Radial Solutions of $\Delta u + f(u) = 0$ with Prescribed Numbers of Zeros*

KEVIN McLEOD, W. C. TROY, † AND F. B. WEISSLER ‡

Department of Mathematics, University of Wisconsin, Milwaukee, Wisconsin 53201;
†Department of Mathematics and Statistics, University of Pittsburgh, Pittsburgh, Pennsylvania 15260; and Département de Mathématiques, Université de Nantes, 44072 Nantes Cedex 03, France

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1. INTRODUCTION

We consider the initial-boundary value problem

$$u'' + \frac{n-1}{r} u' + f(u) = 0 \quad \text{for } r > 0$$  \hspace{1cm} (1.1)

$$u'(0) = 0, \quad u(r) \to 0 \quad \text{as } r \to \infty.$$  \hspace{1cm} (1.2)

Here $n > 1$ is a real parameter, and the function $f$ satisfies the following hypotheses:

(f1) $f: \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous,

(f2) $u \cdot f(u) < 0$ for $|u|$ small, $u \neq 0$,

(f3) There are values $\beta > 0$ and $\beta' < 0$ such that

$$F(u) < 0 \quad \text{on } (0, \beta), \quad f(u) > 0 \quad \text{on } [\beta, \infty)$$

$$F(u) < 0 \quad \text{on } (\beta', 0), \quad f(u) < 0 \quad \text{on } (-\infty, \beta'],$$

where $F(u) = \int_0^u f(s) \, ds$,

(f4) $f(u) = k(u) \, |u|^{p-1} \, u + g(u)$, where

$$1 < p < \frac{n+2}{n-2} \quad (1 < p < \infty \text{ if } n \leq 2),$$

$$k(u) = \begin{cases} k_+ > 0 & (u > 0) \\ k_- > 0 & (u < 0) \end{cases}$$

and

$$|g(u)| = o(|u|^p) \quad \text{as } |u| \to \infty.$$  \hspace{1cm} (f4)

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It should be noted that (f1) and (f2) force \( f(0) = 0 \), while (f3) and (f4) imply that \( F(u) \) has exactly one positive and one negative zero. Clearly, we may take \( \beta \) to be the positive zero of \( F(u) \), and we shall assume this in future. Similarly, we will take \( \beta' \) to be the unique negative zero of \( F(u) \).

Our main result is the following theorem.

**Theorem 1.** Let \( m \geq 0 \) be an integer. If \( f \) satisfies (f1)-(f4) there is a solution of (1.1)-(1.2) which has exactly \( m \) zeros in \((0, \infty)\).

Problem (1.1)-(1.2) has been studied extensively since it arises in the search for radially symmetric solutions of

\[
\Delta u + f(u) = 0 \quad \text{in } \mathbb{R}^n
\]

\[
u \to 0 \quad \text{as} \quad |x| \to \infty
\]

(1.3)

or of standing wave solutions for non-linear Schrödinger equations. (Of course, in these applications, \( n \) must be an integer.) Strauss [8] and Berestycki and Lions [1] have proved the existence of infinitely many radially symmetric solutions of (1.3) by variational methods when \( f(u) \) is an odd function. These arguments apply to very general non-linearities, but they do not give detailed information on the shape of the solution and in particular it was an open question for some years as to whether solutions exist with prescribed numbers of zeros. This question was answered in the affirmative by Jones and Küpper [3] using a dynamical systems approach and an application of the Conley index. Their assumptions on \( f \) are in some respects less general than those of [1], but do cover the important model cases such as \( f(u) = -u + |u|^{p-1}u \). They also do not require \( f \) to be an odd function.

The present paper was motivated partly by the wish to find a "simple" ODE proof of the Jones–Küpper result. It turns out that our conditions on \( f \) are not identical with those of Jones and Küpper, although there is naturally some overlap. We have been able to relax slightly the regularity required, as well as the asymptotic behaviour as \( |u| \to \infty \). We also do not require \( f'(0) < 0 \), replacing it with hypothesis (f2); we can therefore treat examples such as \( f(u) = -|u|^{q-1}u + |u|^{p-1}u \) \((1 < q < p)\) for which \( f'(0) = 0 \). Like Jones and Küpper, we do not require \( f \) to be odd. On the other hand hypothesis (f3), which seems to be necessary to the shooting argument we employ, means that our result is not strictly more general than that of [3].

The plan of the paper is as follows. In Section 2 we prove some elementary facts about Eq. (1.1). They are mostly standard (see for example [4, 6, 7]) but are included for completeness. We also show that there are solutions of (1.1) with arbitrarily large numbers of zeros. We prove this by
a scaling argument: we denote by $u(r, a)$ the solution of (1.1) together with the initial conditions

$$u(0) = a > 0, \quad u'(0) = 0.$$  \hspace{1cm} (1.4)

(It is shown in [6] that this initial value problem has a unique solution.) The functions $u(r, a)$, appropriately scaled, converge as $a \to \infty$ to the solution of

$$w'' + \frac{n-1}{r} w' + k(w) |w|^{p-1} w = 0$$

$$w(0) = 1, \quad w'(0) = 0$$  \hspace{1cm} (1.5) \hspace{1cm} (1.6)

and it is known that, for $1 < p < (n+2)/(n-2)$, $w$ has infinitely many zeros. (This is the only place where the condition on $p$ is used.) In case $k(u) = 1$, this result may be found in [2]. The proof may be easily adapted to the present case, or we may apply a more general result, proved in [5]. In Section 3 we prove our main result. Finally, in Section 4 we state some results for the Dirichlet problem in finite balls and mention some other generalizations of our results. The authors thank J. Serrin for his advice and encouragement with this paper; in particular for showing us a simplification of the proof of Lemma 4.

2. Elementary Results and Scaling

We begin by noting that if $u$ is a solution of (1.1) then $u$ attains a relative minimum at values of $r > 0$ for which $u' = 0$ and $f(u) < 0$; similarly, if $f(u) > 0$ at a critical point, that critical point must be a maximum. (By the uniqueness theorem for solutions of initial value problems a solution of (1.1) cannot have a critical point where $f(u) = 0$ unless it is a constant.)

Next, we multiply (1.1) by $u'$ and obtain

$$\left( \frac{1}{2} u'^2 + F(u) \right)' = -\frac{n-1}{r} u'^2 \leq 0.$$  \hspace{1cm} (2.1)

The quantity $\frac{1}{2} u'^2 + F(u)$ will be called the energy of the solution and denoted $Q(r)$ or $Q(r, a)$ in case $u$ is the solution of (1.1)-(1.4). Since the critical points of a non-constant solution are isolated, we see that $Q(r)$ is strictly decreasing.

Suppose now that a non-constant solution of (1.1) has a critical point, say at $r = r_0 \geq 0$. Suppose $f(u(r_0)) > 0$. Then $u(r_0)$ is a local maximum for
u and so \( u(r) < u(r_0) \) for \( r \) slightly larger than \( r_0 \). Suppose that at some subsequent value of \( r \), say \( r_1 \), we have \( u(r_1) = u(r_0) \). Then we would have

\[
Q(r_1) = \frac{1}{2} u'(r_1)^2 + F(u(r_1)) \geq F(u(r_0)) = Q(r_0),
\]

contradicting the fact that \( Q \) strictly decreases. It follows that \( u(r) < u(r_0) \) for all \( r > r_0 \). Similarly, if \( f(u_0) < 0 \) we would have deduced that \( u(r) < u(r_0) \) for all \( r > r_0 \). The same type of argument also shows that any solution of (1.1) is bounded and must therefore be defined for all \( r > 0 \).

We observe finally that if \( u(r) = 0 \) for some finite value \( r > 0 \) then \( Q(r) = \frac{1}{2} u'(r)^2 > 0 \), while if \( u(r) \to 0 \) as \( r \to 0 \) then \( Q(r) \to 0 \). Thus, if the energy of a solution ever goes negative, that solution cannot subsequently change sign or decay to zero at \( \infty \). (It is easy to see that it must approach a zero of \( f \) as \( r \to \infty \).) In particular, since \( Q(0) = F(u(0, a)) \leq 0 \) for \( 0 < a \leq \beta \), we have

**Lemma 1.** If \( 0 < a \leq \beta \) then \( u(r, a) > 0 \) for all \( r > 0 \) and cannot satisfy \( u(r, a) \to 0 \) as \( r \to \infty \).

Lemma 1 tells us that there are solutions of (1.1) with no zeros at all. We now show that if \( a \) is large \( u(r, a) \) has many zeros; indeed the number of zeros becomes arbitrarily large as \( a \to \infty \). This clearly follows from the following two Lemmas.

**Lemma 2.** For \( \lambda > 0 \), define

\[
v(r, \lambda) = \lambda^{2/\phi(\phi-1)} u \left( \frac{r}{\lambda^{\phi(\phi-1)}} \right).
\]

(2.2)

Then as \( \lambda \to \infty \), \( v(r, \lambda) \to w(r) \) uniformly on compact subsets of \([0, \infty)\), where \( w \) is the solution of (1.5)–(1.6).

**Lemma 3.** For \( 1 < p < (n + 2)/(n - 2) \) \((1 < p < \infty \text{ if } n \leq 2)\) the solution of (1.5)–(1.6) has infinitely many zeros.

The proof of Lemma 3 may be found in [2] or [5]; we prove Lemma 2 here.

**Proof of Lemma 2.** A calculation shows that \( v(r, \lambda) \) satisfies

\[
v'' + \frac{n-1}{r} v' + \lambda^{-2p/(\phi-1)} f(\lambda^{2/(\phi-1)} v) = 0
\]

(2.3)

\[
v(0) = 1, \quad v'(0) = 0
\]

(2.4)
We can define the energy of \( v(r, \lambda) \) as we did for \( u(r, a) \):

\[
E(r, \lambda) = \frac{1}{2} v'(r, \lambda)^2 + \lambda^{-2p/(p-1)} \int_0^{v(r, \lambda)} f(\lambda^{2/(p-1)} s) \, ds.
\] (2.5)

\( E(r, \lambda) \) is strictly decreasing, so for \( \lambda > 0 \),

\[
E(r, \lambda) < E(0, \lambda) = \int_0^1 f(\lambda^{2/(p-1)} s) \, ds
\]

\[
= \lambda^{-(2p + 2)/(p - 1)} F(\lambda^{2/(p - 1)}).
\]

As \( \lambda \to \infty \), hypothesis (f4) shows that this last quantity remains bounded. We deduce that \( E(r, \lambda) \) is bounded above, independently of \( r \) and \( \lambda \). Since also

\[
\int_0^r f(\lambda^{2/(p-1)} s) \, ds = \lambda^{-(2p + 2)/(p - 1)} F(\lambda^{2/(p - 1)} v) \to + \infty
\]
as \( |v| \to \infty \), uniformly in \( \lambda \), at least for \( \lambda \geq 1 \), it follows from (2.5) that \( v(r, \lambda) \) is also bounded:

\[
|v(r, \lambda)| \leq M, \quad \text{for all } r > 0, \quad \lambda \geq 1.
\] (2.6)

Now observe that (2.3)–(2.4) are equivalent to

\[
r^{-1} v'(r, \lambda) = -\lambda^{-2p/(p-1)} \int_0^r s^{n-1} f(\lambda^{2/(p-1)} v(s, \lambda)) \, ds.
\] (2.7)

Using (2.6) and (f4), we see that for \( \lambda \geq 1 \)

\[
|f(\lambda^{2/(p-1)} v(s, \lambda))| \leq \text{const.} \cdot (1 + |v|^{2/(p-1)})
\]

\[
\leq \text{const.} \cdot (1 + \lambda^{2p/(p-1)} M^p)
\]

\[
\leq \text{const.} \cdot \lambda^{2p/(p - 1)}.
\]

Hence, by (2.7),

\[
|v'(r, \lambda)| \leq \lambda^{-2p/(p-1)} \cdot \text{const.} \cdot \lambda^{2p/(p - 1)} \cdot \frac{1}{r^{n-1}} \int_0^r s^{n-1} \, ds
\]

\[
\leq \text{const.} \cdot r
\]

so that \( |v'(r, \lambda)| \) is bounded, independently of \( \lambda \), on compact subsets of \([0, \infty)\). By the Arzela–Ascoli theorem and a standard diagonal argument.
there is a sequence $\lambda_k \to \infty$ and a continuous function $w(r) : [0, \infty) \to \mathbb{R}$ such that

$$v(r, \lambda_k) \to w(r) \quad \text{uniformly on compacta.} \tag{2.8}$$

Now we see that the right hand side of (2.7) converges to

$$-\int_0^r s^{n-1} k(w) |w(s)|^{\rho-1} w(s) \, ds.$$

In particular, the derivatives $v'(r, \lambda_k)$ converge (point-wise) to a function $\phi(r)$. Since $w$ is continuous, so is $\phi$. Since $v'(r, \lambda_k)$ is bounded on compacta, we can apply the dominated convergence theorem to

$$v(r, \lambda_k) = 1 + \int_0^r v'(s, \lambda_k) \, ds$$

and deduce that $\phi(r) = w'(r)$. Consequently

$$w \in C([0, \infty)) \cap C^1((0, \infty)), \quad w(0) = 1, \quad w'(0) = 0 \quad \text{and}$$

$$r^{n-1}w'(r) = -\int_0^r s^{n-1} k(w) |w(s)|^{\rho-1} w(s) \, ds.$$

It now follows easily that $w$ satisfies (1.5).

### 3. Proof of Theorem 1

One of the main tools we need for the proof of our theorem is the following Lemma.

**Lemma 4.** Let $\bar{a} > \beta$ be a value for which $u(r, \bar{a})$ has exactly $k$ zeros ($k \geq 0$) and also $u(r, \bar{a}) \to 0$ as $r \to \infty$. If $|a - \bar{a}|$ is sufficiently small then $u(r, a)$ has at most $k + 1$ zeros on $[0, \infty)$.

Before proving Lemma 4, we will show how it is used to obtain Theorem 1. First, define the set $A_0 = \{a > 0 \mid u(r, a) \text{ has no zeros}\}$. We have shown that $(0, \beta] \subseteq A_0$ so that $A_0$ is non-empty. Also, Lemmas 2 and 3 imply that $A_0$ is bounded above. We define $a_0 = \sup A_0$ and claim that the solution $u(r, a_0)$ satisfies

$$u(r, a_0) > 0 \quad \text{for all } r > 0 \quad \text{and}$$

$$u(r, a_0) \to 0 \quad \text{as } r \to \infty. \tag{3.2}$$
If \( u(r, a_0) \) has a zero at some finite \( r \) then continuity of \( u(r, a) \) on \( a \) implies that \( u(r, a) \) has a finite zero for \( a_0 - a > 0 \) and sufficiently small, contradicting the definition of \( a_0 \). Thus (3.1) must hold. Next, suppose that \( u'(r_0, a_0) < 0 \) at some first \( r_0 > 0 \). Then \( u''(r_0, a_0) > 0 \) so \( u'(r, a_0) > 0 \) for \( r \) slightly larger than \( r_0 \). Continuity implies that if \( a - a_0 \) is sufficiently small, \( u(r, a) \) has a strict local minimum at some first \( r_0(a) \). By the energy arguments of Section 2, \( u(r, a) \) cannot have a zero, again contradicting the definition of \( a_0 \). Thus it must be the case that \( u'(r, a_0) < 0 \) for all \( r > 0 \), so that \( \lim_{r \to \infty} u(r, a_0) = \bar{u} \geq 0 \) exists. Assume for contradiction that \( \bar{u} > 0 \). We recall that the energy \( Q(r, a_0) \) is decreasing, so that \( \lim_{r \to \infty} Q(r, a_0) = \bar{Q} \) also exists. But then

\[
u'(r, a_0)^2 = 2(\bar{Q} - F(u)) \to 2(\bar{Q} - F(\bar{u}))
\]

so \( \lim_{r \to \infty} u'(r, a_0) \) exists. Clearly, this last limit must be 0. From (1.1) we have

\[
u'' = -\frac{n-1}{r} u' - f(u) \to -f(\bar{u})
\]

so we need \( f(\bar{u}) = 0 \). Hypothesis (f3) implies \( \bar{Q} = F(\bar{u}) < 0 \) and it follows that for \( a_0 - a \) sufficiently small, \( Q(r, a) \) becomes negative before the first zero of \( u(r, a) \). By the arguments of Section 2, \( u(r, a) \) can have no zero at all, which again contradicts the definition of \( a_0 \). It therefore follows that \( \bar{u} = 0 \), so (3.2) holds. We now know that \( u(r, a_0) \) is a positive solution of (1.1)–(1.2).

Next, we define the set \( A_1 = \{ a > a_0 \mid u(r, a) \text{ has at most one zero} \} \). It follows from the definition of \( a_0 \) and Lemmas 2 and 4 that \( A_1 \) is bounded above and non-empty. We set \( a_1 = \sup A_1 \) and observe that \( u(r, a) \) has exactly one zero for \( a \in (a_0, a_1) \). We can now show, by arguments similar to those used to prove (3.1) and (3.2), that \( u(r, a_1) \) is a solution of (1.1)–(1.2) with exactly one zero. Proceeding inductively, we can produce solutions with any given number of zeros, proving Theorem 1.

**Proof of Lemma 4.** We will give the proof in the case \( k = 0 \); it will be clear how the details should be modified for \( k > 0 \). The key element in the proof is a differential equation satisfied by \( Q(r, a) \). Recall that \( Q \) was defined by

\[
u = \frac{1}{2} u'^2 + F(u)
\]

so that

\[
u' = -\frac{n-1}{r} u'^2.
\]
Eliminating $u'^2$ between these two equations gives
\[ Q' + \frac{2n-2}{r} Q = \frac{2n-2}{r} F(u). \] (3.3)

We can multiply by the integrating factor $r^{2n-2}$ and write (3.2) as
\[ (r^{2n-2} Q)' = (2n-2) r^{2n-3} F(u), \] (3.4)

an identity used by Peletier and Serrin in [6].

We now assume that $a$ is a close to $\bar{a}$ and that $u(r, a)$ has a zero; say
\[ u(R_1, a) = 0, \quad u > 0 \quad \text{in} \quad [0, R_1). \] (3.5)

Lemma 4 will follow (in case $k = 0$) if we can show that $u(r, a)$ does not have a second zero. This is turn will follow if we can show that $Q(r, a)$ goes negative before any possible second zero of $u$. In fact, we will show:

for any $\gamma < 0$, $Q(r, a)$ has gone negative before $u(r, a) = \gamma$, provided $a$ is sufficiently close to $\bar{a}$. (3.6)

Thus, we fix $\gamma \in (\beta', 0)$ and let $\delta > 0$ be such that
\[ |F(u)| \geq \delta \quad \text{when} \quad u \in (\gamma, \frac{1}{2} \gamma). \] (3.7)

Let $R_2$ and $R_3$ be the first values of $r > R_1$ at which $u(r) = \frac{1}{2} \gamma$ and $u(r) = \gamma$, respectively, and assume for contradiction that $Q(R_3) \geq 0$. Integrating (3.4) from $R_2$ to $R_3$ gives
\[ R_3^{2n-2} Q(R_3) - R_2^{2n-2} Q(R_2) = (2n-2) \int_{R_2}^{R_3} S^{2n-3} F(u(s)) \, ds \]
\[ \leq - (2n-2) \delta (R_3 - R_2) \min(R_3^{2n-3}, R_2^{2n-3}) \]

and, if $Q(R_3) \geq 0$, we obtain
\[ R_2^{2n-2} Q(R_2) \geq (2n-2) \delta (R_3 - R_2) \min(R_3^{2n-3}, R_2^{2n-3}). \]

Since $u'$ is bounded (at least for a close to $\bar{a}$), $R_3 - R_2$ is bounded away from 0. Also, the assumption $Q(R_3) \geq 0$ implies $Q(r) \geq 0$ for $r \leq R_3$; by (3.7), $u'$ is bounded away from 0 in $(R_2, R_3)$, so that the distance $R_3 - R_2$ is bounded. As $a \to \bar{a}$, continuity with respect to $a$ shows that $R_1$, $R_2$, and $R_3$ become arbitrarily large so that $R_2/R_1 \to 1$ and $\min(R_3^{2n-2}, R_2^{2n-2}) \geq \frac{1}{2} R_2^{2n-3}$. Thus, for $a$ sufficiently close to $\bar{a}$, we see that $R_2^{2n-2} Q(R_2) \geq C R_2^{2n-3}$, or
\[ Q(R_2) \geq \frac{C}{R_2}. \] (3.8)
We now let \( \tilde{R} \) be the first value of \( r \) at which \( u(r, \tilde{a}) = \beta \), and we choose a value \( \sigma \in (\tilde{R}, R_1) \), so that \( u(\sigma, \tilde{a}) < \beta \). (\( \tilde{R} \) is a fixed value, so for a close to \( \tilde{a} \) we certainly have \( R_1 > \tilde{R} \).) By the Mean-Value Theorem,

\[
\frac{1}{2} \gamma - \frac{u(\sigma, a)}{R_2 - \sigma} = u'(\zeta, a), \quad \zeta \in (\sigma, R_2).
\]

(3.9)

Since \( Q > C/\sqrt{R_2} \) in \( (\sigma, R_2) \) and \( F(u) < 0 \) in \( (\sigma, R_2) \), we have \( \frac{1}{2} u'' > C/\sqrt{R_2} \), so \( |u'| > C/\sqrt{R_2} \) in \( (\sigma, R_2) \). Using this in (3.9) and observing that the numerator on the left hand side of (3.9) is fixed, we find \( R_2 - \sigma < C \sqrt{R_2} \). For a close to \( \tilde{a} \), \( R_2 \) is large, and so \( \sigma > \frac{1}{2} R_2 \), or \( R_2 < 2\sigma \). Since \( \sigma \) was fixed, this gives a contradiction as \( a \to \tilde{a} \). The contradiction shows that (3.6) is true, and so completes the proof of Lemma 4.

4. ADDITIONAL RESULTS

The scaling argument of Section 2 gives rather more information than was used there. Indeed, let \( z_k(a) \) denote the \( k \)th zero of \( u(r, a) \). For \( a \) sufficiently large, \( z_k(a) \) will exist and be a continuous function of \( a \). Furthermore, as \( a \to \infty \) we see from (2.2) that

\[
a^{(p-1)/2} z_k(a) = k \text{th zero of } v(r, a^{(p-1)/2}) \\
\rightarrow k \text{th zero of } w(r),
\]

where \( w \) satisfies (1.5)–(1.6). In particular, we have

\[
z_k(a) \to 0 \quad \text{as } a \to \infty.
\]

(4.1)

This observation leads immediately to our second theorem, which may be interpreted as given radial solutions for the Dirichlet problem in a ball.

**Theorem 2.** Let \( m \geq 0 \) be an integer, and fix \( R \in (0, \infty) \). If \( f \) satisfies (f1)–(f4) there is a solution of the problem

\[
u'' + \frac{n-1}{r} u' + f(u) = 0
\]

(4.2)

\[
u'(0) = 0, \quad u(R) = 0
\]

(4.3)

which has exactly \( m \) zeros in \( (0, R) \).

Our methods may also be applied to more general equations. For example, it causes no great difficulty to replace the non-linear term \( f(u) \) in
(1.1) by a term $r^\sigma f(u)$ where $\sigma > \max(-2, 2 - 2n)$. If we make the substitution $s = r^{(\sigma + 2)/2}$ the equation will become

$$\frac{d^2 u}{ds^2} + \frac{2n + \sigma - 2}{\sigma + 2} \frac{1}{s} \frac{du}{ds} + f(u) = 0,$$

(4.4)

and our analysis can proceed as before with $n$ replaced throughout by $n'$, where

$$n' - 1 = \frac{2n + \sigma - 2}{\sigma + 2}$$

so that

$$n' = \frac{2n + 2\sigma}{\sigma + 2} > 1,$$

by our assumptions on $\sigma$. More general terms $f(r, u)$ do not seem amenable to our treatment, due to the difficulty of defining a suitable energy function.

Finally, we remark that the coefficient of $u'$ in (1.5) may obviously be replaced by a more general function $g(r)$. The energy may be defined as before, and will satisfy

$$Q' + 2g(r)Q = 2g(r)F(u).$$

We will not write down a complete list of hypotheses on $g$ which will enable the proof to go through, but it can be checked that

$$g(r) = \frac{r}{2} + \frac{n - 1}{r}$$

will work. The corresponding equation

$$u'' + \left(\frac{r}{2} + \frac{n - 1}{r}\right) u' + f(u) = 0$$

occurs in the study of self-similar, radially symmetric solutions of semilinear heat equations [9].

J. Serrin (personal communication) has pointed out that our methods will also apply to the $p$-Laplacian.

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