Radially Symmetric Solutions of $\Delta w - |w|^{p-1}w = 0$

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Abstract
We investigate the existence and asymptotic behavior of radially symmetric singular solutions of $w'' + \frac{N-1}{r} w' - |w|^{p-1}w = 0$, $r \geq 0$, which are unbounded at $r = 0$. We focus on the parameter regime $N > 2$ and $1 < p < \frac{N}{N-2}$ where the equation has the singular solution $w_1 = \left( \frac{4 - 2(N-2)(p-1)}{(p-1)^2} \right)^{\frac{1}{p-1}} r^{\frac{2}{p-1}}$, $r > 0$. Given the above stated parameter regime, we prove the existence and analyze the asymptotic behavior of new positive singular solutions.

Keywords: radially symmetric, singular solution
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1. Introduction
We investigate the behavior of solutions of
\begin{equation}
\Delta w - |w|^{p-1}w = 0, \tag{1.1}
\end{equation}
where $w = w(x_1,..,x_N)$, $N > 1$ and $p > 1$. Solutions of (1.1) are time independent solutions of the nonlinear heat equation
\begin{equation}
\frac{\partial w}{\partial t} = \Delta w - |w|^{p-1}w. \tag{1.2}
\end{equation}
Our focus is on radially symmetric solutions of (1.1) that have the form $w = w(r)$, where $r = (x_1^2 + \cdots + x_N^2)^{1/2}$, and satisfy
\begin{equation}
w'' + \frac{N-1}{r} w' - |w|^{p-1}w = 0, \quad r > 0. \tag{1.3}
\end{equation}
We distinguish two classes of solutions of (1.3). The first consists of nonsingular solutions which are bounded at $r = 0$, and satisfy $(w(0), w'(0)) = (w_0, 0)$,
where \( w_0 \) is finite. The second class consists of solutions of (1.3) that are unbounded, i.e. singular, at \( r = 0 \).

Equation (1.3) has the positive singular solution

\[
   w_1(r) = \left( \frac{4 - 2(N - 2)(p - 1)}{(p - 1)^2} \right)^{\frac{1}{p-1}} r^{\frac{2}{p-1}}, \ N > 2, \ 1 < p < \frac{N}{N - 2}. \tag{1.4}
\]

This solution has the following ‘local integrability’ property: for each \( L > 0 \),

\[
   0 < \int_0^L r^{N-1} w_1^q(r) \, dr < \infty \quad \text{iff} \quad 0 < q < \frac{N(p - 1)}{2}. \tag{1.5}
\]

For example, when \( N = 3 \) and \( q = 1 \),

\[
   0 < \int_0^L r^2 w_1(r) \, dr < \infty \quad \text{when} \quad \frac{5}{3} < p < 3. \tag{1.6}
\]

However, \( w_1(r) \) is not globally integrable for any \( q > 0 \). That is,

\[
   \int_0^\infty r^{N-1} w_1^q(r) \, dr = \infty \quad \forall q > 0, \ N > 2, \ 1 < p < \frac{N}{N - 2}. \tag{1.7}
\]

A related equation. A second, widely studied nonlinear heat equation is

\[
   \frac{\partial v}{\partial t} = \Delta v + |v|^{p-1}v. \tag{1.8}
\]

This equation also has a radially symmetric solution which is positive and singular at \( r = 0 \), namely

\[
   v_1(r) = \left( \frac{2(N - 2)(p - 1) - 4}{(p - 1)^2} \right)^{\frac{1}{p-1}} r^{\frac{2}{p-1}}, \ N > 2, \ \frac{N}{N - 2} < p < \frac{N + 2}{N - 2}. \tag{1.9}
\]

This well known solution has played a central role in the analysis of blowup of solutions of (1.8). For example, when \( v(x_1, \ldots, x_N, 0) \) is appropriately chosen, similarity solutions methods show how \( v(x_1, \ldots, x_N, t) \to cv_1(r) \) as \( t \to \infty \), where \( c > 0 \) is a constant [2, 3, 6]. In a separate paper [7] we proved the existence and asymptotic behavior of positive singular solutions other than \( v_1(r) \). These may also play an important role in the analysis of (1.8).

Specific Aims. In this paper we extend the analysis in [7] and investigate whether (1.3) has positive singular solutions other than \( w_1(r) \). We assume
throughout that $N > 2$ and $1 < p < \frac{N}{N - 2}$, and investigate the following fundamental issues:

(I) Do positive singular solutions exist which are different from $w_1(r)$? What is their asymptotic behavior as $r \to 0^+$, and as $r \to \infty$? How are their integrability properties similar to, or different from, those of $w_1(r)$?

(II) What is the relationship between singular solutions and nonsingular solutions?

Our Approach. Because (1.3) is non-autonomous, it is difficult to use such methods such as topological shooting, or Pohozaev identity arguments, to prove the existence of new singular solutions. Thus, our approach to investigating the questions raised in (I) - (II) is to derive a related equation which is autonomous and more amenable to analysis. For this, let $w(r)$ denote any solution of (1.3), and define

$$h(\tau) = \frac{w(\exp(\tau))}{w_1(\exp(\tau))}, \quad -\infty < \tau < \infty.$$  \hfill (1.10)

Then $h(\tau)$ solves

$$\frac{d^2h}{d\tau^2} + \frac{N - 2}{p - 1} \left( p - \frac{N + 2}{N - 2} \right) \frac{dh}{d\tau} + \frac{2(N - 2)}{(p - 1)^2} \left( p - \frac{N}{N - 2} \right) (|h|^{p-1} - 1)h = 0.$$  \hfill (1.11)

Because (1.11) is autonomous, we can apply phase plane techniques to obtain determine the behavior of solutions. We then use the ‘inverse’ formula

$$w(r) = w_1(r)h(\ln(r)), \quad 0 < r < \infty.$$  \hfill (1.12)

to obtain corresponding solutions of the $w$ equation (1.3). In particular, we will prove that there is a heteroclinic orbit leading from $(0, 0)$ to $(1, 0)$ in the $(h, h')$ phase plane (see Figure 2). We then use (1.12) to show that there is a value $D > 0$ such that the corresponding solution, $w_2(r)$, of (1.3) is a positive singular solution, with

$$\frac{w_2(r)}{w_1(r)} \to 1 \text{ as } r \to 0^+ \quad \text{and} \quad \frac{w_2(r)}{w_1(r)} \sim Dr^{-\left(\frac{N - 2}{p - 1}\right)}(p - \frac{N}{N - 2}) \to 0 \text{ as } r \to \infty.$$  \hfill (1.13)

Thus, $w_2(r) \sim w_1(r)$ as $r \to 0^+$, but $w_2(r) \to 0$ faster than $w_1(r)$ as $r \to \infty$.
2. The Main Result

In this section we analyze solutions of the $h$ equation (1.11), and the corresponding solutions of the $w$ equation (1.3) when $N > 2$ and $1 < p < \frac{N}{N-2}$. Our goals:

**I.** In Theorem 2.1 we classify the behavior of solutions of (1.11). We focus on solutions whose trajectories in the $(h, \frac{dh}{d\tau})$ phase plane form the stable and unstable manifolds of solutions leading to, or from, the constant solutions (0,0) and ($\pm1,0$).

**II.** Following the proof of Theorem 2.1 we analyze its implications for solutions of the $w$ equation (1.3). In particular, we show that the solution of (1.11) described in part (ii) of the theorem generates a new, strictly positive singular solution of (1.3).

**III.** Figure 2 illustrates the behavior of solutions when $(N, p) = (3,2)$.

**Theorem 2.1.** Let $N > 2$ and $1 < p < \frac{N}{N-2}$. Then

(i) There is a two dimensional unstable manifold of solutions of (1.11) that lead from the constant solution $(0,0)$ into the $(h, \frac{dh}{d\tau})$ phase plane.

(ii) There is a one dimensional stable manifold of solutions leading to $(1,0)$ in the $(h, \frac{dh}{d\tau})$ phase plane. One component, $B_1$, points into the $h < 1, \frac{dh}{d\tau} > 0$ region of the phase plane, and its negative counterpart, $B_2$, points into the $h > -1, \frac{dh}{d\tau} < 0$ region (see Figure 2, upper left). If $(h(\tau), h'(\tau)) \in B_1$, then

\[
0 < h(\tau) < 1 \quad \text{and} \quad 0 < h'(\tau) < \frac{N-2}{p-1} \left(\frac{N}{N-2} - p\right) h(\tau) \quad \forall \tau \in \mathbb{R},
\]

\[
\lim_{\tau \to -\infty} (h(\tau), h'(\tau)) = (0,0), \quad \text{and} \quad \lim_{\tau \to -\infty} \frac{h'(\tau)}{h(\tau)} = \frac{N-2}{p-1} \left(\frac{N}{N-2} - p\right).
\]

\[
\lim_{\tau \to -\infty} (h(\tau), h'(\tau)) = (1,0).
\]

**Proof.** (i) A linearization of (1.11) around the constant solution $h \equiv 0$ gives

\[
\frac{d^2 h}{d\tau^2} + \frac{N-2}{p-1} \left(p - \frac{N+2}{N-2}\right) \frac{dh}{d\tau} - \frac{2(N-2)}{(p-1)^2} \left(p - \frac{N}{N-2}\right) h = 0.
\]

The eigenvalues associated with (2.4) are

\[
\mu_1 = \frac{N-2}{p-1} \left(\frac{N}{N-2} - p\right) > 0 \quad \text{and} \quad \mu_2 = \frac{2}{p-1} > 0.
\]
Note that $\mu_2 - \mu_1 = N - 2 > 0$. The existence of the two dimensional unstable manifold of solutions is an immediate consequence of the Stable Manifold Theorem [1]. We will make use of the observation that (1.11) can be written as

$$\frac{d^2 h}{d\tau^2} - (\mu_1 + \mu_2) \frac{dh}{d\tau} + \mu_1 \mu_2 = \mu_1 \mu_2 |h|^{p-1} h.$$

(ii) Linearizing (1.11) about $h \equiv 1$ results in

$$\frac{d^2 h}{d\tau^2} + \frac{N - 2}{p - 1} \left( p - \frac{N + 2}{N - 2} \right) \frac{dh}{d\tau} + \frac{2(N - 2)}{p - 1} \left( p - \frac{N}{N - 2} \right) (h - 1) = 0. \quad (2.7)$$

Define $k = -\frac{2}{p - 1}$. Then (2.7) becomes

$$\frac{d^2 h}{d\tau^2} + \gamma \frac{dh}{d\tau} + 2(\gamma - k)(h - 1) = 0,$$

where

$$\gamma = \frac{N - 2}{p - 1} \left( p - \frac{N + 2}{N - 2} \right) < 0 \quad \text{and} \quad \gamma - k = \frac{N - 2}{p - 1} \left( p - \frac{N}{N - 2} \right) < 0. \quad (2.9)$$

Thus, the eigenvalues associated with (2.7) and (2.8) satisfy

$$\lambda_1 = \frac{-\gamma - \sqrt{\gamma^2 - 8(\gamma - k)}}{2} < 0 \quad \text{and} \quad \lambda_2 = \frac{-\gamma + \sqrt{\gamma^2 - 8(\gamma - k)}}{2} > 0. \quad (2.10)$$

It follows from (2.10) and the Stable Manifold Theorem that there is a one dimensional stable manifold of solutions leading to $(1, 0)$ in the $(h, \frac{dh}{d\tau})$ phase plane. Additionally,

$$\lim_{\tau \to -\infty} \frac{h'(\tau)}{h(\tau) - 1} = \lambda_1 \quad (2.11)$$

for $(h(\tau), h'(\tau))$ on the stable manifold. Thus, solutions on the stable manifold leading to $(1, 0)$ satisfy $h(\tau) > 1$ if $h'(\tau) < 0$ and $h(\tau) < 1$ if $h'(\tau) > 0$. Let $B_1$ denote the component of the stable manifold with $h(\tau) < 1$ and $h'(\tau) > 0$ (see top row of Figure 2). Assume throughout that $(h(0), h'(0)) \in B_1$, and hence (2.3) holds.
It remains to show that (2.1) and (2.2) hold. Because of (2.11) and translation invariance of (2.6), we can choose \(1 - h(0) > 0\) and \(h'(0) > 0\) sufficiently small so that

\[
h(\tau) - 1 < 0 < h'(\tau) \text{ on } [0, \infty),
\]  
(2.12)

and

\[
0 < h(\tau) < 1 \text{ and } 0 < h'(\tau) < \mu_1 h(\tau) \text{ on } [0, \infty).
\]  
(2.13)

The definition of \(B_1\), together with (2.12), imply that the maximal interval of existence is of the form \((\tau_{\text{min}}, \infty)\) where \(\tau_{\text{min}} < 0\).

Next, we show that \(B_1 \subset U^o\) where \(U^o\) is the bounded open triangular region

\[
U^o = \{(h_1, h_2) \mid 0 < h_1 < 1 \text{ and } 0 < h_2 < \mu_1 h_1\}.
\]  
(2.14)

Figure 2 (upper right) shows an example of \(U^o\) with \(N = 3\) and \(p = 2\). Because of (2.13), it is sufficient to show that \((h(\tau), h'(\tau)) \in U^o \forall \tau \in (\tau_{\text{min}}, 0]\). For contradiction, assume that \((h(\tau), h'(\tau))\) leaves \(U^o\) at some point in \((\tau_{\text{min}}, 0]\).

Define the functional

\[
H = \frac{dh}{d\tau} - \mu_1 h.
\]  
(2.15)

It follows from (2.6) that \(H\) satisfies

\[
H' - \mu_2 H = \mu_1 \mu_2 |h|^{p-1} h.
\]  
(2.16)

Suppose that \((h(\tau), h'(\tau))\) leaves \(U^o\) across the line \(H = 0\). Then (see Fig-

![Figure 1](image1.png)

Figure 1: Left panel illustrates \(H(\tau_0) = h'(\tau_0) - \mu_1 h(\tau_0) = 0\) for some \(\tau_0 \in (\tau_{\text{min}}, 0]\). Right panel shows \(h'(\tau_1) = 0\) for some \(\tau_1 \in (\tau_{\text{min}}, 0]\).
dictating (2.19). We conclude that (τ₀, 0), and H(τ₀) = 0. (2.17)

Note that h(τ₀) = 0 and (2.15) imply h'(τ₀) = 0, contradicting uniqueness of the constant solution (h(τ), h'(τ)) ≡ (0, 0). Thus, h(τ₀) > 0. Also, (2.17) implies that

\[ H'(τ₀) \leq 0. \] (2.18)

The fact that h(τ₀) > 0, combined with (2.16), results in

\[ H'(τ₀) = μ₁μ₂(h(τ₀))^p > 0, \]

a contradiction of (2.18).

Thus, (h(τ), h'(τ)) can only leave U° across the line segment h' = 0, 0 < h < 1. If so, then there is a τ₁ ∈ (τ_min, 0) such that 0 < h(τ) < 1 and h'(τ) > 0 on (τ₁, 0), and h'(τ₁) = 0 < h(τ₁) (see Figure 1, right panel). Hence,

\[ h''(τ₁) \geq 0. \] (2.19)

It follows from (2.6) that h''(τ₁) = μ₁μ₂((h(τ₁))p⁻¹)h(τ₁) < 0, contradicting (2.19). We conclude that (h(τ), h'(τ)) cannot leave U° on (τ_min, ∞), hence B₁ ⊂ U° as claimed. Furthermore, since (h(τ), h'(τ)) is bounded, then τ_min = −∞ follows from standard ODE theory. Thus, (h(τ), h'(τ)) ∈ U° on R, implies that h'(τ) > 0 on R, hence, (2.1) is proved.

Proof of (2.2). First, we prove that h → 0⁺ as τ → −∞. Since h'(τ) > 0 and 0 < h(τ) < 1 on R, then 0 ≤ h < 1 where h = \[ \lim_{τ \to -\infty} h. \] To obtain a contradiction suppose that \( \bar{h} > 0 \). Then 0 < \( \bar{h} < 1 \) results in

\[ \frac{d²h}{dτ²} - (μ₁ + μ₂)\frac{dh}{dτ} = μ₁μ₂(h^{p⁻¹} - 1)\bar{h} < 0 \] as τ → −∞. (2.20)

Hence, h'(τ) − (μ₁ + μ₂)h(τ) → ∞ as τ → −∞ which contradicts the fact that U° is bounded and (h(τ), h'(τ)) ∈ U° for all τ ∈ R. Thus, h(τ) → 0⁻ as τ → −∞. Next, we show that h'(τ) → 0⁺ as τ → −∞. Note that 0 < h'(τ) < μ₁h(τ) on (−∞, 0] is an immediate consequence of H'(τ) < 0 and h'(τ) > 0 on (−∞, 0]. Therefore, h'(τ) → 0⁺ as τ → −∞ follows from the fact that h(τ) → 0⁺ as τ → −∞.

Proof of (2.3). Finally, we need to prove that \( ρ = \frac{h'}{h} \) → μ₁ as τ → −∞. The definition of ρ together with (2.6) results in

\[ ρ' + ρ² - (μ₁ + μ₂)ρ = μ₁μ₂(h^{p⁻¹} - 1). \] (2.21)
We show that $\rho \to \mu_1$ monotonically as $\tau \to -\infty$. Differentiating (2.21) yields

$$\rho'' + (2\rho - \mu_1 - \mu_2)\rho' = \mu_1\mu_2(p-1)h^{p-2}h'.$$

Hence, if $\rho'(\tau_*) = 0$ for some $\tau_* \in \mathbb{R}$, then $\rho''(\tau_*) = \mu_1\mu_2(p-1)h^{p-2}(\tau_*)h'(\tau_*) > 0$. This means that $\rho'$ has at most one zero on $\mathbb{R}$. Furthermore, $0 < \rho(\tau) = \frac{h'(\tau)}{h(\tau)} < \mu_1$ for all $\tau \in \mathbb{R}$ is a result of (2.14) together with $(h(\tau), h'(\tau)) \in U^o$ for all $\tau \in \mathbb{R}$. Thus, $\bar{\rho} = \lim_{\tau \to -\infty}\rho$ exists and $0 \leq \bar{\rho} \leq \mu_1$. Furthermore, the fact that $\bar{\rho}$ is finite ensures the existence of a sequence $\{\tau_n\}$ such that $\lim_{\tau_n \to -\infty}\rho'(\tau_n) = 0$. Substituting $\lim_{\tau_n \to -\infty}\rho'(\tau_n) = 0 = \lim_{\tau_n \to -\infty}h(\tau_n)$ and $\bar{\rho} = \lim_{\tau_n \to -\infty}\rho$ into (2.21) results in

$$\bar{\rho}^2 - (\mu_1 + \mu_2)\bar{\rho} + \mu_1\mu_2 = 0. \quad (2.22)$$

The bound $0 \leq \bar{\rho} \leq \mu_1$ and (2.22) imply that $\bar{\rho} = \mu_1$. Thus, $\rho \to \mu_1$ as $\tau \to -\infty$ as claimed. This completes the proof of the theorem.

References


Figure 2: Subcritical case: $n=3$, $p=2$. First row: trajectories of the unstable manifold ($A_1$ and $A_2$) and stable manifold ($B_1$ and $B_2$) leading from (0,0) in the ($h, \frac{dh}{d\tau}$) phase plane. Second and third rows: the $h$ components of solutions along $A_1, A_2, B_1, B_2$ (left column) and corresponding $w$ components along $A_1$ and $B_1$ (right column): $w_0(r)$ is bounded at $r=0$, $w_1(r) = \left(\frac{2}{5}\right)^{1/3} r^{-\frac{2}{3}}$ is the known singular solution, and $w_2(r)$ denotes the new, positive singular solution corresponding to $B_1$. 

[7] Troy, W. & WeKrisner, E. Radially Symmetric Solutions of $\Delta w - |w|^{p-1}w = 0$ preprint, 2010