \frac{4}{n - 4 - 2\sqrt{n - 1}}.
\]

Therefore it follows that

\[
 (1.11') \quad p_c = \begin{cases} 
 1 + \frac{4}{n - 4 - 2\sqrt{n - 1}} & \text{if } n > 10, \\
 \infty & \text{if } n \leq 10.
\end{cases}
\]

Note that \( \lambda_1 = \lambda_2 \) when \( p = p_c \). Next we discuss the notions of stability and weak asymptotic stability. To this end we introduce a scale of weighted norms as follows. For \( \lambda > 0 \), we define

\[
 (1.12) \quad \|\psi\|_{\lambda} = \sup_{x \in \mathbb{R}^n} \left(1 + |x|\right)^{\lambda} |\psi(x)|,
\]

and

\[
 (1.13) \quad \|\psi\|_{\lambda} = \sup_{x \in \mathbb{R}^n} \left(1 + |x|\right)^{\lambda} \frac{|\psi(x)|}{\log(2 + |x|)},
\]

where \( \psi \) is a non-negative continuous function in \( \mathbb{R}^n \). We say that a steady state \( u_\alpha \) of (1.1) is stable with respect to the norm \( \|\cdot\|_{\lambda} \) if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for \( \|\varphi - u_\alpha\|_{\lambda} < \delta \) we always have \( \|u(\cdot, t; \varphi) - u_\alpha\|_{\lambda} < \varepsilon \) for all \( t > 0 \); \( u_\alpha \) is said to be weakly asymptotically stable with respect to \( \|\cdot\|_{\lambda} \) if \( u_\alpha \) is stable with respect to \( \|\cdot\|_{\lambda} \) and there exists \( \delta > 0 \) such that for \( \|\varphi - u_\alpha\|_{\lambda} < \delta \) we have \( \|u(\cdot, t; \varphi) - u_\alpha\|_{\lambda'} \to 0 \) as \( t \to \infty \) for all \( \lambda' < \lambda \). Similarly we define the stability and weak asymptotic stability with respect to the norm \( \|\cdot\|_{\lambda} \).

Our main results may now be stated as follows. (In the rest of this paper \( u_\alpha \) will always denote the radial positive steady state of (1.1) with \( u_\alpha(0) = \alpha \) given by Proposition B.)

**Theorem 1.14.** Suppose that \( p_c > p \geq \frac{n + 2}{n - 2} \). Then the following conclusions hold.
(i) If \( \varphi \leq u_\alpha \) and \( \varphi \neq u_\alpha \) for some \( \alpha > 0 \), then \( \|u(\cdot, t; \varphi)\|_{L^\infty(\mathbb{R}^n)} \to 0 \) as \( t \to \infty \).

(ii) If \( \varphi \geq u_\alpha \) and \( \varphi \neq u_\alpha \) for some \( \alpha > 0 \), then the solution \( u(x, t; \varphi) \) must blow up in finite time.

**Theorem 1.15.**

(i) If \( p = p_c \) then any positive steady state \( u_\alpha \) of (1.1) is stable with respect to the norm \( \|\cdot\|_{m+\lambda_1} \) and is weakly asymptotically stable with respect to the norm \( \|\cdot\|_{m+\lambda_1} \).

(ii) For \( p > p_c \), any positive steady state \( u_\alpha \) of (1.1) is stable with respect to the norm \( \|\cdot\|_{m+\lambda_1} \) and is weakly asymptotically stable with respect to the norm \( \|\cdot\|_{m+\lambda_2} \).

(Recall that \( \lambda_1 = \lambda_2 \) when \( p = p_c \).) Theorems 1.14 and 1.15 imply that for \( \frac{n+2}{n-2} \leq p < p_c \), all the positive radial steady states of (1.1) are unstable in any "reasonable" sense and, as \( p \) becomes bigger than or equal to \( p_c \), they become stable and weakly asymptotically stable in some appropriate sense. The fact that the transition from instability to stability takes place at \( p = p_c \) is closely related to the drastic change of the "collective" behavior of all the positive radial steady states of (1.1); that is, any two positive radial solutions of (1.6) must intersect each other when \( p_c > p \geq \frac{n+2}{n-2} \), while for \( p \geq p_c \), no two of them can intersect each other. (See Proposition 2.16 below.)

Theorem 1.14 follows from a parabolic comparison argument in [16] and an elliptic comparison lemma in [7]. However, more important and new ideas are needed in proving Theorem 1.15. First, a comparison argument devised in [7] is used to construct a sequence of pairs of super- and sub-solutions of (1.6) for each \( u_\alpha \) in such a way that \( u_\alpha \) is the only solution included in each of these pairs. Then precise asymptotic expansions of those super- and sub-solutions and \( u_\alpha \) are needed and properties of the coefficients of the terms in these expansions are studied in order to obtain the stability and weak asymptotic stability.

It will be clear that our results in this paper extend to more general nonlinearities including \( |x|^{\ell} u^p \) (for \( \ell > -2 \)) as in [16]. However, to keep our presentation transparent we shall only consider the simple case (1.1).

Parabolic problems in entire space (or unbounded domains) have not been treated as successfully as their counterparts for bounded domains, partly due to the lack of compactness. Various types of stabilities on entire space problems have been studied besides the works mentioned earlier. In particular, \( L^\infty \)-stability has been studied by H. Matano in [14] for the following problem

\[
\begin{aligned}
\left\{ 
\begin{array}{l}
u_t &= \Delta u + f(u) \quad \text{in } \mathbb{R}^n \times (0, \infty), \\
u(x, 0) &= \varphi(x) \quad \text{in } \mathbb{R}^n,
\end{array}
\right.
\end{aligned}
\]
where \( f(0) = 0 \) and \( f'(0) < 0 \). The main result in [14] claims that for a steady state \( v \) of (1.16) with \( \lim_{x \to \infty} v(x) = 0 \) if there exist a sequence of time-independent strict super-solutions \( \psi_1 > \psi_2 > \cdots \to v \) and a sequence of time-independent strict sub-solutions \( \phi_1 < \phi_2 < \cdots \to v \), then \( v \) is stable with respect to \( L^\infty \) norm. The key observation in [14] is that the condition \( f'(0) < 0 \) takes care of the possible difficulties at \( x = \infty \), since, if \( v \) is a steady state of (1.16) with \( |v(x)| < \delta \) for \( x \) near \( \infty \) where \( \delta > 0 \) is sufficiently small, then, not only \( v \to 0 \) at \( \infty \), \( v \) must actually decay exponentially at \( \infty \). For equation (1.1), this is clearly not the case as is easily seen from Theorem C and Theorems 1.14 and 1.15.

This paper is organized as follows. In Section 2 we recall some known results and give detailed asymptotic expansions of solutions as well as appropriate super- and sub-solutions of (1.6) following the work of Yi Li in [13]. Theorem 1.4 is then proved in Section 3 and the proof of Theorem 1.15 is included in Section 4.

2. Preliminaries

First we give the asymptotic expansions of positive radial solutions of the equation

\[
\Delta u + Ku^p = 0
\]

in \( \mathbb{R}^n \), where \( K \) is a positive radially symmetric function with \( K \equiv 1 \) near \( \infty \), say, in \( (R, \infty) \), and \( p \geq p_c \).

It is straightforward to show that for \( n > 10 \) there exists a finite sequence \((p_c =)p_1(n) < p_2(n) < \cdots < p_N(n)\) such that \( \lambda_2(n, p) = k \lambda_1(n, p) \) if and only if \( p = p_k(n) \), where

\[
N = \left\lfloor \frac{(\sqrt{n-2} + \sqrt{n-10})^2}{8} \right\rfloor
\]

and \( \lfloor a \rfloor \) = the largest integer which is smaller than \( a \). It is not hard to see that

\[
p_k(n) = \frac{n + 2 - z_k}{n - 2 - z_k}, \quad k = 1, 2, \ldots, N,
\]

where \( z_k \) is the only zero of \( f(z) - k = 0 \) and the function

\[
f(z) = \left[ \frac{z + \sqrt{z^2 - 4z - 4(n - 2)}}{4(z + n - 2)} \right]^2, \quad z \in \left[ n - 2 - \frac{4}{p_c - 1}, n - 2 \right]
\]
is strictly increasing in \([n - 2 - \frac{4}{p_c - 1}, \infty)\). It is also possible to give a more explicit expression for \(p_k(n)\). To this end we set \(q = n - 2 - 2m\). Then \(\lambda_2 = k\lambda_1\) if and only if

\[
k = \frac{q + \sqrt{Q(q)}}{q - \sqrt{Q(q)}},
\]

which is equivalent to

\[
\frac{k - 1}{k + 1} = \frac{\sqrt{Q(q)}}{q} = \left[1 - \frac{4}{q} - \frac{4(n - 2)}{q^2}\right]^{1/2},
\]

where \(Q\) is defined in Section 1 immediately after (1.11). Squaring both sides of (2.4) and multiplying by \(q^2\) we obtain

\[
\left[1 - \left(\frac{k - 1}{k + 1}\right)^2\right]q^2 - 4q - 4(n - 2) = 0.
\]

Now, \(p_k(n)\) may be obtained by solving \(q\) explicitly. Incidentally, the fact that \(k \leq N\) also follows easily from (2.4) since \(q < n - 2\) and then

\[
\frac{k - 1}{k + 1} < \frac{\sqrt{Q(n - 2)}}{n - 2} = \left(\frac{n - 10}{n - 2}\right)^{1/2}.
\]

It follows from Theorem 1 in [13] that if \(u\) is a positive radial solution of (2.1), then \(\lim_{r \to \infty} r^m u(r)\) must always exist. A more detailed asymptotic expansion of \(u\) was also obtained by Yi Li; see [13], Lemmas 4.3 and 4.4. The following theorem is a slight extension of Li’s original result.

**Theorem 2.5.** Let \(u\) be a positive radial solution of (2.1) with \(p \geq p_c\) and \(\lim_{r \to \infty} r^m u(r) > 0\), where \(m = \frac{2}{p - 1}\) and \(K\) is a positive radial function with \(K \equiv 1\) in \((R, \infty)\). Then the following statements hold.

(i) For \(p = p_k(n), k = 1, \ldots, N\), we have \(\lambda_2 = k\lambda_1\) and, near \(\infty\),

\[
u(r) = \frac{L}{r^m} + \frac{a_1}{r^{m + \lambda_1}} + \cdots + \frac{a_{k-1}}{r^{m + (k-1)\lambda_1}}
\]

\[
+ \frac{a_k \log r}{r^{m + k\lambda_1}} + \frac{b_1}{r^{m + \lambda_2}} + \cdots + O\left(\frac{1}{r^{n-2+\epsilon}}\right).
\]

(ii) For \(p_k(n) < p < p_{k+1}(n), k = 1, \ldots, N\) (with the convention that \(p_{N+1}(n) = \infty\)), we have \(k\lambda_1 < \lambda_2 < (k+1)\lambda_1\) and, near \(\infty\),

\[
u(r) = \frac{L}{r^m} + \frac{a_1}{r^{m + \lambda_1}} + \cdots + \frac{a_k}{r^{m + k\lambda_1}}
\]

\[
+ \frac{b_1}{r^{m + \lambda_2}} + \frac{c}{r^{m + (k+1)\lambda_1}} + \cdots + O\left(\frac{1}{r^{n-2+\epsilon}}\right).
\]
The constant $L$ is given by (1.8) and is independent of the particular solution $u$. The coefficients $a_2, a_3, \ldots, a_N$ are uniquely determined once $a_1$ is determined. Moreover, once $a_1$ and $b_1$ are determined then all the coefficients in (2.6) and (2.7) are uniquely determined.

Before giving the proof of Theorem 2.5, a few remarks are in order. First of all, Theorem 2.5 is stated in such a way that the forms of expansions (2.6) and (2.7) are exactly the ones that will be needed in the proof of Theorem 1.15. In particular, it will be clear from the proof below what the missing terms in (2.6) and (2.7) are. (See, e.g., (2.12) below.) Moreover, it will also be clear from the proof below that the expansions (2.6) and (2.7) do not have to stop at $O(r^{2-n-\varepsilon})$; they can go on to an arbitrarily high order.

Proof of Theorem 2.5: We start with the proof of (ii). We follow closely the method developed in [13] but, since the constants in [13] are not handled carefully enough for our purposes, we have to repeat part of the arguments in [13]. First, observe that

$$\lim_{r \to \infty} r^m u(r) = L$$

by Theorem 1 in [13]. Setting $W(s) = r^m u(r) - L$ where $s = \log r$, we see that $W$ satisfies the equation

$$W_{ss} + (n - 2 - 2m)W_s + 2(n - 2 - m)W + g(W) = 0$$

in $s \geq S_0 = \log R$ and $g(\tau) = (\tau + L)^p - L^p - pL^{p-1}\tau$. By standard arguments (e.g., (4.29) in [13] or, (8.7) in [8], p. 64) it follows that

$$W(s) = a \exp{-\lambda_1 s} + b \exp{-\lambda_2 s}$$

$$+ \frac{1}{\lambda_2 - \lambda_1} \int_{S_0}^{s} \left( \exp{\lambda_2 (s' - s)} \right) - \exp{\lambda_1 (s' - s)} \right) g \left( W(s') \right) \, ds'$$

where $a, b$ are two constants. Notice that $\lambda_1, \lambda_2$ are the roots of the characteristic polynomial of the linear part of (2.8). For each positive integer $M \geq 2$, $g(\tau)$ admits the following expansion

$$g(\tau) = d_2 \tau^2 + d_3 \tau^3 + \cdots + d_M \tau^M + O(\tau^{M+1})$$

near $\tau = 0$, where the constants $d_2, \ldots, d_M$ depend only on $p$ and $n$. Then the first step in our proof is provided by (4.28) in [13]; i.e., when $k = 1$ we have

$$W(s) = a_1 \exp{-\lambda_1 s} + O(\exp{-\lambda_2 s})$$
near $s = \infty$ (since the case $k = 1$ corresponds to the case $m_1 < m < m_0$ there). Substituting this and (2.10) (with $M = 2$) into (2.9) we obtain (using the fact that $\int_{S_0}^3 = \int_{S_0}^\infty - \int_3^\infty$)

$$W(s) = a_1 \exp \{-\lambda_1 s\} + b' \exp \{-\lambda_2 s\}$$

$$= a_1 \exp \{-\lambda_1 s\} + b' \exp \{-\lambda_2 s\} + a_2 \exp \{-2\lambda_1 s\} + c_1 \exp \{-2(\lambda_1 + \lambda_2)(s')\}$$

(2.11)

where the constants $a_2, b', b_1$ and $b_1$ are defined by the equalities. Note that $a_2 = d_2 a_1^2 c(\lambda_1, \lambda_2)$ where the constant $c(\lambda_1, \lambda_2)$ depends only on $\lambda_1, \lambda_2$, thus $a_2$ depends only on $a_1, p$ and $n$. Now, substituting this expansion for $W$ and (2.10) (with $M = 3$) into (2.9), by similar computation, we have

$$W(s) = a_1 \exp \{-\lambda_1 s\} + b_1 \exp \{-\lambda_2 s\} + a_2 \exp \{-2\lambda_1 s\}$$

$$+ c_{11} \exp \{-(\lambda_1 + \lambda_2)s\} + b_2 \exp \{-2\lambda_2 s\}$$

$$+ a_3 \exp \{-3\lambda_1 s\} + O(\exp \{-(\lambda_1 + \lambda_2)s\})$$

for $s \geq S_0$ where $a_i = a_i(a_1, p, n), b_2 = b_2(b_1, p, n)$ and $c_{11} = c_{11}(a_1, b_1, p, n)$. Iterating this process, after finitely many steps (with the integer $M$ in (2.10) getting larger each time) we arrive at, for each positive integer $\ell$,

$$W(s) = \sum_{i=1}^{\ell+k} a_i \exp \{-i\lambda_1 s\} + \sum_{j \in J} b_j \exp \{-j\lambda_2 s\}$$

(2.12)

$$+ \sum_{(i, j) \in I} c_{ij} \exp \{-(i\lambda_1 + j\lambda_2)s\} + O(\exp \{-(\ell\lambda_1 + \lambda_2)s\})$$

where $k = 1$ and

$$J = \{j \in \mathbb{Z} \mid j \geq 1 \text{ and } j\lambda_2 < \ell\lambda_1 + \lambda_2\},$$

$$I = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} \mid i \geq 1, j \geq 1 \text{ and } i\lambda_1 + j\lambda_2 < \ell\lambda_1 + \lambda_2\},$$

and $a_i$'s depend only on $a_1, p, n$, $b_j$'s depend only on $b_1, p, n$, and $c_{ij}$'s depend only on $a_1, b_1, p, n$. (Here $\mathbb{Z}$ is the set of all integers.) Taking $\ell$ large enough (e.g., $\ell > (n - 2)/\lambda_1$) we obtain (2.7) for the case $k = 1$. 
For $k > 1$, the proof of (2.7) is similar. Our starting point still is (4.28) in [13] which says that in case $k > 1$

$$W(s) = a_1 \exp \{-\lambda_1 s\} + O(\exp \{-2\lambda_1 s\})$$

near $s = \infty$. As before, substituting this and (2.10) into (2.9) we have

$$W(s) = a_1 \exp \{-\lambda_1 s\} + a_2 \exp \{-2\lambda_1 s\} + O(\exp \{-\min \{3\lambda_1, \lambda_2\} s\})$$

(2.13) near $s = \infty$, where $a_2$ depends only on $a_1$, $p$, and $n$ (but is independent of $b$). (Here we ought to point out that although the derivation of (2.13) is similar to that of (2.11), an additional trick that $\int_{S_0}^s = \int_0^s - \int_0^{S_0}$ is needed in handling the first part of the integral in (2.9) while the second part of that integral can be handled by $\int_{S_0}^s = \int_s^\infty - \int_s^{\infty} as before.) Substituting (2.13), (2.10) into (2.9) and iterating this process, after $(k - 1)$ steps we arrive at

$$W(s) = a_1 \exp \{-\lambda_1 s\} + a_2 \exp \{-2\lambda_1 s\} + \cdots + a_k \exp \{-k\lambda_1 s\} + O(\exp \{-\lambda_2 s\})$$

(2.14) near $s = \infty$. Repeating this process once more, we obtain

$$W(s) = a_1 \exp \{-\lambda_1 s\} + \cdots + a_k \exp \{-k\lambda_1 s\} + b_1 \exp \{-\lambda_2 s\} + O(\exp \{-(k + 1)\lambda_1 s\})$$

(2.15) near $s = \infty$. Now iterating the above process with (2.15), (2.10), and (2.9), after finitely many steps we reach (2.12) and (2.7) is thus established.

Part (i) may be proved similarly by the arguments above together with the proof of Lemmas 4.3 and 4.4 in [13]. We omit the details here.

In studying the parabolic problem (1.1), it is actually very important to understand the global qualitative behavior of its steady states and their mutual relations. Proposition B already gives us a fairly good understanding of them. We shall need much more, however, in order to study their stability properties. Besides Theorem 2.5, we shall also make use of the following result.

**Proposition 2.16.**

(i) For $p_c > p \geq \frac{p+2}{n+2}$, every positive radial solution of (1.6) must intersect the singular solution $U$ (defined in (1.2)), and every two positive radial solutions of (1.6) must intersect each other.

(ii) For $p \geq p_c$, no two positive radial solutions of (1.6) can intersect each other.
Although this result might have been known earlier, we can only find it stated and proved explicitly in [16], Propositions 3.5 and 3.7.

One of the key ingredients used in the proofs of our main results Theorems 1.14 and 1.15 is a comparison argument first devised by Gui in [7]. To describe it we begin with a standard definition.

**Definition 2.17.** A function $v$ is a super-solution of the equation

$$\Delta u + f(x,u) = 0$$

in an open set $\Omega$ in $\mathbb{R}^n$ if $\Delta v + f(x,v) \leq 0$ in $\Omega$. And we say that $v$ is a sub-solution of (2.18) in $\Omega$ if $\Delta v + f(x,v) \geq 0$ in $\Omega$.

Gui's comparison lemma concerns super- and sub-solutions of the linear equation

$$\Delta u + k(x)u = 0$$

in $B_R$, the ball of radius $R$ centered at the origin.

**Lemma 2.20.** Suppose that $w_1$ is a positive radial super-solution of (2.19) in $B_R$ and $w_2$ is a radial sub-solution of (2.19) in $B_R$ with $w_2(0) > 0$. Then

$$w_2(r) \geq \frac{w_2(0)}{w_1(0)} w_1(r)$$

for all $0 \leq r \leq R$. Moreover,

$$w_2(R) > \frac{w_2(0)}{w_1(0)} w_1(R)$$

if one of the functions $w_1, w_2$ is not a solution of (2.19).

**Proof:** Our first step is to show that $w_2 > 0$ in $B_R$. Suppose that this is not the case, i.e., there exists $0 < \rho < R$ such that $w_2 > 0$ in $B_{\rho}$ and $w_2(\rho) = 0$. Then $w'_2(\rho) < 0$ by Hopf's boundary point lemma. (See, e.g., [6]. Note that no sign condition on $k$ is needed here.) On the other hand, it follows from Green's identity that

$$n\omega_n \rho^{n-1} w_1(\rho) w'_2(\rho) = [w_1(\rho) w'_2(\rho) - w_2(\rho) w'_1(\rho)] n\omega_n \rho^{n-1}$$

$$= \int_{\partial B_\rho} \left( w_1 \frac{\partial w_2}{\partial \nu} - w_2 \frac{\partial w_1}{\partial \nu} \right) = \int_{B_R} (w_1 \Delta w_2 - w_2 \Delta w_1)$$

$$\geq \int_{B_R} [w_1(-k w_2) + w_2 k w_1] = 0$$

since both $w_1, w_2$ are positive in $B_\rho$. Thus $w'_2(\rho) \geq 0$, a contradiction.
Next, to establish (2.21) we again begin with Green's identity in $B_r$, $0 < r < R$, as follows.

\[ n \omega_n r^{n-1} \left[ w_1(r)w'_2(r) - w_2(r)w'_1(r) \right] = \int_{B_r} \left( w_1 \Delta w_2 - w_2 \Delta w_1 \right) \geq 0, \]

which implies that for all $0 < r < R$

\[ \frac{w'_2(r)}{w_2(r)} \geq \frac{w'_1(r)}{w_1(r)}. \]

Integrating (2.24) from 0 to $r$, we obtain (2.21). The strict inequality (2.22) follows from the observation that the strict inequalities in (2.23) and (2.24) hold for some $r < R$ if one of the functions $w_1, w_2$ is not a solution of (2.19).

The basic tool we shall use to carry the information from steady states to solutions of the parabolic problem (1.1) is the super- and sub-solution method for parabolic equations. We include a brief discussion here. Consider the Cauchy problem

\[ \begin{cases} u_t = \Delta u + f(u) & \text{in } \mathbb{R}^n \times (0, T], \\ u(x, 0) = \varphi(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{2.25} \]

where $f$ is locally Lipschitz continuous. We say that $u$ is a continuous weak super-solution of (2.25) if

(i) $u$ is continuous on $\Omega_T = \mathbb{R}^n \times [0, T]$ and $u(\cdot, 0) \geq \varphi$;

(ii) for any $\eta \in C^{2,1}(\Omega_T)$ with $\eta \geq 0$ and supp $\eta(\cdot, t)$ being compact in $\mathbb{R}^n$ for all $t \in [0, T]$, we have

\[ \int_{\mathbb{R}^n} u(x, t)\eta(x, t) \, dx \bigg|_{t=0}^{t=T'} \geq \int_0^{T'} \int_{\mathbb{R}^n} \left[ u(x, s)(\Delta \eta + \eta_t)(x, s) + \eta(x, s)f(u(x, s)) \right] \, dx \, ds \]

for all $T' \in [0, T]$.

Continuous weak sub-solutions may be defined in a similar way by reversing the inequalities in (i) and (2.26) above. Note also that the above definitions may be applied to time-independent functions (by setting $u(x, t) \equiv u(x)$ for all $t \geq 0$) to obtain the weak super- and sub-solutions of the corresponding elliptic equation $\Delta u + f(u) = 0$. The following result is basic to our approach.
PROPOSITION 2.27.

(i) If \( \bar{u} \) and \( u \) are bounded continuous weak super- and sub-solutions of (2.25) respectively, then \( \bar{u} \geq u \) on \( \Omega_T \) and (2.25) has a unique classical solution \( u \) with \( \bar{u} \geq u \geq u \) on \( \Omega_T \).

(ii) If the initial value \( \varphi \) in (2.25) is a bounded continuous weak super-solution (sub-solution) of the elliptic equation \( \Delta u + f(u) = 0 \) in \( \mathbb{R}^n \), then the solution of (2.25) is strictly decreasing (increasing, respectively) in \( t \) as long as it exists provided that \( \varphi \) is not a steady state of (2.25).

(iii) If \( \varphi \) is radially symmetric, so is \( u \) in \( x \)-variable.

This proposition is well known in the classical case. In the present form, part (i) is a special case of Lemmas 1.2 and 1.3 in [16], and, parts (ii) and (iii) may be proved by the same arguments used in the proof of Lemma 2.6 in [16].

In the case where \( p < p_c \), the following result plays an important role in the proof of Theorem 1.14.

PROPOSITION 2.28. Suppose that \( p_c > p > \frac{n}{n-2} \).

(i) If \( \varphi \leq \psi \) in \( \mathbb{R}^n \) where \( \psi \) is a radial continuous weak super-solution but not a solution of (1.6), then the solution \( u(x, t; \varphi) \) of (1.1) exists globally in time with \( u \leq \psi \) and \( \|u(\cdot, t; \varphi)\|_{L^\infty(\mathbb{R}^n)} \to 0 \) as \( t \to \infty \).

(ii) If \( \varphi \geq \psi \) in \( \mathbb{R}^n \) where \( \psi \) is a radial continuous weak sub-solution but not a solution of (1.6), then the solution \( u(x, t; \varphi) \) of (1.1) blows up in finite time.

The proof of Proposition 2.28 may be found as special cases of Theorems 3.6 and 3.10 in [16].

Finally, we recall the well-known Pohozaev identity. The following version may be found in [4], Lemma 3.7.

LEMMA 2.29. Let \( u \) be a positive solution of the equation \( \Delta u + f(x, u) = 0 \) in a bounded smooth domain \( \Omega \) in \( \mathbb{R}^n \). Then

\[
\int_{\Omega} \left[ nF(x, u) - \frac{n-2}{2} uf(u) + x \cdot \nabla F(x, u) \right] \, dx
\]

\[
= \int_{\partial\Omega} \left[ (x \cdot \nabla u) \frac{\partial u}{\partial \nu} - (x \cdot \nu) \frac{\|\nabla u\|^2}{2} + (x \cdot \nu)F(x, u) + \frac{n-2}{2} u \frac{\partial u}{\partial \nu} \right]
\]

where \( F(x, u) = \int_0^u f(x, z) \, dz \), \( \nabla F(x, u) = \frac{\partial}{\partial x} F(x, u) \) and \( \nu \) is the unit outer normal to \( \partial\Omega \).
A useful consequence of the above identity is the following

**Corollary 2.30.** Suppose that \( p > \frac{n+2}{n-2} \) and \( h \) is a radial smooth function which satisfies

\[
1 + h > 0 \quad \text{and} \quad \left( \frac{n}{p+1} - \frac{n-2}{2} \right)(1 + h) + \frac{x \cdot \nabla h}{p+1} \leq 0
\]

in \( \mathbb{R}^n \). Then for each \( \beta > 0 \) the problem

\[
\begin{cases}
u'' + \frac{n-1}{r}v' + (1 + h)v^p = 0, \\
v(0) = \beta, \ v'(0) = 0,
\end{cases}
\]

always has a positive solution \( v_\beta \) in \([0, \infty)\).

**Proof:** By standard arguments one sees that (2.32) always has a unique solution \( v_\beta \) near \( r = 0 \) and the solution is decreasing wherever it exists and remains positive. Suppose that \( v_\beta(R) = 0 \) and \( v_\beta > 0 \) in \([0, R)\) for some \( R \). Then from Lemma 2.29 we conclude

\[
\int_{B_R} \left[ \left( \frac{n}{p+1} - \frac{n-2}{2} \right)(1 + h) + \frac{x \cdot \nabla h}{p+1} \right] v_\beta^{p+1} = \frac{R}{2} \int_{\partial B_R} \left( \frac{\partial v_\beta}{\partial \nu} \right)^2.
\]

Since \( \frac{\partial v_\beta}{\partial \nu} > 0 \) at \( |x| = R \) by Hopf's boundary point lemma, this contradicts (2.31). Thus \( v_\beta \) must be positive in \([0, \infty)\).

### 3. Proof of Theorem 1.14

An important step in proving Theorem 1.14 lies in the study of the first intersection points of nearby radial solutions of the elliptic equation (1.6). We set \( Z(\alpha, \beta) \) to be the first zero of \( u_\alpha - u_\beta \) where \( \alpha > \beta > 0 \). Then \( Z(\alpha, \beta) < \infty \) for all \( \alpha > \beta > 0 \) when \( p_c > p \geq \frac{n+2}{n-2} \) by Proposition 2.16. Moreover, \( Z(\alpha, \beta) \) has the following monotonicity property.

**Lemma 3.1.** Assume that \( p_c > p \geq \frac{n+2}{n-2} \). Then for every fixed \( \alpha > 0 \), \( Z(\alpha, \beta) \) is strictly decreasing in \( \beta \) for \( \alpha > \beta > 0 \). In fact, we have

\[
Z(\alpha, \beta) < \min\{Z(\alpha, \gamma), Z(\beta, \gamma)\}
\]

for \( \alpha > \beta > \gamma > 0 \).
Proof: Setting \( w_1 = u_\alpha - u_\beta \) we have \( w_1 > 0 \) in \( 0 \leq r < Z(\alpha, \beta) \), and \( \Delta w_1 + k_1w_1 = 0 \) where
\[
k_1 \equiv \frac{u_\alpha^p - u_\beta^p}{u_\alpha - u_\beta} > pu_\beta^{p-1}
\]
in \( |x| < Z(\alpha, \beta) \). Next, setting \( w_2 = u_\beta - u_\gamma \) we have similarly that \( w_2 > 0 \) in \( 0 \leq r < Z(\beta, \gamma) \) and \( \Delta w_2 + k_2w_2 = 0 \) where
\[
k_2 \equiv \frac{u_\beta^p - u_\gamma^p}{u_\beta - u_\gamma} < pu_\beta^{p-1}
\]
in \( |x| < Z(\beta, \gamma) \). Suppose for contradiction that \( Z(\beta, \gamma) \leq Z(\alpha, \beta) \). Applying Lemma 2.20 (with \( k = pu_\beta^{p-1} \) and \( R = Z(\beta, \gamma) \) there) we see from (2.22) that \( w_2 > 0 \) at \( r = R = Z(\beta, \gamma) \). This contradicts the definition of \( Z(\beta, \gamma) \). Therefore \( Z(\alpha, \beta) < Z(\beta, \gamma) \) which automatically guarantees that \( Z(\alpha, \beta) < Z(\alpha, \gamma) \) since \( \alpha > \beta > \gamma \).

We now begin to prove part (i) of Theorem 1.14. Without loss of generality we may assume that \( \varphi < u_\alpha \) in \( \mathbb{R}^n \). For, the assumption that \( \varphi \leq u_\alpha \) and \( \neq u_\alpha \) together with the strong maximum principle for parabolic equations immediately imply that \( u(x, t; \varphi) < u_\alpha(x) \) for all \( x \in \mathbb{R}^n \) and \( t > 0 \). Thus we may replace \( \varphi \) by \( u(\cdot, \varepsilon; \varphi) \) for some \( \varepsilon > 0 \) if necessary.

Next, observe that by Proposition 2.28 it suffices to construct a radial continuous weak super-solution \( \psi \) of (1.6) which is not a solution of (1.6) such that \( \varphi < \psi \) in \( \mathbb{R}^n \). To this end we first observe that \( u_\beta \to u_\alpha \) uniformly in \([0, Z(\alpha, \frac{\alpha}{2})]\) as \( \beta \to \alpha \), since \( Z(\alpha, \frac{\alpha}{2}) < \infty \). Thus there exists \( \frac{\alpha}{2} < \beta' < \alpha \) such that \( \varphi < u_{\beta'} \) in \([0, Z(\alpha, \frac{\alpha}{2})]\). Setting
\[
\psi(r) = \begin{cases} 
  u_{\beta'}(r) & \text{if } r \leq Z(\alpha, \beta') \\
  u_\alpha(r) & \text{if } r \geq Z(\alpha, \beta')
\end{cases}
\]
we see that \( \varphi(x) < \psi(|x|) \leq u_\alpha(x) \) for all \( x \in \mathbb{R}^n \) since \( Z(\alpha, \beta') < Z(\alpha, \frac{\alpha}{2}) \) by Lemma 3.1. On the other hand, it is standard to verify that \( \psi \) is a continuous weak super-solution of (1.6). (See, e.g., the proof of Proposition 3.8 in [16].) This completes the proof of part (i).

Part (ii) of Theorem 1.14 may be handled in a similar fashion. As before, we may assume without loss of generality that \( \varphi > u_\alpha \). Since \( u_\beta \to u_\alpha \) uniformly in \([0, Z(\alpha, \frac{\alpha}{2})]\), there exists \( \tilde{\beta} > \alpha \) such that \( \varphi > u_{\tilde{\beta}} \) in \([0, Z(\alpha, \frac{\alpha}{2})]\). From (3.2) it follows that \( Z(\tilde{\beta}, \alpha) < Z(\alpha, \frac{\alpha}{2}) \). Thus, setting
\[
\tilde{\psi}(r) = \begin{cases} 
  u_{\tilde{\beta}}(r) & \text{if } r \leq Z(\tilde{\beta}, \alpha) \\
  u_\alpha(r) & \text{if } r \geq Z(\tilde{\beta}, \alpha)
\end{cases}
\]
we have $\varphi(x) > \tilde{\psi}(|x|) \geq u_\alpha(x)$ for all $x \in \mathbb{R}^n$. Since $\tilde{\psi}$ is a continuous weak sub-solution of (1.6) (cf. the proof of Proposition 3.8 in [16]), our conclusion follows from Proposition 2.28.

4. Proof of Theorem 1.15

First we show that $u_\alpha$ is stable with respect to the norm $\| \cdot \|_{m+\lambda_1}$ in case $p > p_c$. This is an easy consequence of Theorem 2.5 and part (ii) of Proposition 2.16. For, from Theorem 2.5, it follows that for any $\varepsilon > 0$ there exists an $R_{\alpha, \varepsilon}$ such that in $[R_{\alpha, \varepsilon}, \infty)$ we have

$$u_\alpha(r) = \frac{L}{r^m} + \frac{a_{1, \alpha}}{r^{m+\lambda_1}} + E_\alpha(r)$$

where $|E_\alpha(r)| \leq \varepsilon/r^{m+\lambda_1}$ in $[R_{\alpha, \varepsilon}, \infty)$. For $\beta$ near $\alpha$, say, $|\beta - \alpha| < \alpha/2$, since

$$u_\beta(r) = \frac{\beta}{\alpha} u_\alpha \left( \left( \frac{\beta}{\alpha} \right)^{1/m} r \right),$$

we conclude that there exist constants $R_\varepsilon$ and $C$, both independent of $\beta$, such that

$$u_\beta(r) = \frac{L}{r^m} + \frac{a_{1, \beta}}{r^{m+\lambda_1}} + E_\beta(r)$$

in $[R_\varepsilon, \infty)$ where $|E_\beta(r)| \leq C\varepsilon/r^{m+\lambda_1}$ in $[R_\varepsilon, \infty)$ and $a_{1, \beta} = (\frac{\beta}{\alpha})^{\lambda_1/m} a_{1, \alpha}$. Thus, in $[R_\varepsilon, \infty)$ we have

$$|r^{m+\lambda_1}(u_\beta - u_\alpha)| \leq |a_{1, \beta} - a_{1, \alpha}| + 2C\varepsilon.$$

Since $(1+r)^{m+\lambda_1}(u_\beta - u_\alpha) \to 0$ uniformly in $[0, R_\varepsilon]$ as $\beta \to \alpha$, it follows that

$$\limsup_{\beta \to \alpha} \|u_\beta - u_\alpha\|_{m+\lambda_1} \leq 4C\varepsilon.$$

Since $\varepsilon$ is arbitrary, we conclude that

$$\lim_{\beta \to \alpha} \|u_\beta - u_\alpha\|_{m+\lambda_1} = 0.$$

This in particular implies that for any given $\varepsilon > 0$, there exists $\eta \in (0, \frac{\alpha}{2})$ such that $\|u_{\alpha+\eta} - u_\alpha\|_{m+\lambda_1} < \varepsilon$. For this $\eta$, we claim that there exists $\delta > 0$ such that if $\|\varphi - u_\alpha\|_{m+\lambda_1} < \delta$ then $u_{\alpha-\eta} \leq \varphi \leq u_{\alpha+\eta}$. From this assertion and Proposition 2.27 (i) our conclusion that $u_\alpha$ is stable under the norm $\| \cdot \|_{m+\lambda_1}$ follows immediately. We now proceed to prove this assertion. Since $p > p_c$, 
Proposition 2.16 (ii) guarantees that \( u_{\alpha+\eta} > u_\alpha \) and therefore \( a_{1,\alpha+\eta} > a_{1,\alpha} \). Thus it follows from our arguments above that for any \( \varepsilon' > 0 \)

\[
 r^{m+\lambda_i}(u_{\alpha+\eta} - u_\alpha) = (a_{1,\alpha+\eta} - a_{1,\alpha}) + [E_{\alpha+\eta}(r) - E_\alpha(r)] r^{m+\lambda_i} \\
 \geq (a_{1,\alpha+\eta} - a_{1,\alpha}) - 2C \varepsilon'
\]

in \([R_\varepsilon', \infty)\). If we choose \( \delta \leq \frac{1}{2}(a_{1,\alpha+\eta} - a_{1,\alpha}) \) and \( \varepsilon' < (a_{1,\alpha+\eta} - a_{1,\alpha})/(4C) \), then in \([R_\varepsilon', \infty)\) we have, for any \( \|\varphi - u_\alpha\|_{m+\lambda_i} < \delta \),

\[
 r^{m+\lambda_i}(u_{\alpha+\eta} - \varphi) \geq r^{m+\lambda_i}(u_{\alpha+\eta} - u_\alpha) - |r^{m+\lambda_i}(u_\alpha - \varphi)| \\
 \geq (a_{1,\alpha+\eta} - a_{1,\alpha}) - 2C \varepsilon' - \delta > 0.
\]

On the other hand, we can always choose \( \delta \) even smaller if necessary so that \( \varphi < u_{\alpha+\eta} \) in \([0, R_\varepsilon')\). Hence \( \varphi < u_{\alpha+\eta} \) in \([0, \infty)\). The other inequality that \( \varphi > u_{\alpha-\eta} \) in \([0, \infty)\) may be derived by similar arguments, and our assertion is established.

The case \( p = p_c \) can be handled in a similar fashion, we therefore omit the details.

The proof of the weak asymptotic stability of \( u_\alpha \) is, however, far more involved. One of the main ingredients in our proof is supplied by the following result.

**Theorem 4.1.** Suppose that \( p \geq p_c \). Then for each fixed positive radial solution \( u_\alpha \) of (1.6) there exist a sequence of radial strict super-solutions \( \tilde{u}_{\alpha}^{(1)} > \tilde{u}_{\alpha}^{(2)} > \cdots > u_\alpha \) and a sequence of radial strict sub-solutions \( u_{\alpha}^{(1)} < u_{\alpha}^{(2)} < \cdots < u_\alpha \) such that \( u_\alpha \) is the only solution of (1.6) in the ordered interval \( u_{\alpha}^{(k)} < u_\alpha < U_{\alpha}^{(k)} \) for every \( k \). Moreover,

\[
(4.2) \quad \lim_{k \to \infty} \tilde{u}_{\alpha}^{(k)} = u_\alpha = \lim_{k \to \infty} u_{\alpha}^{(k)}.
\]

**Proof:** For each \( p \geq p_c \), there exists a non-negative nontrivial smooth function \( h \) such that both \( h \) and \( -h \) satisfy (2.31) with \( \text{supp } h \subseteq B_1 \), the unit ball. Denoting the solution of the problem

\[
\begin{cases}
 v'' + \frac{n-1}{r}v' + (1 \pm h)v^p = 0 \quad \text{in } (0, \infty), \\
v(0) = \beta > 0, \, v'(0) = 0,
\end{cases}
\]

by \( v_\beta^\pm \), respectively, we see by Corollary 2.30 that both \( v_\beta^\pm \) exist and are positive in \([0, \infty)\). Obviously, \( v_\beta^+ \) is a strict super-solution of (1.6) and \( v_\beta^- \) is
a strict sub-solution of (1.6). We shall use $v^\pm_\beta$ to construct the required $\bar{u}^{(k)}_\alpha$ and $u^{(k)}_\alpha$. The construction is divided into five steps. Roughly speaking, we observe that $u^\beta_{_{\beta}} \geq v^+_\beta$ in Step 1, while in Step 2, it is established that there exist small $\beta_1 > \alpha_1 > 0$ such that $v^+_\beta_1 > u^\alpha_{_{\beta_1}}$. In Step 3 we prove that there exist small $0 < \gamma_1 < \alpha_1$ such that $u_{_{\alpha_1}} > v^-_{\gamma_1}$. Then $\bar{u}^{(1)}_\alpha$ and $u^{(1)}_\alpha$ are constructed in Step 4 and $\bar{u}^{(k)}_\alpha$ and $u^{(k)}_\alpha$, $k = 2, 3, \ldots$, are constructed in the last step.

**Step 1:** $u^\beta_{_{\beta}} \geq v^+_\beta$ for all $\beta > 0$.

It suffices to show that $u^\delta > v^+_\beta$ for every $\delta > \beta$. Suppose that this is not true, i.e., there exist $\delta > \beta$ and $R > 0$ such that $w^2_\delta \equiv u^\delta - v^+_\beta > 0$ in $[0, R)$ and $w^2_\delta(R) = 0$. Then $w^2$ satisfies

$$\begin{cases} 
\Delta w^2 + k^2 w^2 \geq 0 & \text{in } B_R, \\
w^2(0) = \delta - \beta > 0 
\end{cases}$$

where

$$k^2 \equiv \frac{u^p_\delta - (v^+_{\beta})^p}{u^\delta - v^+_\beta} < p u^{p-1}_\delta$$

in $B_R$. Choosing $\eta > \delta$ and setting $w^1_\delta \equiv u^\eta - u^\delta$, we have $w^1 > 0$ in $\mathbb{R}^n$ (by Proposition 2.16 (ii)) and $\Delta w^1 + k^1 w^1 = 0$ in $\mathbb{R}^n$ where

$$k^1 \equiv \frac{u^p_\eta - u^p_\delta}{u^\eta - u^\delta} > p u^{p-1}_\delta.$$

But then Lemma 2.20 (with $k \equiv p u^{p-1}_\delta$ in (2.19)) implies that $w^2_\delta(R) > 0$, a contradiction.

**Step 2:** For every sufficiently small $\beta_1 > 0$, there exists $0 < \alpha_1 < \beta_1$ such that $v^+_\beta_1 > u^\alpha_{_{\beta_1}}$ in $\mathbb{R}^n$.

Since $h$ is supported in $B_1$,

$$h \leq \frac{p}{2} \left[ \left( \frac{u^1}{u^\beta_1} \right)^{p-1} - 1 \right]$$

in $\mathbb{R}^n$ for all $\beta_1 > 0$ sufficiently small. For each such small $\beta_1$ setting $\alpha_1 = \frac{1}{2} \min_{B_1} v^+_\beta_1$ (which is smaller than $\frac{1}{2} \beta_1$) and $w^3_\beta = v^+_\beta_1 - u^\alpha_{_{\beta_1}}$, we have

$$w^3 \geq v^+_\beta_1 - \alpha_1 \geq \frac{1}{2} v^+_\beta_1 \quad \text{in } B_1.$$
Suppose that there exists an $R > 0$ such that $w_3 > 0$ in $B_R$ and $w_3(R) = 0$. Then $w_3$ satisfies $\Delta w_3 + k_3 w_3 = 0$ in $B_R$ where

$$k_3 \equiv \frac{\left( v_{\beta_1}^+ \right)^p - u_{\alpha_1}^p}{v_{\beta_1}^+ - u_{\alpha_1}} + h \frac{\left( v_{\beta_1}^+ \right)^p}{v_{\beta_1}^+ - u_{\alpha_1}}.$$

Thus, for $0 \leq r < R$

$$k_3 < \begin{cases} p \left( v_{\beta_1}^+ \right)^{p-1} + h \cdot 2 \left( v_{\beta_1}^+ \right)^{p-1} & \text{if } r \leq 1, \\ p \left( v_{\beta_1}^+ \right)^{p-1} & \text{if } 1 < r < R, \\ \leq (p + 2h)u_{\beta_1}^{p-1} & \end{cases}$$

by (4.3) and Step 1, which in turn implies that $k_3 < pu_{\alpha_1}^{p-1}$ in $B_R$ by the choice of $\beta_1$. Therefore $\Delta w_3 + (pu_{\alpha_1}^{p-1})w_3 \leq 0$ in $B_R$.

On the other hand, if we set $w_4 = u_2 - u_1$ then

$$\Delta w_4 + \left( pu_{\alpha_1}^{p-1} \right) w_4 \leq \Delta w_4 + k_4 w_4 = 0$$

where $k_4 \equiv (u_2^p - u_{\alpha_1}^p)/(u_2 - u_1)$ in $\mathbb{R}^n$. Since $w_4 > 0$ in $\mathbb{R}^n$ (by Proposition 2.16) which is a super-solution of (2.19) with $k \equiv pu_{\alpha_1}^{p-1}$, and $w_3(0) > 0$, Lemma 2.20 applies and we conclude that $w_3(R) \geq w_3(0)w_4(R) > 0$, a contradiction, and Step 2 is established.

**Step 3:** For each $\alpha_1$ in Step 2, there exists $\gamma_1 > 0$ such that $u_{\alpha_1} > v_{\gamma}^-$ for all $0 < \gamma \leq \gamma_1$.

We set $\gamma_1 = \frac{1}{2} \min_{B_i} u_{\alpha_1}$ and $w_5 = u_{\alpha_1} - v_{\gamma}^-$ where $0 < \gamma < \gamma_1$. If there exist $R > 0$ such that $w_5 > 0$ in $B_R$ and $w_5(R) = 0$, then $w_5$ satisfies $\Delta w_5 + k_5 w_5 = 0$ in $B_R$ where

$$k_5 \equiv \frac{u_{\alpha_1}^p - (v_{\gamma}^-)^p}{u_{\alpha_1} - v_{\gamma}^-} + h \frac{(v_{\gamma}^-)^p}{u_{\alpha_1} - v_{\gamma}^-}$$

$$< \begin{cases} pu_{\alpha_1}^{p-1} + h \frac{(v_{\gamma}^-)^p}{u_{\alpha_1} - v_{\gamma}^-} & \text{if } r \leq 1, \\ pu_{\alpha_1}^{p-1} & \text{if } 1 \leq r \leq R \\ \leq pu_{\alpha_1}^{p-1} & \end{cases}$$

if $\gamma$ is chosen even smaller, say, $\gamma \leq \gamma_1$. This implies that $\Delta w_5 + (pu_{\alpha_1}^{p-1})w_5 \geq 0$ in $B_R$ and $w_5(0) > 0$. Comparing with $w_4$ we see immediately that $w_5(R) \geq$
Step 4: For each \( \alpha > 0 \) there exist a radial strict super-solution \( \bar{u}_{\alpha}^{(1)} \) of (1.6) and a radial strict sub-solution \( \underline{u}_{\alpha}^{(1)} \) of (1.6) such that

\[
\bar{u}_{\alpha}^{(1)} > u_{\alpha} > \underline{u}_{\alpha}^{(1)}
\]

in \( \mathbb{R}^n \). Moreover, \( u_{\alpha} \) is the only solution of (1.6) which satisfies (4.4).

From Steps 2 and 3 it follows that there exist small \( \beta_1 > \alpha_1 > \gamma_1 > 0 \) such that \( v_{\beta_1}^+ > u_{\alpha_1} > v_{\gamma_1}^- \) in \( \mathbb{R}^n \). Now, fix \( \beta_1 \) and \( \gamma_1 \), and define

\[
\alpha_1' = \sup \{ \alpha \in (\gamma_1, \beta_1) \mid v_{\beta_1}^+ > u_{\alpha} > v_{\gamma_1}^- \text{ in } \mathbb{R}^n \}
\]

and

\[
\alpha_1'' = \inf \{ \alpha \in (\gamma_1, \beta_1) \mid v_{\beta_1}^+ > u_{\alpha} > v_{\gamma_1}^- \text{ in } \mathbb{R}^n \}
\]

Obviously we have

\[
v_{\beta_1}^+ \geq u_{\alpha_1'} \geq u_{\alpha_1} \geq u_{\alpha_1''} \geq v_{\gamma_1}^-
\]

in \( \mathbb{R}^n \). Then, for each given \( \alpha > 0 \) we set

\[
\bar{u}_{\alpha}^{(1)}(r) = \frac{\alpha}{\alpha_1'} v_{\beta_1}^+ \left( \left( \frac{\alpha}{\alpha_1'} \right)^{\frac{n-1}{2}} r \right) \quad \text{and} \quad \underline{u}_{\alpha}^{(1)}(r) = \frac{\alpha}{\alpha_1''} v_{\gamma_1}^- \left( \left( \frac{\alpha}{\alpha_1''} \right)^{\frac{n-1}{2}} r \right).
\]

By standard scaling arguments (cf. Proposition B) we have

\[
\bar{u}_{\alpha}^{(1)}(r) \geq \frac{\alpha}{\alpha_1'} u_{\alpha_1'} \left( \left( \frac{\alpha}{\alpha_1'} \right)^{\frac{n-1}{2}} r \right) = u_{\alpha}(r)
\]

by (4.5), and similarly \( u_{\alpha} \geq u_{\alpha}^{(1)} \). Since \( h \geq 0 \) and \( \alpha \neq 0 \), \( u_{\alpha}^{(1)} \) and \( u_{\alpha}^{(1)} \) are strict super- and sub-solutions of (1.6) respectively. Hence \( \bar{u}_{\alpha}^{(1)} > u_{\alpha} > \underline{u}_{\alpha}^{(1)} \) in \( \mathbb{R}^n \) by the strong maximum principle.

It remains to show that \( u_{\alpha} \) is the only solution of (1.6) which satisfies (4.4). Suppose for contradiction that there exists \( \beta \neq \alpha \) such that \( \bar{u}_{\alpha}^{(1)} > u_{\beta} > \underline{u}_{\alpha}^{(1)} \) in \( \mathbb{R}^n \). Without loss of generality we may assume that \( \beta > \alpha \). From \( \bar{u}_{\alpha}^{(1)} > u_{\beta} \) it follows that

\[
v_{\beta_1}^+ (r) > \frac{\alpha'}{\alpha} u_{\beta} \left( \left( \frac{\alpha'}{\alpha} \right)^{\frac{n-1}{2}} r \right) = u_{\beta} \left( \left( \frac{\alpha'}{\alpha} \right)^{\frac{n-1}{2}} r \right) > u_{\alpha_1'}(r) \geq v_{\gamma_1}^-(r)
\]
since $\frac{\rho_{\alpha_1}}{\alpha} > \alpha_1$. This, however, contradicts the definition of $\alpha_1$. Hence $u_\alpha$ is the only solution of (1.6) satisfying (4.4).

*Step 5:* Setting

$$h_\alpha(r) = h \left( \left( \frac{\alpha}{\alpha_1} \right)^{\frac{\rho - 1}{2}} r \right) \quad \text{and} \quad h_\alpha(r) = -h \left( \left( \frac{\alpha}{\alpha_1} \right)^{\frac{\rho - 1}{2}} r \right),$$

we have immediately that

$$\Delta u^{(1)}_\alpha + \left( 1 + h_\alpha \right) \left( \bar{u}^{(1)}_\alpha \right)^p = 0$$

and

$$\Delta u^{(1)}_\alpha + \left( 1 + h_\alpha \right) \left( \bar{u}^{(1)}_\alpha \right)^p = 0$$

in $\mathbb{R}^n$. Now, considering the equation

$$(4.7)_k \quad \Delta u + \left( 1 + \frac{h_\alpha}{k} \right) u^p = 0$$

in $\mathbb{R}^n$ we see that for $k = 2$, $\bar{u}^{(1)}_\alpha$ is a strict super-solution of $(4.7)_2$ and $u_\alpha$ is a strict sub-solution of $(4.7)_2$. Thus $(4.7)_2$ has a radial solution $\bar{u}^{(2)}_\alpha$ with $u_\alpha < \bar{u}^{(2)}_\alpha < \bar{u}^{(1)}_\alpha$ by the usual barriers method. (See, e.g., the arguments used in Theorem 2.10 in [15].) Iterating this argument, we obtain a sequence of radial strict super-solutions of (1.6) $\bar{u}^{(1)}_\alpha > \bar{u}^{(2)}_\alpha > \cdots > u_\alpha$ in $\mathbb{R}^n$. Similarly, a sequence of radial strict sub-solutions $\bar{u}^{(1)}_\alpha < \bar{u}^{(2)}_\alpha < \cdots < u_\alpha$ may be constructed by using $h_\alpha$ and the corresponding equations

$$(4.7)_k \quad \Delta u + \left( 1 + \frac{h_\alpha}{k} \right) u^p = 0$$

in $\mathbb{R}^n$. Since $u_\alpha$ is the only solution of (1.6) satisfying (4.4), it must be the only solution of (1.6) with the property that $\bar{u}^{(k)}_\alpha > u_\alpha > \bar{u}^{(k)}_\alpha$, and our construction is complete.

Finally, we shall conclude our proof by establishing (4.2). Since the sequence $\{\bar{u}^{(k)}_\alpha \mid k = 1, 2, \ldots \}$ is bounded and monotonically decreasing, from standard elliptic estimates it follows that its limit $\bar{u}$ must be a (classical) solution of (1.6). (Note that $\frac{h_\alpha}{k} \to 0$ as $k \to \infty$ in $C^2$ norm.) Since $\bar{u} \geq u_\alpha$, $\bar{u}$ must also satisfy (4.4) and, $\bar{u} \equiv u_\alpha$ by the uniqueness. Thus $\bar{u}^{(k)}_\alpha \to u_\alpha$ as $k \to \infty$. Similarly, $\bar{u}^{(k)}_\alpha \to u_\alpha$ as $k \to \infty$ and (4.2) is established.

The rest of the proof of Theorem 1.15 is still technical and lengthy. To make our arguments more transparent, we begin with the case $p_1(n) < p < p_2(n)$. (Other cases are similar but more complicated.)
Our next goal is to use Theorem 2.5 to obtain the asymptotic expansions of the super- and sub-solutions $u^{(k)}_a$, $u^{(k)}_b$, $k = 1, 2, \ldots$, obtained in Theorem 4.1 as well as the solution $u_\alpha$ of (1.6).

Since $u^{(k)}_a$ is a solution of (4.7) with $1 + \frac{\beta}{k} = 1$ outside a finite ball (which is independent of $k$) and $u^{(k)}_a \geq u_\alpha$, Theorem 2.5 applies and we have

$$u^{(k)}_a(r) = \frac{L}{r^m} + a^{(k)}_a + b^{(k)}_a + \cdots + O\left(\frac{1}{r^{n-2+\varepsilon}}\right)$$

near $\infty$ by (2.7). Similarly,

$$u_\alpha(r) = \frac{L}{r^m} + a_\alpha + b_\alpha + \cdots + O\left(\frac{1}{r^{n-2+\varepsilon}}\right)$$

near $\infty$. For sub-solutions $u^{(k)}_b$, Theorem 2.5 still applies and

$$u^{(k)}_b(r) = \frac{L}{r^m} + a^{(k)}_b + b^{(k)}_b + \cdots + O\left(\frac{1}{r^{n-2+\varepsilon}}\right)$$

near $\infty$. For, if

$$\lim_{r \to \infty} r^m u^{(k)}_b(r) = 0$$

for a particular $k$, then $\lim_{r \to \infty} r^{n-2} u^{(k)}_b(r)$ exists and is finite by Theorem 1 in [13]. For this particular $u^{(k)}_b$, considering the solution $u_{\alpha/2}(r)$ of (1.6), since $\lim_{r \to \infty} r^m u_{\alpha/2}(r) = L$, we see that $u_{\alpha/2} > u^{(k)}_b$ near $\infty$, say, in $[R, \infty)$. Since $u_\beta \to u_\alpha$ uniformly in $[0, R]$ and $u_\alpha > u^{(k)}_b$ in $[0, R]$, there exists $\frac{\alpha}{2} < \beta < \alpha$ such that $u_\beta > u^{(k)}_b$ in $[0, \infty)$. Since $u_\beta > u_{\alpha/2}$ in $\mathbb{R}^n$, we conclude that $u_\alpha > u_\beta > u^{(k)}_b$ in $\mathbb{R}^n$ which contradicts the uniqueness of $u_\alpha$ in Theorem 4.1. Thus $\lim_{r \to \infty} r^m u^{(k)}_b(r) > 0$ by Theorem 1 in [13] and our Theorem 2.5 applies and gives (4.10).

It is necessary for our purposes to understand the relations between the coefficients $\tilde{a}^{(k)}_{1,\alpha}$, $a^{(k)}_{1,\alpha}$, $\tilde{b}^{(k)}_{1,\alpha}$, $b^{(k)}_{1,\alpha}$, and $\tilde{b}^{(k)}_{1,\alpha}$. Lemmas 4.11, 4.13, and 4.16 below are essential to our weak asymptotic stability considerations.

**Lemma 4.11.** For every $k$ we have $(\tilde{a}^{(k)}_{1,\alpha} - a^{(k)}_{1,\alpha})^2 + (\tilde{b}^{(k)}_{1,\alpha} - b^{(k)}_{1,\alpha})^2 > 0$ and $(a^{(k)}_{1,\alpha} - a^{(k)}_{1,\alpha})^2 + (\tilde{b}^{(k)}_{1,\alpha} - b^{(k)}_{1,\alpha})^2 > 0$. Furthermore, for every $\ell \neq k$, we have $(\tilde{a}^{(k)}_{1,\alpha} - a^{(k)}_{1,\alpha})^2 + (\tilde{b}^{(k)}_{1,\alpha} - b^{(k)}_{1,\alpha})^2 > 0$ and $(a^{(k)}_{1,\alpha} - a^{(\ell)}_{1,\alpha})^2 + (\tilde{b}^{(k)}_{1,\alpha} - b^{(\ell)}_{1,\alpha})^2 > 0$.

**Proof:** We shall only prove the first inequality since the others may be handled in a similar fashion.
Suppose for some \( k \) that \( \bar{a}^{(k)}_{1,\alpha} = a_{1,\alpha} \) and \( \bar{b}^{(k)}_{1,\alpha} = b_{1,\alpha} \). Then for this particular \( k \) all the coefficients in the expansions (4.8) and (4.9) are the same by Theorem 2.5. Thus

\[
\bar{u}^{(k)}_\alpha(r) - u_\alpha(r) = O \left( \frac{1}{r^{n-2+\epsilon}} \right)
\]

near \( \infty \). Since \( \Delta (\bar{u}^{(k)}_\alpha - u_\alpha) < 0 \) in \( \mathbb{R}^n \) by (4.7) and that \( \bar{u}^{(k)}_\alpha > u_\alpha \), it follows from standard arguments that

\[
\left( \bar{u}^{(k)}_\alpha - u_\alpha \right)(r) \geq C r^{2-n}
\]

near \( \infty \) for some positive constant \( C \). (See the proof of Theorem 3.8 in [15].) This contradicts (4.12) and finishes the proof.

**Lemma 4.13.** \( a^{(k)}_{1,\alpha} = a_{1,\alpha} = \bar{a}^{(k)}_{1,\alpha} < 0 \) for all \( k \) and \( \alpha \).

**Proof:** Since \( u_\alpha(r) = \alpha u_1(\alpha r^{-\frac{1}{\alpha}}) \), we deduce from (4.9) that

\[
\frac{L}{r^m} + \frac{a_{1,\alpha}}{r^{m+\lambda_1}} + \frac{b_{1,\alpha}}{r^{m+\lambda_2}} + \cdots + O \left( \frac{1}{r^{n-2+\epsilon}} \right)
\]

\[
= \frac{\alpha L}{(\alpha^{1/m} r)^m} + \frac{\alpha a_{1,1}}{(\alpha^{1/m} r)^{m+\lambda_1}}
\]

\[
+ \frac{\alpha b_{1,1}}{(\alpha^{1/m} r)^{m+\lambda_2}} + \cdots + O \left( \frac{1}{r^{n-2+\epsilon}} \right)
\]

near \( \infty \) where \( m = \frac{2}{p-1} \). It then follows that

\[
a_{1,\alpha} = \alpha^{-\frac{1}{\alpha}} a_{1,1} \quad \text{and} \quad b_{1,\alpha} = \alpha^{-\frac{1}{\alpha}} b_{1,1}.
\]

By Proposition 2.16 (ii) we conclude that \( u_\alpha \) is monotonically increasing as \( \alpha \) increases. This implies that \( a_{1,\alpha} \) is nondecreasing in \( \alpha \), which in turn implies that \( a_{1,1} \leq 0 \), and \( a_{1,\alpha} \leq 0 \) by (4.15).

From (4.8)–(4.10) it follows easily that \( a^{(k)}_{1,\alpha} \geq a_{1,\alpha} \geq a^{(k)}_{1,\alpha} \) since \( \bar{u}^{(k)}_\alpha > u_\alpha > u^{(k)}_\alpha \). If \( \bar{a}^{(k)}_{1,\alpha} > a_{1,\alpha} \) then \( \bar{a}^{(k)}_{1,\alpha} > a_{1,\beta} \) for all \( \beta \) sufficiently close to \( \alpha \). We then infer from (4.8) and the asymptotic expansion for \( u_\beta \) (with \( \alpha \) replaced by \( \beta \) in (4.9)) that for every \( \beta > \alpha \) and sufficiently close to \( \alpha \) there exists \( R(\beta) \) such that \( \bar{u}^{(k)}_\alpha > u_\beta \) in \( [R(\beta), \infty) \). Since \( u_\beta \) is decreasing in \( \beta \), \( R(\beta) \) may be chosen independent of \( \beta \) if \( \beta \) is sufficiently close to \( \alpha \). That is, \( \bar{u}^{(k)}_\alpha > u_\beta \) in \( [R, \infty) \) for all \( \beta \) sufficiently close to \( \alpha \). On the other hand, \( u_\beta \to u_\alpha \) uniformly on \( [0, R] \) (since \( u_\beta \to u_\alpha \) monotonically as \( \beta \) decreases to \( \alpha \) and \( u_\alpha \) is continuous),
thus $\tilde{u}^{(k)}_\alpha > u_\beta$ on $[0, R]$ and therefore in $\mathbb{R}^n$ for all $\beta$ sufficiently close to $\alpha$. This contradicts the uniqueness assertion in Theorem 4.1. Hence $a^{(k)}_{1,\alpha} = a_{1,\alpha}$.

Similarly we have $a_{1,\alpha} = a^{(k)}_{1,\alpha}$.

It remains to show that $a_{1,\alpha} < 0$. Suppose for contradiction that $a_{1,\alpha} = 0$ for some $\alpha > 0$. Then $a_{1,\alpha} = 0$ for all $\alpha$ by (4.15), and therefore $\tilde{a}^{(k)}_{1,\alpha} = a_{1,\alpha} = a^{(k)}_{1,\alpha}$ for all $k$ by what we have just proved.

We can now repeat the arguments in the previous paragraph (which lead to the conclusion that $\tilde{a}^{(k)}_{1,\alpha} = a_{1,\alpha} = a^{(k)}_{1,\alpha}$) to conclude that $\tilde{b}^{(k)}_{1,\alpha} = b_{1,\alpha} = b^{(k)}_{1,\alpha}$ which clearly gives rise to a contradiction to Lemma 4.11. Therefore $a_{1,\alpha} < 0$ for all $\alpha > 0$ and our proof is complete.

**Lemma 4.16.** $\tilde{b}^{(k)}_{1,\alpha}$ is strictly decreasing to $b_{1,\alpha}$ and $\tilde{b}^{(k)}_{1,\alpha}$ is strictly increasing to $b_{1,\alpha}$ as $k \to \infty$.

Proof: Since $\tilde{u}^{(k)}_\alpha$ is a decreasing sequence with limit $u_\alpha$ and $\tilde{a}^{(k)}_{1,\alpha} = a_{1,\alpha}$ for all $k$ by Lemma 4.13, $\tilde{b}^{(k)}_{1,\alpha}$ must be strictly decreasing in view of Lemma 4.11. Similarly $\tilde{b}^{(k)}_{1,\alpha}$ is strictly increasing in $k$. It remains to show that these two sequences have the same limit $b_{1,\alpha}$. This follows directly from the fact that $\tilde{u}^{(k)}_\alpha \to u_\alpha$ and $\tilde{u}^{(k)}_\alpha \to u_\alpha$ as $k \to \infty$ once we show that the error terms in the expansions (4.8)–(4.10) have a uniform bound in $k$. To obtain such a uniform bound we proceed as follows.

As in Section 2 it is convenient to do the estimates in the variable $s = \log r$ and for the function $W^{(k)}_\alpha(s) = r^m \tilde{u}^{(k)}_\alpha(r) - L$. Since $W^{(k)}_\alpha(s)$ is uniformly bounded above and below respectively by $W^{(1)}_\alpha$ and $W^{(1)}_\alpha(s) \equiv r^m u_\alpha(r) - L$ for all $k$ and $a^{(k)}_{1,\alpha} = a_{1,\alpha}$, we have, from (4.8) and (4.9) that

$$a_{1,\alpha} \exp\{-\lambda_1 s\} - C_1 \exp\{-\lambda_2 s\} \leq W_\alpha(s) \leq W^{(k)}_\alpha(s) \leq W^{(1)}_\alpha(s)$$

$$\leq a_{1,\alpha} \exp\{-\lambda_1 s\} + C_1 \exp\{-\lambda_2 s\}$$

in $s \geq S_0$ where $C_1 > 0$ and $S_0$ are independent of $k$. That is, for $\delta > 0$ sufficiently small,

$$-\delta < a_{1,\alpha} \exp\{-\lambda_1 s\} - C_1 \exp\{-\lambda_2 s\}$$

(4.17) $$\leq W^{(k)}_\alpha(s)$$

$$\leq a_{1,\alpha} \exp\{-\lambda_1 s\} + C_1 \exp\{-\lambda_2 s\} < 0$$
in $s \geq S_0$ if $S_0$ is large enough. (Recall that $a_{1,\alpha} < 0$ by Lemma 4.13.) By the
definition of $g$ in Section 2 (between (2.8) and (2.9)) we know that $g(\tau)$ is
decreasing in $(-\delta, 0)$ for $\delta > 0$ sufficiently small. Thus, for $s \geq S_0$, by (2.9)

\begin{equation}
(4.18) \quad g \left( W^{(k)}_\alpha(s) \right) \leq g(a_{1,\alpha} \exp\{-\lambda_1 s\} - C_1 \exp\{-\lambda_2 s\}) \leq C_2 \exp\{-2\lambda_1 s\}
\end{equation}

where $C_2 > 0$ is also independent of $k$. Similarly it holds that

\begin{equation}
(4.19) \quad g \left( W^{(k)}_\alpha(s) \right) \geq -C_3 \exp\{-2\lambda_1 s\}
\end{equation}

for $s \geq S_0$ and the constant $C_3 > 0$ is independent of $k$. Substituting (4.18) and (4.19) into (2.8) we obtain, after some computations as in Section 2 or in [13], that

\begin{equation}
(4.20) \quad \left| W^{(k)}_\alpha(s) - a^{(k)}_{1,\alpha} \exp\{-\lambda_1 s\} - b^{(k)}_{1,\alpha} \exp\{-\lambda_2 s\} \right| \leq C_4 \exp\{-2\lambda_1 s\}
\end{equation}

for $s \geq S_0$, where the constant $C_4 > 0$ is independent of $k$. (See the derivation
of (2.10) and (2.11).)

Now suppose that $\lim_{k \to \infty} b^{(k)}_{1,\alpha} \neq b_{1,\alpha}$. Then there exists $\varepsilon > 0$ such that

$b^{(k)}_{1,\alpha} > b_{1,\alpha} + \varepsilon$ for $k$ large. From (4.20) it follows that for $s \geq S_0$

\begin{align}
\left| \left( b^{(k)}_{1,\alpha} - b_{1,\alpha} \right) \exp\{-\lambda_2 s\} \right|
& \leq \left| a^{(k)}_{1,\alpha} \exp\{-\lambda_1 s\} + b^{(k)}_{1,\alpha} \exp\{-\lambda_2 s\} - W^{(k)}_\alpha(s) \right|
& \quad + \left| W^{(k)}_\alpha(s) - W_\alpha(s) \right|
& \quad + \left| W_\alpha(s) - (a_{1,\alpha} \exp\{-\lambda_1 s\} + b_{1,\alpha} \exp\{-\lambda_2 s\}) \right|
& \leq C_5 \exp\{-2\lambda_1 s\} + \left| W^{(k)}_\alpha(s) - W_\alpha(s) \right|
\end{align}

(4.21)

where the constant $C_5 > 0$ is independent of $k$. Since $\lambda_2 < 2\lambda_1$ (by our
assumption $p_1(n) < p < p_2(n)$), there exists a number $s_1 > S_0$ such that

$\exp\{(2\lambda_1 - \lambda_2)s_1\} > C_5\varepsilon^{-1}$. Letting $k \to \infty$ in (4.21) we obtain

$\varepsilon \exp\{-\lambda_2 s_1\} \leq C_5 \exp\{-2\lambda_1 s_1\}$,

which contradicts the choice of $s_1$. Therefore $b^{(k)}_{1,\alpha} \to b_{1,\alpha}$ as $k \to \infty$. Similarly,

$b^{(k)}_{1,\alpha} \to b_{1,\alpha}$ as $k \to \infty$, and our proof is complete.

Now the stability of $u_\alpha$ in the norm $\| \cdot \|_{m+\lambda_2}$ is easily established by using
Lemma 4.16. For, given $\varepsilon > 0$, Lemma 4.16 and estimate (4.20) guarantee
that there exists $k'$ such that if $u_{\alpha}^{(k')} \leq v \leq u_{\alpha}^{(k')}$, then $\|v - u_{\alpha}\|_{m+\lambda_2} < \epsilon$. On the other hand, for this $k'$, since $u_{\alpha}^{(k')} > u_{\alpha} > u_{\alpha}^{(k')}$ in $\mathbb{R}^n$ and $b_{1,\alpha}^{(k')} > b_{1,\alpha} > b_{1,\alpha}^{(k')}$, there exists $\delta > 0$ such that if $\|\varphi - u_{\alpha}\|_{m+\lambda_2} < \delta$ then $u_{\alpha}^{(k')} > \varphi > u_{\alpha}^{(k')}$ in $\mathbb{R}^n$. Then Proposition 2.27 implies that $u_{\alpha}^{(k')} > u(\cdot, t; \varphi) > u_{\alpha}^{(k')}$. 

Thus $u_{\alpha}$ is stable with respect to the norm $\|\cdot\|_{m+\lambda_2}$.

To establish the weak asymptotic stability of $u_{\alpha}$ with respect to the norm $\|\cdot\|_{m+\lambda_2}$, it remains to show that there exists $\delta > 0$ such that for $\|\varphi - u_{\alpha}\|_{m+\lambda_2} < \delta$ we always have $\|u(\cdot, t; \varphi) - u_{\alpha}\|_{\lambda'} \to 0$ as $t \to \infty$ for every $\lambda' < m + \lambda_2$. This follows from Theorem 4.1 and Lemma 4.13 almost immediately. For we may choose $\delta > 0$ so small that if $\|\varphi - u_{\alpha}\|_{m+\lambda_2} < \delta$ then $u_{\alpha}^{(1)} \leq \varphi \leq u_{\alpha}^{(1)}$ in $\mathbb{R}^n$. Then Proposition 2.27 implies that

$$u_{\alpha}^{(1)} < u(\cdot, t; u_{\alpha}^{(1)}) < u \left(\cdot, t; \varphi\right) < u \left(\cdot, t; u_{\alpha}^{(1)}\right) < u_{\alpha}^{(1)}$$

in $\mathbb{R}^n$. Since $u_{\alpha}$ is the only steady state satisfying (4.4) and both $u(\cdot, t; u_{\alpha}^{(1)})$ and $u(\cdot, t; u_{\alpha}^{(1)})$ are monotone in $t$, we must have

$$\lim_{t \to \infty} u \left(x, t; u_{\alpha}^{(1)}\right) = u_{\alpha}(x) = \lim_{t \to \infty} u \left(x, t; u_{\alpha}^{(1)}\right).$$

Therefore $u(\cdot, t; \varphi) \to u_{\alpha}$ as $t \to \infty$. Then, for every $\lambda' < m + \lambda_2$ and every $R > 0$ it follows from (4.22) and the expansions (4.8)–(4.10) that

$$\left|(1 + |x|)^{\lambda'}(u(\cdot, t; \varphi) - u_{\alpha}(x))\right| 
\leq \begin{cases} 
C(1 + |x|)^{\lambda'}|x|^{-(m+\lambda_2)} & \text{if } |x| \geq R, \\
(1 + R)^{\lambda'}\|u(\cdot, t; \varphi) - u_{\alpha}\|_{L^\infty(B_R)} & \text{if } |x| < R, \\
CR^{\lambda'-\lambda_2} & \text{if } |x| \leq R.
\end{cases}$$

Letting $t \to \infty$ we obtain

$$\limsup_{t \to \infty} \|u(\cdot, t; \varphi) - u_{\alpha}\|_{\lambda'} \leq CR^{\lambda'-\lambda_2}. $$

Since $R$ is arbitrary, we conclude that $\|u(\cdot, t; \varphi) - u_{\alpha}\|_{\lambda'} \to 0$ as $t \to \infty$. Therefore $u_{\alpha}$ is weakly asymptotically stable with respect to the norm $\|\cdot\|_{m+\lambda_2}$ and the proof of Theorem 1.15 in the case $(p_c =) p_1(n) < p < p_2(n)$ is complete.

The rest of part (ii) of Theorem 1.15 can be handled in an analogous way. One simply notes that in Theorem 2.5 the coefficients $a_1, \ldots, a_N$ are uniquely
determined by \( a_1 \) and thus create no extra difficulties in extending Lemmas 4.11, 4.13, and 4.16 to the more general case \( p > p_1(n) = p_c \).

In proving part (i) of Theorem 1.15 by the above arguments, we first notice that now the two independent terms are \( a_1 r^{-(m + \lambda_i)} \log r \) and \( b_1 r^{-(m + \lambda_i)} \) (which accounts for the slightly different norms used in (i)). As a result of this difference, (4.15) now takes a new form

\[
a_{1,\alpha} = a_{1,1}^{-\frac{\lambda_i}{m}} a_{1,1} \quad \text{and} \quad b_{1,\alpha} = a_{1,1}^{-\frac{\lambda_i}{m}} (b_{1,1} + a_{1,1} \log \alpha).
\]

Since the explicit form of \( b_{1,\alpha} \) in (4.15) was never used in our proof of part (ii), this also causes no additional problem, and part (i) of Theorem 1.15 can now be established by the same arguments we used to handle the case \( p_1(n) < p < p_2(n) \) earlier in this section.

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**Bibliography**


[13] Li, Yi, *Asymptotic behavior of positive solutions of equation \( \Delta u + K(x)u^\rho = 0 \) in \( \mathbb{R}^n \)*, J. Diff. Eqns. 95, 1992, pp. 304–330.


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