Noncooperative optimizations of controls for time-periodic Navier-Stokes systems with multiple solutions

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Time-periodic systems governed by differential equations are somewhat difficult to consider in the numerical setting because they may possess many solutions. The number of solutions of such systems may be finite or infinite. Further, some trajectories which are exactly time-periodic over a given period might only approximately solve the governing equation, whereas nearby trajectories which exactly solve the governing equation might only be approximately time-periodic over the given period. The difficulty of the time-periodic setting is compounded in the case of systems governed by the Navier-Stokes equation, as the solutions of such systems in the time-evolving setting may be chaotic and multiscale. When considering the optimization of controls for such systems in the time-periodic setting, the situation is thus particularly delicate, as one doesn't know a priori which time-periodic solution (or approximate solution) one should design the controls for.

In the present work, the idea of noncooperative optimization is applied in an attempt to develop a tractable framework to solve the problem of optimization of controls for time-periodic Navier-Stokes systems. The noncooperative aspect of the optimization, however, is somewhat nonstandard: the best controls are found for the worst (of the many) time-periodic solutions of the governing equation. As the number of solutions may be finite, we have employed a technique developed by Barbu (1998) of first looking at a suitable approximation of the time-periodic system of interest with an infinite number of solutions, finding the solution to this approximate system with a gradient-based algorithm leveraging an adjoint analysis, then refining the level of approximation until we have solved (with a sufficient level of accuracy) the optimization problem we are actually interested in. The present brief note motivates this work, presents the structure of our analysis, and outlines the resulting numerical algorithm. A future paper (under preparation) will describe our mathematical proofs of the associated theorems in detail and present some preliminary numerical results.

Introduction

An essential ingredient which is fundamental to the numerical study of the very physical problem of near-wall turbulence is the very artificial assumption of spatial periodicity (see, e.g., Kim, Moin, & Moser 1987). It is well known in the numerical simulation literature that even though this assumption is highly artificial, it has little to no effect on the quantities of interest in the system (that is, the statistics of the near-wall turbulence) if the problem is formulated correctly (that is, if the computational box is chosen to be large as compared with the correlation length scales of the turbulence).

A possible generalization of this technique is to assume time periodicity in the numerical model of the physical system over a time period which is long with respect to the correlation time scales of the turbulence. Such a technique has the attractive feature that spectral methods may be used in time, affording a high degree of accuracy with a small number of discretization points in time and obviating the need for a CFL constraint on the timestep to insure numerical stability. However, this assumption converts a smaller problem which evolves parabolically in time into a much larger problem which is elliptic in both space and time, necessitating a large, stationary, four-dimensional problem to be solved with a multigrid-type strategy. For this reason, in addition to the several disadvantages mentioned in the first paragraph of the abstract, the time-periodic framework has not found favor in the turbulence simulation literature.

Many fluid-mechanical systems of physical interest, such as jets and wakes, are dominated by the approximately time-periodic phenomenon of vortex shedding. Flow systems dominated by such behavior are typically characterized in numerical simulations by marching the governing equations in time (from random initial conditions) until the flow reaches an approximately time-periodic statistical steady state. For representative examples of such numerical simulations, see the cylinder wake flow simulation of Kravchenko & Moin (2000) and the turbulent round jet flow simulation of Freund (2001). Time-periodic simulations for periods which are large with respect to the shedding period, if they could be made numerically tractable, would certainly be able to capture the quantities of interest in such systems. In fact, one might even hypothesize that time-periodic simulations which are only a few integer multiples of the shedding period might also capture these systems with adequate fidelity.

In the setting of the iterative optimization of controls for flow systems dominated by time-periodic behavior (a setting which is receiving a growing amount of interest for a variety of engineering systems), an important new consideration is introduced: that is, after each small update of the controls, a very good initial guess for the entire trajectory of the controlled flow system is known (*i.e.*, the flow solution before the control was updated). Unfortunately, the time-evolving numerical model can not easily take advantage of this information. To evaluate the effect of the control update on the system using the time-evolving model, the entire system must again be marched in time towards statistical steady state.

Fortunately, the time-periodic numerical model *can* take advantage of the knowledge of this nearby periodic orbit. A very small number of multigrid cycles (perhaps a single W cycle, depending on the scheme implemented) would be needed to update the entire flow solution (over the whole domain of space-time under consideration) when the change to the control distribution is small. This represents a distinct advantage for the time-periodic numerical model when it is to be used as the core of an iterative optimization algorithm. This advantage may indeed tip the scales in favor of such a numerical model in future numerical optimization of controls for such systems.

However, the mathematical infrastructure for the adjoint-based optimization of controls for time-periodic Navier-Stokes systems is not yet in place. In fact, in the standard setting for adjoint analysis of fluid systems (see, e.g., the seminal work of Abergel & Temam 1990), it is not obvious even how to formulate the present problem. We believe that valuable insight into this very practical problem can be gained via mathematical analysis before jumping into large numerical simulations which may or may not converge. The insight we seek includes how to approximate the present optimization problem in order to make it manageable, how to compute the relevant gradient information, how to select which flow solution to optimize for (recall that time-periodic systems in general have multiple solutions), how to refine the level of approximation, and how to insure convergence to a relevant and useful solution.

Complete mathematical analysis of this problem is beyond the scope of this note, and will appear elsewhere. What appears below is a brief skeleton of this analysis with all mathematical proofs and much of the precise mathematical characterization removed. We hope that such a brief presentation might be useful to introduce to the aeronautics and astronautics engineering community a summary of where we are going with this new class of Navier-Stokes optimization problems.

1 Mathematical setting

We are controlling the worst case that appears due to the nonuniqueness of the solutions to the time-periodic Navier-Stokes equation. More precisely, for the cost functional

$$J(u,\phi) = \frac{1}{2}\int_Q (C_1 u(x,t))^2\,dxdt + \int_0^T h(\phi(t))dt \tag{1.1}$$

we compute the

$$\operatorname*{Inf}_{\phi\in L^2(Q)}\ \operatorname*{Sup}_{u\in L^2(Q)}\ J(u,\phi) \tag{1.2}$$

subject to

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u - \nu\Delta u + \nabla p = B_2\phi + f_0 \text{ in } \Omega\times\mathbf{R} \tag{1.3}$$

$$\nabla\cdot u = 0 \text{ in } \Omega\times\mathbf{R};\ \ u = 0 \text{ in } \partial\Omega\times\mathbf{R}$$

$$u(x,t) = u(x,t+T),\ \forall(x,t)\in\Omega\times\mathbf{R}.$$

Here $Q = \Omega\times(0,T)$, Ω is an open bounded subset of $\mathbf{R}^2$ with smooth boundary $\partial\Omega$, $f_0 \in L^2(\mathbf{R};L^2(\Omega))$ is a T-periodic source field, $u(x,t) = (u_1(x,t),u_2(x,t))$ is the velocity vector, p stands for the pressure, while $\phi\in L^2_{loc}(\mathbf{R};L^2(\Omega))$ is a T-periodic input. We denote by U the real Hilbert space of controlers, B_2 is a linear continuous operator from U to $L^2(\Omega)^2$ and h a lower semicontinuous, convex function on U. Finally, C_1 is a unbounded operator in $L^2(\Omega)^2$ satisfying

$$\int_\Omega (C_1 u(x))^2\,dx \le \alpha\int_\Omega u(x)^2 dx + \beta\int_\Omega (\nabla u(x))^2 dx. \tag{1.4}$$

In particular,

$$C_1 = d_1 I \Rightarrow \text{ regulation of turbulent kinetic energy, or}$$

$$C_1 = d_2 \nabla \times \;\Rightarrow \text{regulation of the square vorticity.}$$

We shall briefly recall the setting of (1.3) as an infinite-dimensional differential equation (see [7]-[10]). Let V be the divergence free subspace of $H_0^1(\Omega)^2$, i.e.

$$V = \{u \in H_0^1(\Omega)^2;\; \nabla \cdot u = 0\}$$

and

$$H = \{u \in L^2(\Omega)^2;\; \nabla \cdot u = 0 \text{ in } \Omega;\; n \cdot u = 0 \text{ in } \partial\Omega\}.$$

The space H is endowed with the usual $L^2(\Omega)^2$-norm denoted $|\cdot|$ and V with the norm $\|\cdot\|$ defined

$$\|u\|^2 = \sum_{1 \le i \le 2} \int_\Omega |\nabla u_i|^2 dx,\; u = (u_1, u_2).$$

If we denote V^* the dual of V and identify H with its own dual, we have $V \subset H \subset V^*$. Let $A \in L(V, V^*)$ and $b : V \times V \times V$ be defined by

$$(Au, v) = \int_0^T \nabla u_i \cdot \nabla v_i dx, \qquad \forall u, v \in V \tag{1.5}$$

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j dx, \qquad \forall u, v, w \in V \tag{1.6}$$

and $B : V \times V \to V^*$ given by

$$(B(u, v), w) = b(u, v, w) \qquad \forall u, v, w \in V$$

$$B(u) = B(u, u), \qquad \forall\, u \in V.$$

We set $D(A) = \{u \in V;\; Au \in H\}$ and denote again by A the restriction of A to H. Recall that b defined in (1.6) is a trilinear continuous functional satisfying (see [9]-[10])

$$b(u, v, w) = -b(u, w, v) \quad \forall u, v, w \in V,$$

$$|b(u, v, w)| \le C\left(|u|\|u\||w|\|w\|\right)^{1/2} \|v\| \quad \forall u, v, w \in V$$

$$|b(u, v, w)| \le C\left(|u|\|u\|\|v\||Av|\right)^{1/2} |w| \quad \forall u \in V,\; v \in D(A),\; w \in H.$$

Let $f(t) = Pf_0(t)$ and $\mathcal{B}_2 \in L(U, H)$ be given by $\mathcal{B}_2 = PB_2$, where $P : L^2(\Omega)^2 \to H$ is the projection on H. Now we can write the state equation (1.3) as

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) + \nu Au(t) + Bu(t) = \mathcal{B}_2\phi(t) + f(t) \qquad t \in (0, T) \tag{1.7}$$

$$u(0) = u(T).$$

Problem (P)
We shall confine to solutions u in (1.7) which satisfy the condition

$$u \in W^{1,2}([0, T]; H),\; Au \in L^2(0, T; H),\; Bu \in L^2(0, T; H)$$

and we may reformulate the problem (1.2) as

$$\operatorname*{Inf}_{\phi} \operatorname*{Sup}_{u} \int_0^T \left(\frac{1}{2}|C_1 u(t)|^2 + h(\phi(t))\right) dt \tag{P}$$

over $(u, \phi) \in \left(W^{1,2}([0, T]; H) \cap L^2(0, T; D(A))\right) \times L^2(0, T; U)$ subject to (1.7). We have denoted by $W^{1,2}([0, T]; H)$ the space of all absolutely continuous functions $u : [0, T] \to H$ such that $u' = \mathrm{d}u/\mathrm{d}t \in L^2(0, T; H)$. We have

$$W^{1,2}([0, T]; H) \cap L^2(0, T; D(A)) \subset C([0, T]; V).$$

2 Existence of optimal solutions

We consider first the inner problem

$$\text{(2.1)} \qquad \underset{u}{\text{Maximize}} \int_0^T \frac{1}{2} |C_1 u(t)|^2 \, dt$$

for all $u \in W^{1,2}([0,T];H) \cap L^2(0,T;D(A))$ satisfying (1.7), while $\phi \in L^2(0,T;U)$ is fixed.

Proposition 2.1 *Problem* (2.1) *has at least one solution* $u_* \in W^{1,2}([0,T];H) \cap L^2(0,T;D(A))$.

Proof. See section 7. □

We shall study now the existence in problem (P). Assume that

(i) The function $h : U \to \mathbf{R}$ is convex, lower semi-continuous and satisfies the coercivity condition

$$h(\phi) \geq \omega |\phi|_U^2 + \beta, \quad \forall \phi \in U$$

for some $\omega > 0$, $\beta \in \mathbf{R}$.

Theorem 2.1 *Under hypotheses* (i), (1.4) *problem* (P) *has at least one solution.*

Proof. See section 7. □

3 Approximation of Problem (2.1)

Due to the genericity properties of the time-periodic Navier-Stokes equation (see [12], [11]), it is difficult to derive a gradient algorithm based on adjoint field information for problem (P). The main obstacle is this: the application $\phi \to u(\phi)$ is neither differentiable, nor continuous. In fact, the result mentioned above states that, for a dense set in $L^2(0,T;H) \ni f + \mathcal{B}_2\phi$, equation (1.7) has a finite number of solutions, which is constant on every connected part of this set. Therefore, for a small variation of ϕ, the number of solutions to (1.7) may vary to infinity.

Thus, let consider first the approximate inner problem, which in the appropriate limit, approaches the solution of (2.1). To accomplish this, consider the maximization of

$$\text{(3.1)} \qquad \int_0^T \left(\frac{1}{2} |C_1 u|^2 - \frac{1}{2\varepsilon} |\xi|^2 \right) dt$$

over $u \in W^{1,2}([0,T];H) \cap L^2(0,T;D(A))$, $\xi \in L^2(0,T;H)$ subject to

$$\text{(3.2)} \qquad \frac{\mathrm{d}u}{\mathrm{d}t}(t) + \nu A u(t) + B u(t) = \mathcal{B}_2 \phi(t) + f(t) + \xi(t) \qquad t \in (0,T)$$

$$u(0) = u(T).$$

Lemma 3.1 *For each* $\varepsilon > 0$ *sufficiently small problem* (3.1) *has at least one solution* $(u_\varepsilon, \xi_\varepsilon)$.

Proof. See section 7. □

Proposition 3.1 *For* $\varepsilon \to 0$, *we have*

$$u_\varepsilon \to u_* \text{ strongly in } L^2(0,T;V) \cap C([0,T];H)$$

$$\varepsilon^{-1/2} \xi_\varepsilon \to 0 \text{ weakly in } L^2(0,T;H)$$

$$\lim_{\varepsilon \to 0} \underset{u,\xi}{\text{Sup}}(3.1) = \underset{u}{\text{Sup}}(2.1).$$

Proof. See section 7. □

Proposition 3.2 *If* $(u_\varepsilon, \xi_\varepsilon)$ *is an optimal pair in problem* (3.1), *then there is* $q_\varepsilon \in W^{1,2}([0,T];H) \cap L^2(0,T;D(A))$ *such that*

$$\text{(3.3)} \qquad -q_\varepsilon'(t) + \nu A q_\varepsilon(t) + B'(u_\varepsilon(t))^* q_\varepsilon(t) = C_1^* C_1 u_\varepsilon(t), \ \ a.e. \ t \in (0,T)$$

$$q_\varepsilon(0) = q_\varepsilon(T)$$

$$\text{(3.4)} \qquad \xi_\varepsilon(t) = \varepsilon q_\varepsilon(t), \ \ a.e. \ t \in (0,T).$$

Here $B'(u_\varepsilon(t)),\ B'(u_\varepsilon(t))^* \in L(V,V^*)\cap L(D(A),H)$ are defined by

$$(3.5)\qquad \begin{aligned}(B'(u_\varepsilon)z,w) &= b(z,u_\varepsilon,w)+b(u_\varepsilon,z,w),\ \forall z\in D(A), w\in H\\ (B'(u_\varepsilon)^*q,w) &= b(w,u_\varepsilon,q)+b(u_\varepsilon,w,q),\ \forall q\in D(A), w\in H.\end{aligned}$$

Proof. See section 7. □

Remark. From Propositions 3.1, 3.2 we have

$$\operatorname*{Sup}_u(2.1) = \lim_{\varepsilon\to 0}\operatorname*{Sup}_{u,\xi}(3.1) = \lim_{\varepsilon\to 0}\int_0^T \left(\frac{1}{2}|C_1u_\varepsilon|^2 - \frac{\varepsilon}{2}|q_\varepsilon|^2\right)dt$$

where $(u_\varepsilon,q_\varepsilon)$ satisfies

$$\begin{cases} u_\varepsilon' + \nu A u_\varepsilon + B(u_\varepsilon) = \mathcal{B}_2\phi + f + \varepsilon q_\varepsilon,\ u_\varepsilon(0)=u_\varepsilon(T)\\ -q_\varepsilon' + \nu A q_\varepsilon + B'(u_\varepsilon)^* q_\varepsilon = C_1^*C_1u_\varepsilon,\ q_\varepsilon(0)=q_\varepsilon(T).\end{cases}$$

Therefore we can develop an algorithm that computes, for a fixed ϕ, a solution of maximum energy in the sense of (2.1) to the time periodic Navier-Stokes equation (1.7).

4 Approximation of Problem (P)

For each $\varepsilon>0$ consider the following optimization problem: minimize

$$(P_\varepsilon)\qquad J_\varepsilon(u_\varepsilon,q_\varepsilon,\phi) = \int_0^T\left(\frac{1}{2}|C_1u_\varepsilon|^2 + h(\phi) - \frac{\varepsilon}{2}|q_\varepsilon|^2\right)dt$$

over $(u_\varepsilon,q_\varepsilon)\in W^{1,2}([0,T];H)\cap L^2(0,T;D(A))$, $\phi\in L^2(0,T;U)$ subject to

$$(4.1)\qquad \begin{cases} u_\varepsilon'(t)+\nu A u_\varepsilon(t) + B(u_\varepsilon(t)) = \mathcal{B}_2\phi(t) + f(t) + \varepsilon q_\varepsilon(t)\\ -q_\varepsilon'(t) + \nu A q_\varepsilon(t) + B'(u_\varepsilon(t))^* q_\varepsilon(t) = C_1^*C_1u_\varepsilon(t)\end{cases},\ t\in(0,T),\quad \begin{matrix}u_\varepsilon(0)=u_\varepsilon(T)\\ q_\varepsilon(0)=q_\varepsilon(T).\end{matrix}$$

By Lemma 3.1 and Proposition 3.2 we know that the system (4.1) has at least one solution $(u_\varepsilon,q_\varepsilon)$, which also solves the Problem (3.1)-(3.2).
Instead of (i) we shall use the following hypothesis.
(i)$'$ The function $h: U\to \mathbf{R}$ is convex, lower semi-continuous and satisfies

$$\omega|\phi|_U^2 + \beta \le h(\phi) \le \omega_1|\phi|_U^2 + \beta_1,\quad \forall \phi\in U$$

for some $\omega,\omega_1>0,\ \beta,\beta_1\in\mathbf{R}$.

Proposition 4.1 *Under hypotheses* (i)$'$, (1.4) *problem* (P_ε) *has at least one solution* $(u_\varepsilon,q_\varepsilon,\phi_\varepsilon)$.

Proof. See section 7. □

Proposition 4.2 *For* $\varepsilon\to 0$ *we have*

$$(4.2)\qquad \lim_{\varepsilon\to 0}\inf_{\phi,u_\varepsilon,q_\varepsilon}(P_\varepsilon) = \inf_{\phi,u_*^\phi}\sup_u(P).$$

Proof. See section 7. □

Recall that if K is a closed convex subset of H, we may define the indicator function $I_K: V\to\overline{\mathbf{R}}$

$$I_K(q) = \begin{cases} 0 & \text{if } q\in K,\\ +\infty & \text{if } q\notin K\end{cases}$$

and the normal cone

$$N_K(q) = \partial I_K(q) = \{q^*\in V^*;\ (q-w,q^*)\ge 0\ \ \forall q\in K\}\qquad \forall q\in K.$$

We denote by $K=\{q\in L^2(0,T;H); \varepsilon\|q\|^2_{L^2(0,T;H)}\le 1\}$. Note that this indicator function is used for convenience in the derivation, by incorporating the restriction of q to K in the cost function. (For more details on convexity and optimization see [3].)

In order to get necessary optimality conditions for the approximate problem $(P_{\varepsilon\lambda})$ we encounter again the obstacle of nonuniqueness for the solution to system (4.1). Let consider instead the following optimization problem: minimize

$$(P_{\varepsilon\lambda}) \qquad J_{\varepsilon\lambda}(u_\varepsilon, q_\varepsilon, \phi, \psi_1, \psi_2) = \int_0^T \left(\frac{1}{2}|C_1 u_\varepsilon|^2 - \frac{\varepsilon}{2}|q_\varepsilon|^2 + I_K(q_\varepsilon) + h(\phi) + \frac{1}{2\lambda}|\psi_1|^2 + \frac{1}{2\lambda}|\psi_2|^2\right) dt$$

over $(u_\varepsilon, q_\varepsilon) \in W^{1,2}([0,T];H) \cap L^2(0,T;D(A))$, $\phi \in L^2(0,T;U)$, $\psi_1, \psi_2 \in L^2(0,T;H)$ subject to

$$(4.3) \qquad \begin{cases} u_\varepsilon'(t) + \nu A u_\varepsilon(t) + B(u_\varepsilon(t)) = \mathcal{B}_2\phi(t) + f(t) + \varepsilon q_\varepsilon(t) + \psi_1(t) \\ -q_\varepsilon'(t) + \nu A q_\varepsilon(t) + B'(u_\varepsilon(t))^* q_\varepsilon(t) = C_1^* C_1 u_\varepsilon(t) + \psi_2(t) \end{cases}, \; t \in (0,T), \quad \begin{matrix} u_\varepsilon(0) = u_\varepsilon(0) \\ q_\varepsilon(0) = q_\varepsilon(T). \end{matrix}$$

Using an argument similar to Proposition 4.1 we get that $(P_{\varepsilon\lambda})$ has at least one solution $(\phi_\lambda, u_{\varepsilon\lambda}, q_{\varepsilon\lambda}, \psi_{1\lambda}, \psi_{2\lambda})$.

Proposition 4.3 *For $\varepsilon > 0$ sufficiently small we have*

$$(4.4) \qquad \lim_{\lambda\to 0} \inf_{\phi, u_\varepsilon, q_\varepsilon, \psi_1, \psi_2} (P_{\varepsilon\lambda}) = \inf_{\phi, u_\varepsilon, q_\varepsilon} (P_\varepsilon).$$

Proof. See section 7. □

By the proof of Proposition 4.3 we see that $I_K(q_{\varepsilon\lambda}) = 0$ (since $q_{\varepsilon\lambda} \in K$, $\forall\lambda$). Therefore the indicator function is needed just for the correct formulation of $(P_{\varepsilon\lambda})$ and can be dropped out in the numerical algorithm.

We note that Propositions 4.2 and 4.3 prove that

$$(4.5) \qquad \lim_{\varepsilon\to 0} \lim_{\lambda\to 0} \inf_{\phi, u_\varepsilon, q_\varepsilon, \psi_1, \psi_2} (P_{\varepsilon\lambda}) = \inf_{\phi, u_*^\phi} \sup_u (P).$$

5 Necessary Conditions for Optimality

Here we shall establish a maximum principle type result for problem $(P_{\varepsilon\lambda})$.

Theorem 5.1 *Under hypotheses* (1.4), *(i)′ if $(\phi_\lambda, u_{\varepsilon\lambda}, q_{\varepsilon\lambda}, \psi_{1\lambda}, \psi_{2\lambda})$ is optimal in problem $(P_{\varepsilon\lambda})$ then there is $U_{\varepsilon\lambda}, Q_{\varepsilon\lambda} \in W^{1,2}([0,T];H) \cap L^2(0,T;D(A))$ such that*

$$(5.1) \qquad \begin{gathered} -U_\lambda' + \nu A U_\lambda + (B'(u_{\varepsilon\lambda}))^* U_\lambda = C_1^* C_1 Q_\lambda - B'(Q_\lambda)^* q_{\varepsilon\lambda} - C_1^* C_1 u_{\varepsilon\lambda}, \quad a.e. \; t \in (0,T) \\ Q_\lambda' + \nu A Q_\lambda + (B'(u_{\varepsilon\lambda})) Q_\lambda = \varepsilon U_\lambda - \varepsilon q_{\varepsilon\lambda}, \quad a.e. \; t \in (0,T) \\ U_\lambda(0) = U_\lambda(T), \quad Q_\lambda(0) = Q_\lambda(T). \end{gathered}$$

$$(5.2) \qquad \psi_{1\lambda} = \lambda U_\lambda, \; \psi_{2\lambda} = \lambda Q_\lambda, \; \mathcal{B}_2^* \phi_\lambda \in \partial h(Q_\lambda), \quad a.e. \; in \; (0,T).$$

Here $\partial h : U \to U$ is the subdifferential of h.
Proof. See section 7. □

Due to the lack of differentiability in application $\phi \to u(\phi)$, we have replaced problem (P) by a sequence of approximating problems $(P_{\varepsilon\lambda})$, for which we can compute necessary conditions for optimality. An algorithm of gradient type is now proposed in order to compute a optimal solution to problem $(P_{\varepsilon\lambda})$. We use iterative processes to solve the inner loop $(P_{\varepsilon\lambda})$ and the outer loop (P_ε).

6 Numerical algorithm

Let us assume that $C_1 \equiv I$ and rewrite the optimality system in the following form

$$(4.3)' \qquad \begin{cases} u'_{\varepsilon\lambda}(t) + \nu A u_{\varepsilon\lambda}(t) + B(u_{\varepsilon\lambda}(t)) = \mathcal{B}_2\phi_\lambda(t) + f(t) + \varepsilon q_{\varepsilon\lambda}(t) + \psi_{1\lambda}(t),\ t \in (0,T),\ u_{\varepsilon\lambda}(0) = u_{\varepsilon\lambda}(T) \\ \mathcal{A}^*_\varepsilon q_{\varepsilon\lambda} = P_{R(\mathcal{A}^*_{\varepsilon\lambda})}(u_{\varepsilon\lambda} + \psi_{2\lambda}) \end{cases}$$

$$(5.1)' \qquad \begin{cases} \mathcal{A}^*_\varepsilon U_\lambda = P_{R(\mathcal{A}^*_\varepsilon)}\left(Q_\lambda - u_{\varepsilon\lambda} - B'(Q_\lambda)^* q_{\varepsilon\lambda}\right) \\ \mathcal{A}_\varepsilon Q_\lambda = P_{R(\mathcal{A}_\varepsilon)}\left(\varepsilon U_\lambda - \varepsilon q_{\varepsilon\lambda}\right). \end{cases}$$

(Here $\mathcal{A}_\varepsilon, \mathcal{A}^*_\varepsilon$ are defined by (7.22), (7.22)$'$ where $u^n_\varepsilon := u_{\varepsilon\lambda}$.)
1. *Initialize* $\varepsilon, \lambda > 0$ and ϕ^0_λ on $t \in [0,T]$.
2. *Initialize* $i = 0$ *and* $(\psi^0_{1\lambda}, \psi^0_{2\lambda})$ *on* $[0,T]$, *where* i *is the iteration index and* $(\phi^i_\lambda, \psi^i_{1\lambda}, \psi^i_{2\lambda})$ *represent the approximation of the control and forcing that we use in* $J_{\varepsilon\lambda}$.
3. *Determine the state* $(u^i_{\varepsilon\lambda}, q^i_{\varepsilon\lambda})$ *on* $[0,T]$ *from the state equation* (4.3)$'$.
4. *Determine the adjoint state* $(U^i_\lambda, Q^i_\lambda)$ *from the adjoint equation* (5.1)$'$.
5. *Determine local expressions for the gradients*

$$\frac{DJ_{\varepsilon,\lambda}}{D\phi}(\phi^i_\lambda, \psi^i_{1\lambda}, \psi^i_{2\lambda}) = \nabla h(\phi^i_\lambda) - \mathcal{B}_2(\psi^i_{1\lambda}),\ \frac{DJ_{\varepsilon,\lambda}}{D\psi_1} = \frac{1}{\lambda}\psi^i_{1\lambda} - U^i_\lambda,\ \frac{DJ_{\varepsilon,\lambda}}{D\psi_2} = \frac{1}{\lambda}\psi^i_{2\lambda} - Q^i_\lambda.$$

6. *Determine the updated control* ϕ^{i+1}_λ *and forcing* $\psi^{i+1}_{1\lambda}, \psi^{i+1}_{2\lambda}$ *with*

$$\phi^{i+1}_\lambda = \phi^i_\lambda - \alpha^i \frac{DJ_{\varepsilon,\lambda}}{D\phi}(\phi^i_\lambda, \psi^i_{2\lambda}),\quad \psi^{i+1}_{1\lambda} = \psi^i_{1\lambda} - \alpha^i \frac{DJ_{\varepsilon,\lambda}}{D\psi_1}(\phi^i_\lambda, \psi^i_{2\lambda}),\quad \psi^{i+1}_{2\lambda} = \psi^i_{2\lambda} - \alpha^i \frac{DJ_{\varepsilon,\lambda}}{D\psi_2}(\phi^i_\lambda, \psi^i_{2\lambda}),$$

where $0 < \alpha^i < 1$.
7. *Increment index* $i = i + 1$. *Repeat from step* 3 *until converged.*
8. *Reset* $\phi^0_\lambda = \phi^i_\lambda, \psi^0_{1\lambda} = \psi^i_{1\lambda}, \psi^0_{2\lambda} = \psi^i_{2\lambda}$.
9. $\lambda = \lambda/2$. *Repeat from step* 3 *until stop criterion in* λ *is satisfied* (e.g., $\|\psi_{1\lambda}\| + \|\psi_{2\lambda}\|$ <Tolerance).
At this point we have solved problem (P$_\varepsilon$).
10. *Reset* $\phi^0_\lambda = \phi^i_\lambda$.
11. $\varepsilon = \varepsilon/2$. *Repeat from step* 2 *until stop criterion in* ε *is satisfied.*

7 Proofs

Proof of Proposition 2.1 We note that the function

$$u \longmapsto \int_0^T |C_1 u|^2 dt$$

is continuous from $C([0,T];H) \cap L^2(0,T;V)$ into $\mathbf{R}$. Moreover, the set of solutions to (1.7) is relatively compact in $C([0,T];H) \cap L^2(0,T;V)$ (see [2]), and therefore (2.1) has at least one solution.
Indeed, let $\{u_n\}_n$ be a sequence of solutions to (1.7). Recalling that $(Bu, u) = 0,\ \forall u \in V$ we have by (1.7) and the Poincaré inequality

$$(7.1) \qquad \begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_n(t)|^2 + \nu\|u_n(t)\|^2 = (\mathcal{B}_2\phi(t) + f(t), u_n(t)) \\ &\le \frac{\nu\lambda_1}{2}|u_n(t)|^2 + \frac{1}{2\nu\lambda_1}\left|\mathcal{B}_2\phi(t) + f(t)\right|^2 \le \frac{\nu}{2}\|u_n(t)\|^2 + \frac{1}{2\nu\lambda_1}\left|\mathcal{B}_2\phi(t) + f(t)\right|^2. \end{aligned}$$

Now integrate over $(0,T)$ and we get by the periodicity condition that

$$\int_0^T \|u_n(t)\|^2 dt \le \frac{1}{\nu^2\lambda_1}\int_0^T \left|\mathcal{B}_2\phi(t) + f(t)\right|^2 dt.$$

If multiply (1.7) by $tu_n(t)$ we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(t|u_n|^2\right) - \frac{1}{2}|u_n|^2 + \nu t\|u_n\|^2 = t\left(\mathcal{B}_2\phi(t) + f(t), u_n(t)\right)$$

$$\leq t\frac{\nu\lambda_1}{2}|u_n(t)|^2 + t\frac{1}{2\nu\lambda_1}\left|\mathcal{B}_2\phi(t)+f(t)\right|^2 \leq t\frac{\nu}{2}\|u_n(t)\|^2 + t\frac{1}{2\nu\lambda_1}\left|\mathcal{B}_2\phi(t)+f(t)\right|^2$$

while integrating on $(0,t)$ we get

$$t|u_n|^2 + \nu\int_0^t s\|u_n(s)\|^2 ds \leq \int_0^t |u_n(s)|^2 ds + \frac{1}{\nu\lambda_1}\int_0^t s\left|\mathcal{B}_2\phi(s)+f(s)\right|^2 ds.$$

Therefore

$$t|u_n(t)|^2 \leq C, \quad \forall t\in(0,T].$$

Since $u_n(0)=u_n(T)$ we infer that $|u_n(0)|=|u_n(T)|\leq C$. Now integrating (7.1) on $(0,t)$ we get from above

$$|u_n(t)|^2 + \int_0^T \|u_n(t)\|^2 dt \leq C, \quad \forall t\in[0,T]. \tag{7.2}$$

(Here and throughout this proof we shall denote by $C=C(\nu,\lambda_1,f,\phi,T)$ several positive constants independent of u_n and n.) Multiplying (1.10) by tAu_n and integrating on $(0,t)$, we get after some calculation involving (7.2) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(t\|u_n(t)\|^2\right) - \frac{1}{2}\|u_n(t)\|^2 + t\nu|Au_n(t)|^2 = t\left(Au_n(t),\mathcal{B}_2\phi(t)+f(t)\right) - tb(u_n(t),u_n(t),Au_n(t))$$

$$\leq t\frac{\nu}{4}|Au_n(t)|^2 + t\frac{1}{\nu}\left|\mathcal{B}_2\phi(t)+f(t)\right|^2 + tC\left(|u_n(t)|\|u_n(t)\|^2|Au_n(t)|\right)^{1/2}|Au_n|$$

$$= t\frac{\nu}{4}|Au_n(t)|^2 + t\frac{1}{\nu}\left|\mathcal{B}_2\phi(t)+f(t)\right|^2 + tC|u_n(t)|^{1/2}\|u_n(t)\||Au_n(t)|^{3/2}$$

$$\leq t\frac{\nu}{4}|Au_n(t)|^2 + t\frac{1}{\nu}\left|\mathcal{B}_2\phi(t)+f(t)\right|^2 + t\frac{\nu}{4}|Au_n(t)|^2 + tC\|u_n(t)\|^4|u_n(t)|^2$$

therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t\|u_n(t)\|^2\right) + t\nu|Au_n(t)|^2 \leq \|u_n(t)\|^2 + t\frac{2}{\nu}\left|\mathcal{B}_2\phi(t)+f(t)\right|^2 + 2tC\|u_n(t)\|^4|u_n(t)|^2.$$

Now integrating on $(0,t)$ and using (7.2) we get

$$t\|u_n(t)\|^2 + \int_0^t s|Au_n(s)|^2 ds \leq C\left(1+\int_0^t s\|u_n(s)\|^4 ds\right), \ \forall t\in[0,T]$$

and by Gronwall's lemma

$$t\|u_n(t)\|^2 \leq C \qquad \forall t\in(0,T].$$

Since $u_n(0)=u_n(T)$ we infer that $\|u_n(0)\|\leq C$. Then multiplying (1.7) by Au_n and integrating on $(0,t)$ we get as above

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_n(t)\|^2 + \nu|Au_n(t)|^2 = (\mathcal{B}_2\phi(t)+f(t),Au_n(t)) - b(u_n(t),u_n(t),Au_n(t))$$

$$\leq (\mathcal{B}_2\phi(t)+f(t),Au_n(t)) + |u_n(t)|^{1/2}\|u_n(t)\|\,|Au_n(t)|^{3/2}$$

$$\leq \frac{\nu}{4}|Au_n(t)|^2 + \frac{1}{\nu}|\mathcal{B}_2\phi(t)+f(t)|^2 + \frac{\nu}{4}|Au_n(t)|^2 + C\|u_n(t)\|^4.$$

Integrate over $(0,t)$ we see that

$$\|u_n(t)\|^2 + \int_0^t |Au_n(s)|^2 \leq C\left(\|u_n(0)\|^2 + \int_0^t \|u_n(t)\|^4 ds\right)$$

and therefore

$$\|u_n(t)\|^2 + \int_0^t |Au_n(s)|^2 ds \leq C, \ \forall t\in[0,T].$$

We note that we have

$$|Bu_n| \leq C|u_n|^{1/2}\,\|u_n\|\,|Au_n|^{1/2}, \quad \forall u_n\in D(A)$$

therefore using the above estimates we see

$$\|Bu_n\|^2_{L^2(0,T;H)} = \int_0^T |Bu_n(t)|^2 dt \le C \int_0^T |u_n(t)|\,\|u_n(t)\|^2\,|Au_n| dt \le C$$

and from the equation (1.7) we have then

$$\|u_n'\|_{L^2(0,T;H)} + \|Bu_n\|_{L^2(0,T;H)} \le C.$$

Since the injection of V into H is compact, we infer by the weak formulation of (1.7) that $\{u_n\}_n$ is compact in $C([0,T];H) \cap L^2(0,T;V)$. □

Proof of Theorem 2.1 Let $\{u_*^{\phi_n}, \phi_n\}$ be a minimizing sequence, i.e.

$$\inf_\phi (2.1) \le \frac{1}{2}\int_0^T |C_1 u_*^{\phi_n}|^2 dt + \int_0^T h(\phi_n) dt \le \inf_\phi (2.1) + \frac{1}{n} \tag{7.3}$$

where we have already solved the inner problem (2.1)

$$u_*^{\phi_n\,\prime} + \nu A u_*^{\phi_n} + B(u_*^{\phi_n}) = \mathcal{B}_2\phi_n + f, \text{ a.e. } t \in (0,T)\ ;\ u_*^{\phi_n}(0) = u_*^{\phi_n}(T) \tag{7.4}$$

$$\int_0^T |C_1 u_*^{\phi_n}|^2 dt = \sup_u \left\{ \int_0^T |C_1 u|^2 dt;\ u' + \nu Au + B(u) = \mathcal{B}_2\phi_n + f,\ u(0) = u(T) \right\}. \tag{7.5}$$

By (7.3) it follows that $\{\phi_n\}_n$ is bounded in $L^2(0,T;U)$ and therefore on a subsequence, again denoted n, we have

$$\phi_n \to \phi^* \text{ weakly in } L^2(0,T;U).$$

On the other hand, from (7.3) and the optimality of $u_*^{\phi_n}$ in problem (2.1)

$$\frac{1}{2}\int_0^T |C_1 u^{\phi_n}|^2 dt + \int_0^T h(\phi_n) dt \le \inf_\phi (2.1) + \frac{1}{n} \tag{7.6}$$

for all (u^{ϕ_n}, ϕ_n) satisfying

$$u^{\phi_n\,\prime} + \nu A u^{\phi_n} + B(u^{\phi_n}) = \mathcal{B}_2\phi_n + f, \text{ a.e. } t \in (0,T)\ ;\ u^{\phi_n}(0) = u^{\phi_n}(T). \tag{7.7}$$

As in Proposition 2.1, if multiply the latter by tAu^{ϕ_n}, Au^{ϕ_n} and integrate over $(0,t)$ we obtain after some calculations

$$\|u^{\phi_n}(t)\|^2 + \int_0^t |Au^{\phi_n}(s)|^2 ds \le C,\ \forall t \in [0,T].$$

and

$$\|u^{\phi_n\prime}\|_{L^2(0,T;H)} + \|Bu^{\phi_n}\|_{L^2(0,T;H)} \le C.$$

Since the injection V into H is compact, we infer that $\{u^{\phi_n}\}$ is compact in $C([0,T];H) \cap L^2(0,T;V)$ and therefore, on a selected subsequence, we have

$$u^{\phi_n} \to u^{\phi^*} \text{ strongly in } L^2(0,T;V) \cap C([0,T];H)$$

$$Au^{\phi_n} \to Au^{\phi^*} \text{ weakly in } L^2(0,T;H)$$

$$u^{\phi_n\prime} \to u^{\phi^*\prime} \text{ weakly in } L^2(0,T;H).$$

We have also

$$|(Bu^{\phi_n} - Bu^{\phi^*}, w)| \le |b(u^{\phi_n} - u^{\phi^*}, u^{\phi_n}, w)| + |b(u^{\phi^*}, u^{\phi_n} - u^{\phi^*}, w)|$$
$$\le C\left(\left(|u^{\phi_n} - u^{\phi^*}|\,\|u^{\phi_n} - u^{\phi^*}\|\,\|u^{\phi_n}\|\,|Au^{\phi_n}|\right)^{1/2}\right.$$
$$\left. + \left(|u^{\phi^*}|\,\|u^{\phi^*}\|\,\|u^{\phi_n} - u^{\phi^*}\|\,|Au^{\phi_n} - Au^{\phi^*}|\right)^{1/2}\right)|w|, \quad \forall w \in H$$

and therefore

$$Bu^{\phi_n} \to Bu^{\phi^*} \text{ strongly in } L^2(0,T;H).$$

Then letting n go to ∞ in (7.6), (7.7) we see that

$$\frac{1}{2}\int_0^T |C_1 u^{\phi^*}|^2 dt + \int_0^T h(\phi^*) dt \leq \operatorname*{Inf}_{\phi}(2.1)$$

for all u^{ϕ^*} satisfying

$$u^{\phi^*\,\prime} + B(u^{\phi^*}) + \nu A u^{\phi^*} = \mathcal{B}_2\phi^* + f, \text{ a.e. } t \in (0,T)\ ;\ u^{\phi^*}(0) = u^{\phi^*}(T).$$

This will also be true for $u_*^{\phi^*}$, the solutions to problem (2.1) corresponding to ϕ^*. Now using again the definition of infimum, we obtain that $(u^* \equiv u_*^{\phi^*}, \phi^*)$ is a solution to (P). □

Proof of Lemma 3.1 If multiply (3.2) by u and integrate on $(0,T)$ we see that

$$\int_0^T |\nabla u|^2 dt \leq \frac{2}{\nu^2\lambda_1}\int_0^T |\mathcal{B}_2\phi + f|^2 dt + \frac{2}{\nu^2\lambda_1}\int_0^T |\xi|^2 dt \tag{7.8}$$

while by (1.4) and (3.2) we get

$$\int_0^T \left(\frac{1}{2}|C_1 u|^2 - \frac{1}{2\varepsilon}|\xi|^2\right) dt \leq \left(\frac{\alpha+\beta\lambda_1}{\nu^2\lambda_1^2} - \frac{1}{2\varepsilon}\right)\int_0^T |\xi|^2 dt + \frac{\alpha+\beta\lambda_1}{\nu^2\lambda_1^2}\int_0^T |\mathcal{B}_2\phi + f|^2 dt.$$

So for $\varepsilon > 0$ sufficiently small we have

$$-\frac{1}{2\varepsilon}\int_0^T |\xi|^2 dt \leq \int_0^T \left(\frac{1}{2}|C_1 u|^2 - \frac{1}{2\varepsilon}|\xi|^2\right) dt \leq -C_\varepsilon \int_0^T |\xi|^2 dt + C. \tag{7.9}$$

Hence problem (3.1) is well-posed in the sense that we are maximizing a coercive functional, bounded from above. Let denote by

$$d := \sup_{u,\xi}\int_0^T \left(\frac{1}{2}|C_1 u|^2 - \frac{1}{2\varepsilon}|\xi|^2\right) dt$$

and consider $\{u^n, \xi^n\}$ a maximizing sequence for problem (3.1), i.e.,

$$d - \frac{1}{n} \leq \int_0^T \left(\frac{1}{2}|C_1 u^n|^2 - \frac{1}{2\varepsilon}|\xi^n|^2\right) dt \leq d \tag{7.10}$$

where

$$\frac{\mathrm{d}}{\mathrm{dt}}u^n(t) + \nu A u^n(t) + Bu^n(t) = \mathcal{B}_2\phi(t) + f + \xi^n(t), \quad \text{a.e. } t \in (0,T);\ u^n(0) = u^n(T).$$

By (7.8)-(7.10) we obtain

$$d + \left(\frac{1}{2\varepsilon} - \frac{\alpha+\beta\lambda_1}{\nu^2\lambda_1^2}\right)\int_0^T |\xi^n|^2 dt \leq \frac{1}{n} + \frac{\alpha+\beta\lambda_1}{\nu^2\lambda_1^2}\int_0^T |\mathcal{B}_2\phi + f|^2 dt$$

so $\{\xi^n\}$ is bounded in $L^2(0,T;H)$. Now using an argument similar to that in Theorem 2.1 we can see that $\{u^n\}$ is bounded in $W^{1,2}(0,T;H) \cap L^2(0,T;D(A))$ and on selected subsequences we have

$$u^n \to u_\varepsilon \text{ strongly in } L^2(0,T;V) \cap C([0,T];H)$$

$$Bu^n \to Bu_\varepsilon \text{ strongly in } L^2(0,T;H)$$

$$\xi^n \to \xi_\varepsilon \text{ weakly in } L^2(0,T;H).$$

Letting n go to ∞ in (7.10) we see that $(u_\varepsilon, \xi_\varepsilon)$ satisfies system (3.2) and is a solution to problem (3.1). □

Proof of Proposition 3.1 For $\xi := 0$ we get

$$\operatorname*{Sup}_{u}(2.1) \leq \operatorname*{Sup}_{u,\xi}(3.1). \tag{7.11}$$

Now let $(u_\varepsilon, \xi_\varepsilon) \in \left(W^{1,2}([0,T];H) \cap L^2(0,T;D(A))\right) \times L^2(0,T;H)$ be a solution to problem (3.1), i.e.

$$\operatorname*{Sup}_{u,\xi} \left\{ \int_0^T \left(\frac{1}{2}|C_1 u|^2 - \frac{1}{2\varepsilon}|\xi|^2 \right) dt;\ (u,\xi) \text{ subject to } (3.2) \right\} = \int_0^T \left(\frac{1}{2}|C_1 u_\varepsilon|^2 - \frac{1}{2\varepsilon}|\xi_\varepsilon|^2 \right) dt$$

with

$$(3.2)'' \qquad u_\varepsilon' + \nu A u_\varepsilon + B(u_\varepsilon) = \mathcal{B}_2\phi + f + \xi_\varepsilon \text{ a.e. } t \in (0,T);\ u_\varepsilon(0) = u_\varepsilon(T).$$

If multiply (3.2)″ by u_ε and use the Poincaré inequality we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}|u_\varepsilon(t)|^2 + \nu\|u_\varepsilon(t)\|^2 = (\mathcal{B}_2\phi + f + \xi_\varepsilon, u_\varepsilon) \leq \frac{\nu}{2}\|u_\varepsilon(t)\|^2 + \frac{2}{\nu\lambda_1}\left(|\mathcal{B}_2\phi(t) + f(t)|^2 + |\xi_\varepsilon(t)|^2\right)$$

and integrating on $(0,T)$ we have

$$(7.12) \qquad \int_0^T \|u_\varepsilon(t)\|^2 dt \leq \frac{2}{\nu^2\lambda_1}\int_0^T \left(|\xi_\varepsilon(t)|^2 + |\mathcal{B}_2\phi(t) + f(t)|^2\right) dt.$$

By (7.11), (1.4) we see

$$\operatorname*{Sup}_u (2.1) \leq \frac{1}{2}\int_0^T \left(|C_1 u_\varepsilon(t)|^2 - \frac{1}{\varepsilon}|\xi_\varepsilon(t)|^2 \right) dt \leq \frac{1}{2}\int_0^T \left(\left(\frac{\alpha}{\lambda_1} + \beta \right) \|u_\varepsilon(t)\|^2 - \frac{1}{\varepsilon}|\xi_\varepsilon(t)|^2 \right) dt$$

and therefore (7.12) implies

$$\left(\frac{1}{2\varepsilon} - \frac{\alpha + \beta\lambda_1}{\nu^2\lambda_1^2} \right) \int_0^T |\xi_\varepsilon(t)|^2 dt \leq -\operatorname*{Sup}_u (2.1) + \frac{\alpha + \beta\lambda_1}{\nu^2\lambda_1^2}\int_0^T |\mathcal{B}_2\phi(t) + f(t)|^2 dt.$$

Hence for ε sufficiently small

$$\int_0^T |\xi_\varepsilon(t)|^2 dt \leq C\varepsilon$$

and by (7.12) we get

$$\int_0^T \|u_\varepsilon(t)\|^2 dt \leq C.$$

Now we multiply (3.2)″ by tu_ε and obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left(t|u_\varepsilon(t)|^2\right) - \frac{1}{2}|u_\varepsilon(t)|^2 + \nu t\|u_\varepsilon(t)\|^2 \leq t\frac{\nu}{2}\|u_\varepsilon\|^2 + \frac{t}{\nu\lambda_1}|\mathcal{B}_2\phi(t) + f(t)|^2 + \frac{t}{\nu\lambda_1}|\xi_\varepsilon(t)|^2$$

while integrating on $(0,t)$ we get

$$t|u_\varepsilon(t)|^2 + \nu\int_0^t s\|u_\varepsilon(s)\|^2 ds \leq \int_0^t |u_\varepsilon(s)|^2 ds + \frac{2}{\nu\lambda_1}\int_0^T \left(s\,|\mathcal{B}_2\phi(s) + f(s)|^2 + s|\xi_\varepsilon(s)|^2 \right) ds.$$

Therefore

$$t|u_\varepsilon(t)|^2 \leq C, \quad \forall t \in (0,T]$$

and due to the periodic condition in (3.2)″ we infer that $|u_\varepsilon(0)| = |u_\varepsilon(T)| \leq C$. Integrating on $(0,t)$ the relation that lead to (7.12) we obtain

$$|u_\varepsilon(t)| \leq C, \quad \forall t \in [0,T].$$

On the other hand, multiplying (3.2)″ by tAu_ε we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left(t\|u_\varepsilon(t)\|^2\right) - \frac{1}{2}\|u_\varepsilon(t)\|^2 + t\nu|Au_\varepsilon(t)|^2$$

$$\leq t\frac{\nu}{2}|Au_\varepsilon(t)|^2 + 2\frac{t}{\nu}|\mathcal{B}_2\phi(t) + f(t)|^2 + 2\frac{t}{\nu}|\xi_\varepsilon(t)|^2 + tC\|u_\varepsilon(t)\|^4|u_\varepsilon(t)|^2.$$

Next we integrate on $(0,t)$ and get as above

$$t\|u_\varepsilon(t)\|^2 + \int_0^t s|Au_\varepsilon(s)|^2 ds \le C\left(1+\int_0^t s\|u_\varepsilon(s)\|^4 ds\right).$$

This yields

$$t\|u_\varepsilon(t)\|^2 \le C,\ \forall t\in(0,T]$$

and therefore

$$\|u_\varepsilon(0)\| = \|u_\varepsilon(T)\| \le C,\ \forall \varepsilon > 0.$$

Now if we multiply $(3.2)''$ by Au_ε, integrate on $(0,t)$ and use the above estimates, we get

$$\|u_\varepsilon(t)\|^2 + \int_0^T |Au_\varepsilon(t)|^2 dt \le C,\ \forall t\in[0,T],\ \varepsilon>0.$$

By $(3.2)''$ we have that

$$\int_0^T |u_\varepsilon'(t)|^2 dt \le C,\ \forall\varepsilon>0.$$

Hence on a subsequence convergent to zero, again denoted ε, we have

$$u_\varepsilon \longrightarrow u_0 \text{ strongly in } C([0,T];H)\cap L^2(0,T;V) \tag{7.13}$$

$$Au_\varepsilon \longrightarrow Au_0 \text{ weakly in } L^2(0,T;H)$$

$$u_\varepsilon' \longrightarrow u_0' \text{ weakly in } L^2(0,T;H)$$

$$\xi_\varepsilon \longrightarrow 0 \text{ weakly in } L^2(0,T;H).$$

The same argument from the proof of Theorem 2.1 implies

$$B(u_\varepsilon) \longrightarrow B(u_0) \text{ strongly in } L^2(0,T;H).$$

Clearly by (7.13) we see that u_0 satisfies the state system (3.2), i.e.,

$$u_0' + \nu Au_0 + B(u_0) = \mathcal{B}_2\phi + f \text{ a.e. } t\in(0,T) \tag{7.14}$$

$$u_0(0) = u_0(T).$$

Finally by (7.11), (7.13), (7.14) we have

$$\operatorname*{Sup}_u(2.1) \le \lim_{\varepsilon\to0}\operatorname*{Sup}_{u,\xi}(3.1) = \lim_{\varepsilon\to0}\int_0^T \frac{1}{2}|C_1u_\varepsilon(t)|^2dt - \lim_{\varepsilon\to0}\frac{1}{2\varepsilon}\int_0^T|\xi_\varepsilon(t)|^2dt$$

$$\le \int_0^T \frac{1}{2}|C_1u_0(t)|^2dt \le \operatorname*{Sup}_u(2.1).$$

We infer from the latter that

$$\lim_{\varepsilon\to0}\operatorname*{Sup}_{u,\xi}(3.1) = \operatorname*{Sup}_u(2.1) = \int_0^T |C_1u_0(t)|^2dt,$$

and

$$u_0 = u_*,\ \lim_{\varepsilon\to0}\frac{1}{2\varepsilon}\int_0^T|\xi_\varepsilon(t)|^2dt = 0$$

which completes the proof. □

Proof of Proposition 3.2 The argument is standard (see [2]), so the proof will only be sketched. For $\lambda\in\mathbf{R}, u\in X$ we set

$$\xi^\lambda = (u_\varepsilon+\lambda u)' + \nu A(u_\varepsilon+\lambda u) + B(u_\varepsilon+\lambda u) - \mathcal{B}_2\phi - f$$

where

$$X = \{u\in W^{1,2}([0,T];H)\cap L^2(0,T;D(A)),\ u(0)=u(T)\}. \tag{7.15}$$

We may write

$$\xi^\lambda = \xi_\varepsilon + \lambda(u' + \nu Au + B'(u_\varepsilon)u + \lambda B(u)).$$

Since $(u_\varepsilon, \xi_\varepsilon)$ is optimal in (3.1), we have

$$\frac{1}{2}\int_0^T |C_1(u_\varepsilon + \lambda u)|^2 dt - \frac{1}{2\varepsilon}\int_0^T |\xi^\lambda|^2 dt \le \frac{1}{2}\int_0^T |C_1 u_\varepsilon|^2 dt - \frac{1}{2\varepsilon}\int_0^T |\xi_\varepsilon|^2 dt \quad \forall \lambda \in \mathbf{R},\ u \in X$$

which yields

$$\int_0^T \left((C_1^* C_1 u_\varepsilon, u) - \frac{1}{\varepsilon}(\xi_\varepsilon, u' + \nu Au + B'(u_\varepsilon)u) \right) dt = 0 \tag{7.16}$$

We set $q_\varepsilon = \varepsilon^{-1}\xi_\varepsilon$ and get from above that $q_\varepsilon \in X$ and satisfies (3.3). □

Proof of Proposition 4.1 Let $\{u_\varepsilon^n, q_\varepsilon^n, \phi^n\}$ be a minimizing sequence in problem (P_ε), i.e.,

$$\inf(P_\varepsilon) \le \int_0^T \left(\frac{1}{2}|C_1 u_\varepsilon^n|^2 + h(\phi^n) - \frac{1}{2\varepsilon}|\xi_\varepsilon^n|^2 \right) dt \le \inf(P_\varepsilon) + \frac{1}{n} \tag{7.17}$$

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{dt}} u_\varepsilon^n(t) + \nu A u_\varepsilon^n(t) + B(u_\varepsilon^n(t)) = \mathcal{B}_2\phi^n(t) + f(t) + \xi_\varepsilon^n(t) & \\ -\frac{\mathrm{d}}{\mathrm{dt}} q_\varepsilon^n(t) + \nu A q_\varepsilon^n(t) + B'(u_\varepsilon^n(t))^* q_\varepsilon^n(t) = C_1^* C_1 u_\varepsilon^n(t) & t \in (0,T), \quad \begin{array}{l} u_\varepsilon^n(0) = u_\varepsilon^n(T) \\ q_\varepsilon^n(0) = q_\varepsilon^n(T). \end{array} \\ \xi_\varepsilon^n(t) = \varepsilon q_\varepsilon^n(t) & \end{cases} \tag{7.18}$$

As in the proof of Proposition 3.1 we see that for $\xi^n = 0$

$$\begin{aligned} \inf(P_\varepsilon) + \frac{1}{n} &\ge \int_0^T h(\phi^n(t))dt + \sup_{u^n, \xi^n} \int_0^T \left(\frac{1}{2}|C_1 u^n|^2 - \frac{1}{2\varepsilon}|\xi^n|^2 \right) dt; \\ &\qquad \frac{\mathrm{d}}{\mathrm{dt}} u^n + \nu A u^n + B(u^n) = \mathcal{B}_2\phi^n + f + \xi^n, \text{ a.e. } t \in (0,T); \quad u^n(0) = u^n(T) \\ &\ge \int_0^T \left(h(\phi^n(t)) + \frac{1}{2}|C_1 u_*^n|^2 \right) dt; \ \frac{\mathrm{d}}{\mathrm{dt}} u_*^n + \nu A u_*^n + B(u_*^n) = \mathcal{B}_2\phi^n + f, \text{ a.e. } t \in (0,T); \quad u_*^n(0) = u_*^n(T). \end{aligned}$$

This yields that $\{\phi^n\}_n$ is bounded in $L^2(0,T;U)$. On the other hand, by (7.17) we get

$$\begin{aligned} \inf(P_\varepsilon) &\le \int_0^T \left(\frac{1}{2}|C_1 u_\varepsilon^n|^2 + h(\phi^n) - \frac{1}{2\varepsilon}|\xi_\varepsilon^n|^2 \right) dt \le \int_0^T \left(\frac{\alpha + \beta\lambda_1}{2\lambda_1}\|u_\varepsilon^n\|^2 + \omega_1|\phi^n|_U^2 + \beta_1 - \frac{1}{2\varepsilon}|\xi_\varepsilon^n|^2 \right) dt \\ &\le \int_0^T \left(C|\phi^n|_U^2 + \left(C - \frac{1}{2\varepsilon}|\xi_\varepsilon^n|^2 \right) \right) dt \end{aligned}$$

where $C = C(\nu, \lambda_1, f, T)$. Therefore for $\varepsilon > 0$ sufficiently small we obtain that $\{\xi_\varepsilon^n\}_n$ is bounded in $L^2(0,T;H)$. Using an argument similar to that in Theorem 2.1 we see that $\{u_\varepsilon^n\}_n$ is compact in $C(0,T;H) \cap L^2(0,T;V)$ and consequently, on a selected subsequence, we have

$$u_\varepsilon^n \to u_\varepsilon \text{ strongly in } L^2(0,T;V) \cap C([0,T];H) \tag{7.19}$$

$$\phi^n \to \phi_\varepsilon \text{ weakly in } L^2(0,T;U) \tag{7.20}$$

$$\xi_\varepsilon^n \to \xi_\varepsilon \text{ weakly in } L^2(0,T;H). \tag{7.21}$$

In order to let $n \to \infty$ in (7.17)-(7.18) we need a strong convergence of ξ_ε^n. Let define the operators

$$\mathcal{A}_{\varepsilon n}\zeta = \zeta' + \nu A\zeta + B'(u_\varepsilon^n)\zeta, \quad \forall \zeta \in D(\mathcal{A}_\varepsilon) = X \tag{7.22}$$

$$\mathcal{A}_{\varepsilon n}^*\zeta = -\zeta' + \nu A\zeta + (B'(u_\varepsilon^n))^*(\zeta), \quad \forall \zeta \in D(\mathcal{A}_\varepsilon) = X \tag{7.22$'$}$$

where X was defined in (7.15).

It is readily seen that

$$\int_0^T (\mathcal{A}_{\varepsilon n}^*\zeta, \eta)\, dt = \int_0^T (\mathcal{A}_{\varepsilon n}\eta, \zeta)\, dt, \ \forall \zeta, \eta \in D(\mathcal{A}_{\varepsilon n}) = D(\mathcal{A}_{\varepsilon n}^*) = X.$$

The operators $\mathcal{A}_\varepsilon$ and $\mathcal{A}_\varepsilon^*$ are defined by the same formulae (7.22) and (7.22)$'$ where $u_\varepsilon^n = u_\varepsilon$. Now let recall without proof a result from [2].

Lemma 7.1 *The operators $\mathcal{A}_{\varepsilon n}, \mathcal{A}^*_{\varepsilon n}, \mathcal{A}_\varepsilon, \mathcal{A}^*_\varepsilon$ are closed, densely defined and have closed ranges in $L^2(0,T;H)$. Moreover, dim $N(\mathcal{A}_{\varepsilon n})$, dim $N(\mathcal{A}^*_{\varepsilon n}) \le n_0$, independent of ε, $\mathcal{A}^*_{\varepsilon n}$ is the adjoint of $\mathcal{A}_{\varepsilon n}$ and the following estimates hold*

$$\|\mathcal{A}_{\varepsilon n}^{-1} g\|_{L^2(0,T;D(A))\cap W^{1,2}([0,T];H)} \le C\|g\|_{L^2(0,T;H)},\ \forall g \in R(\mathcal{A}_{\varepsilon n}) \tag{7.23}$$

$$\|(\mathcal{A}^*_{\varepsilon n})^{-1} g\|_{L^2(0,T;D(A))\cap W^{1,2}([0,T];H)} \le C\|g\|_{L^2(0,T;H)},\ \forall g \in R(\mathcal{A}^*_{\varepsilon n}). \tag{7.24}$$

Similarly, the operators $\mathcal{A}_\varepsilon, \mathcal{A}^*_\varepsilon$ are mutually adjoint and estimates (4.8)-(4.9) *remain true for $\mathcal{A}_\varepsilon$ and $\mathcal{A}^*_\varepsilon$.*

Here we have used the symbols N and R to denote the null space and the range of the corresponding operators. We can rewrite the second equation in (7.18) as

$$\mathcal{A}^*_{\varepsilon n} q^n_\varepsilon = C_1^* C_1 u^n_\varepsilon$$

and q^n_ε as $q^{n1}_\varepsilon + q^{n2}_\varepsilon$ where $q^{n1}_\varepsilon \in R(\mathcal{A}_{\varepsilon n})$, $q^{n2}_\varepsilon \in N(\mathcal{A}^*_{\varepsilon n})$. By (7.19) and Lemma 7.1, part (7.24), we know that

$$\|q^{n1}_\varepsilon\|_{L^2(0,T;D(A))\cap W^{1,2}([0,T];H)} \le C,\ \forall n \in \mathbf{N}. \tag{7.25}$$

Hence on a subsequence, again denoted n, we have

$$q^{n1}_\varepsilon \to q^1_\varepsilon \text{ strongly in } C([0,T];H)\cap L^2(0,T;V). \tag{7.26}$$

Because $\{q^{n2}_\varepsilon\} \subset N(\mathcal{A}^*_{\varepsilon n})$ and dim $N(\mathcal{A}^*_{\varepsilon n}) \le n_0$, by (7.21) we see

$$q^{n2}_\varepsilon \to q^2_\varepsilon \text{ strongly in } L^2(0,T;H). \tag{7.27}$$

Now letting n tend to ∞ into (7.17)-(7.18) it follows by (7.19), (7.20), (7.26)-(7.27) that $(u_\varepsilon, q^1_\varepsilon + q^2_\varepsilon, \phi_\varepsilon)$ satisfies (4.1) and $J_\varepsilon(u_\varepsilon, q_\varepsilon, \phi_\varepsilon) = \inf(P_\varepsilon)$. This completes the proof of Proposition 4.1. □

Proof of Proposition 4.2 We have to prove that

$$\inf_{\phi, u^\phi_*} \sup_u \left\{ \int_0^T \left(\frac{1}{2}|C_1 u|^2 + h(\phi) \right) dt; u' + \nu A u + B(u) = \mathcal{B}_2 \phi + f, u(0) = u(T) \right\}$$

$$= \lim_{\varepsilon \to 0} \inf_{\phi, u^\phi_\varepsilon, \xi^\phi_\varepsilon} \sup_{u,\xi} \left\{ \int_0^T \left(\frac{1}{2}|C_1 u|^2 + h(\phi) - \frac{1}{2\varepsilon}|\xi|^2 \right) dt; u' + \nu A u + B(u) = \mathcal{B}_2 \phi + f + \xi, u(0) = u(T) \right\},$$

where

$$u^\phi_* = arg \left\{ \sup_u \int_0^T \frac{1}{2}|C_1 u|^2 dt; u' + \nu A u + B(u) = \mathcal{B}_2 \phi + f, u(0) = u(T) \right\}$$

$$(u^\phi_\varepsilon, \xi^\phi_\varepsilon) = arg \left\{ \sup_{u,\xi} \int_0^T \left(\frac{1}{2}|C_1 u|^2 + h(\phi) - \frac{1}{2\varepsilon}|\xi|^2 \right) dt; u' + \nu A u + B(u) = \mathcal{B}_2 \phi + f + \xi, u(0) = u(T) \right\}.$$

First, for $\xi := 0$ we have

$$\lim_{\varepsilon \to 0} \inf_{\phi, u^\phi_\varepsilon, \xi^\phi_\varepsilon} \sup_{u,\xi} \left\{ \int_0^T \left(\frac{1}{2}|C_1 u|^2 + h(\phi) - \frac{1}{2\varepsilon}|\xi|^2 \right) dt; u' + \nu A u + B(u) = \mathcal{B}_2 \phi + f + \xi, u(0) = u(T) \right\}$$

$$\ge \inf_{\phi, u^\phi_*} \sup_u \left\{ \int_0^T \left(\frac{1}{2}|C_1 u|^2 + h(\phi) \right) dt; u' + \nu A u + B(u) = \mathcal{B}_2 \phi + f, u(0) = u(T) \right\}.$$

On the other hand, taking $\phi := \phi^*$ we get

$$\lim_{\varepsilon \to 0} \inf_{\phi, u^\phi_\varepsilon, \xi^\phi_\varepsilon} \sup_{u,\xi} \left\{ \int_0^T \left(\frac{1}{2}|C_1 u|^2 + h(\phi) - \frac{1}{2\varepsilon}|\xi|^2 \right) dt; u' + \nu A u + B(u) = \mathcal{B}_2 \phi + f + \xi, u(0) = u(T) \right\}$$

$$\le \lim_{\varepsilon \to 0} \sup_{u,\xi} \left\{ \int_0^T \left(\frac{1}{2}|C_1 u|^2 + h(\phi^*) - \frac{1}{2\varepsilon}|\xi|^2 \right) dt; u' + \nu A u + B(u) = \mathcal{B}_2 \phi^* + f + \xi, u(0) = u(T) \right\}$$

$$= \int_0^T h(\phi^*)dt + \lim_{\varepsilon\to 0}\sup_{u,\xi}\left\{\int_0^T \left(\frac{1}{2}|C_1u|^2 - \frac{1}{2\varepsilon}|\xi|^2\right)dt; u' + \nu Au + B(u) = \mathcal{B}_2\phi^* + f + \xi, u(0) = u(T)\right\}$$

which by Proposition 3.1 is equal to

$$\equiv \left\{\int_0^T \left(h(\phi^*) + \frac{1}{2}|C_1u_*^{\phi^*}|^2\right)dt; \frac{d}{dt}u_*^{\phi^*} + \nu Au_*^{\phi^*} + B(u_*^{\phi^*}) = \mathcal{B}_2\phi^* + f, u_*^{\phi^*}(0) = u_*^{\phi^*}(T)\right\} = \inf_{\phi, u_*^{\phi}}\sup_{u}(P).$$

Then by the two above inequalities we obtain (4.2) as claimed. □

Proof of Proposition 4.3 First, by taking $(\psi_1, \psi_2) = 0$, we get

$$\inf_{\phi,u_\varepsilon,q_\varepsilon,\psi_1,\psi_2}\left\{\int_0^T \left(\frac{1}{2}|C_1u_\varepsilon|^2 - \frac{\varepsilon}{2}|q_\varepsilon|^2 + I_K(q_\varepsilon) + h(\phi) + \frac{1}{2\lambda}|\psi_1|^2 + \frac{1}{2\lambda}|\psi_2|^2\right)dt; (4.3)\right\}$$

$$\leq \inf_{\phi,u_\varepsilon,q_\varepsilon}\left\{\int_0^T \left(\frac{1}{2}|C_1u_\varepsilon|^2 - \frac{\varepsilon}{2}|q_\varepsilon|^2 + I_K(q_\varepsilon) + h(\phi)\right)dt; (4.1)\right\} \equiv \inf_{\phi,u_\varepsilon,q_\varepsilon}(P_\varepsilon)$$

because by Proposition 3.1 we see that $I_K(q_\varepsilon) = 0$, for $\varepsilon > 0$ sufficiently small.
On the other hand, by Proposition 4.1 we have

$$\lim_{\lambda\to 0}\inf_{\phi,u_\varepsilon,q_\varepsilon,\psi_1,\psi_2}(P_{\varepsilon\lambda}) = \lim_{\lambda\to 0}\int_0^T \left(\frac{1}{2}|C_1u_{\varepsilon\lambda}|^2 - \frac{\varepsilon}{2}|q_{\varepsilon\lambda}|^2 + h(\phi_\lambda) + \frac{1}{2\lambda}|\psi_{1\lambda}|^2 + \frac{1}{2\lambda}|\psi_{2\lambda}|^2\right)dt \tag{7.28}$$

where

$$\left\{\begin{array}{l} u'_{\varepsilon\lambda} + \nu Au_{\varepsilon\lambda} + B(u_{\varepsilon\lambda}) = \mathcal{B}_2\phi_\lambda + f + \varepsilon q_{\varepsilon\lambda} + \psi_{1\lambda} \\ -q'_{\varepsilon\lambda} + \nu Aq_{\varepsilon\lambda} + B'(u_{\varepsilon\lambda})^*q_{\varepsilon\lambda} = C_1^*C_1u_{\varepsilon\lambda} + \psi_{2\lambda}\end{array}\right., t \in (0,T), \begin{array}{l} u_{\varepsilon\lambda}(0) = u_{\varepsilon\lambda}(T) \\ q_{\varepsilon\lambda}(0) = q_{\varepsilon\lambda}(T)\end{array} \tag{4.3$'$}$$

and

$$\varepsilon\|q_{\varepsilon\lambda}\|^2_{L^2(0,T;H)} \leq 1,\ \forall\lambda > 0.$$

Hence by assumption (i)$'$ we infer

$$\int_0^T \left(|\phi_\lambda|^2 + \frac{1}{2\lambda}|\psi_{1\lambda}|^2 + \frac{1}{2\lambda}|\psi_{2\lambda}|^2\right)dt \leq C,\ \forall\lambda > 0.$$

Using the argument from Theorem 2.1 we have $\{u_{\varepsilon\lambda}\}$ compact in $C([0,T];H) \cap L^2(0,T;H)$ and as in Proposition 4.1 we get for $q_{\varepsilon\lambda} = q^1_{\varepsilon\lambda} + q^2_{\varepsilon\lambda}$ that

$$\|q^1_{\varepsilon\lambda}\|^2_{L^2(0,T;D(A))\cap W^{1,2}([0,T];H)} + \|q^2_{\varepsilon\lambda}\|^2_{L^2(0,T;H)} \leq C,\ \forall\lambda > 0.$$

Therefore selecting further subsequences, if necessary, we obtain

$$\begin{gathered}\phi_\lambda \to \overline{\phi} \text{ weakly in } L^2(0,T;U) \\ (\psi_{1\lambda}, \psi_{2\lambda}) \to 0 \text{ weakly in } L^2(0,T;H)^2 \\ u_{\varepsilon\lambda} \to \overline{u}_\varepsilon \text{ strongly in } C([0,T];H) \cap L^2(0,T;V) \\ {q_{\varepsilon\lambda}}^1 \to \overline{q}^1_\varepsilon \text{ strongly in } C([0,T];H) \cap L^2(0,T;V) \\ {q_{\varepsilon\lambda}}^2 \to \overline{q}^2_\varepsilon \text{ strongly in } L^2(0,T;H).\end{gathered}$$

Now letting λ go to 0 in (4.3)$'$we see that $(\overline{\phi}, \overline{u}_\varepsilon, \overline{q}_\varepsilon)$ satisfies system (4.1) and

$$\lim_{\lambda\to 0}\inf_{\phi,u_\varepsilon,q_\varepsilon,\psi_1,\psi_2}(P_{\varepsilon\lambda}) \geq \inf_{\phi,u_\varepsilon,q_\varepsilon}(P_\varepsilon)$$

which completes the proof. □

Proof of Theorem 5.1 For $\eta \in \mathbf{R}, (u,q) \in X^2, \phi \in L^2(0,T;U)$ we set

$$(7.29)\qquad \begin{aligned} \psi_1^\eta &= (u_{\varepsilon\lambda}+\eta u)' + \nu A(u_{\varepsilon\lambda}+\eta u) + B(u_{\varepsilon\lambda}+\eta u) - \mathcal{B}_2(\phi_\lambda+\eta\phi) - f - \varepsilon(q_{\varepsilon\lambda}+\eta q) \\ \psi_2^\eta &= -(q_{\varepsilon\lambda}+\eta q)' + \nu A(q_{\varepsilon\lambda}+\eta q) + B'(u_{\varepsilon\lambda}+\eta u)^*(q_{\varepsilon\lambda}+\eta q) - C_1^*C_1(u_{\varepsilon\lambda}+\eta u). \end{aligned}$$

We may rewrite ψ_1^η, ψ_2^η as

$$\begin{aligned} \psi_1^\eta &= \psi_{1\lambda} + \eta\Big(u' + \nu Au + B'(u_{\varepsilon\lambda})u + \eta B(u) - \mathcal{B}_2\phi - \varepsilon q\Big) \\ \psi_2^\eta &= \psi_{2\lambda} + \eta\Big(-q' + \nu Aq + B'(u_{\varepsilon\lambda})^*q + B'(u)^*q_{\varepsilon\lambda} + \eta B'(u)^*q - C_1^*C_1u\Big) \end{aligned}$$

and so by the optimality of $(\phi_\lambda, u_{\varepsilon\lambda}, q_{\varepsilon\lambda}, \psi_{1\lambda}, \psi_{2\lambda})$ we have

$$(7.30)\qquad \begin{aligned} &\int_0^T \Bigg((C_1^*C_1u_{\varepsilon\lambda}, u) - \varepsilon(q_{\varepsilon\lambda}, q) + \frac{1}{\eta}\left(I_K(q_{\varepsilon\lambda}+\eta q) - I_K(q_{\varepsilon\lambda})\right) + \frac{1}{\eta}(h(\phi_\lambda+\eta\phi) - h(\phi_\lambda)) \\ &+\frac{1}{\lambda}\left(\psi_{1\lambda}, \mathcal{A}_{\varepsilon\lambda}u + \eta B(u) - \mathcal{B}_2\phi - \varepsilon q\right) + \frac{1}{\lambda}\left(\psi_{2\lambda}, \mathcal{A}^*_{\varepsilon\lambda}q + B'(u)^*q_{\varepsilon\lambda} + \eta B'(u)^*q - C_1^*C_1u\right) \\ &+\frac{\eta}{2\lambda}(|\mathcal{A}_{\varepsilon\lambda}u + \eta B(u) - \mathcal{B}_2\phi - \varepsilon q|^2 + |\mathcal{A}^*_{\varepsilon\lambda}q + B'(u)^*q_{\varepsilon\lambda} + \eta B'(u)^*q - C_1^*C_1u|^2)\Bigg)dt \geq 0. \end{aligned}$$

Here the operators $\mathcal{A}_{\varepsilon\lambda}$ and $\mathcal{A}^*_{\varepsilon\lambda}$ are defined by the same formulae (7.22) and (7.22)$'$ where $u_\varepsilon^n = u_{\varepsilon\lambda}$. We set $U_\lambda = \lambda^{-1}\psi_{1\lambda}, Q_\lambda = \lambda^{-1}\psi_{2\lambda}$. If in (7.30) we take $\phi = 0, q = 0$ we get

$$\int_0^T \Big((C_1^*C_1u_{\varepsilon\lambda}, u) + (U_\lambda, \mathcal{A}_{\varepsilon\lambda}u) + (Q_\lambda, B'(u)^*q_{\varepsilon\lambda} - C_1^*C_1u)\Big)dt = 0.$$

Hence $U_\lambda \in D(\mathcal{A}^*_{\varepsilon\lambda})$ and

$$(7.31)\qquad \mathcal{A}^*_{\varepsilon\lambda}U_\lambda = C_1^*C_1Q_\lambda - B'(Q_\lambda)^*q_{\varepsilon\lambda} - C_1^*C_1u_{\varepsilon\lambda}.$$

If in (7.30) we take $\phi = 0, u = 0$ we also get

$$\int_0^T \left(\varepsilon(q_{\varepsilon\lambda}, q) + \frac{1}{\eta}\left(I_K(q_{\varepsilon\lambda}+\eta q) - I_K(q_{\varepsilon\lambda})\right) + (U_\lambda, -\varepsilon q) + (Q_\lambda, \mathcal{A}^*_{\varepsilon\lambda}q) + \frac{\eta}{2\lambda}(|-\varepsilon q|^2 + |\mathcal{A}^*_{\varepsilon\lambda}q|^2)\right)dt = 0.$$

Let assume that in $(P_{\varepsilon\lambda})$ and (7.30) we have instead of I_K the Moreau-Yosida approximation I_K^ϵ, $\epsilon > 0$, which is convex and Fréchet differentiable (see [3]). By letting $\eta \to 0$ we obtain from above that

$$\int_0^T \Big((\varepsilon q_{\varepsilon\lambda} - \varepsilon U_\lambda, q) + (Q_\lambda, \mathcal{A}^*_{\varepsilon\lambda}q) + (\partial I_K^\epsilon(q_{\varepsilon\lambda}), q)\Big)dt = 0.$$

Hence $Q_\lambda \in D(\mathcal{A}_{\varepsilon\lambda})$ and

$$(7.32)\qquad \mathcal{A}_{\varepsilon\lambda}Q_\lambda = \varepsilon U_\lambda - \varepsilon q_{\varepsilon\lambda} - \partial I_K^\epsilon(q_{\varepsilon\lambda}).$$

Now using once again (7.30) we see that

$$\int_0^T \Big(h'(\phi_\lambda, \phi) - (\mathcal{B}_2U_\lambda, \phi)_U\Big)dt \geq 0$$

where h' is the directional derivative of h. This yields the optimality condition (5.2)

$$\mathcal{B}_2^*\phi_\lambda \in \partial h(U_\lambda), \quad \text{a.e. in } (0,T).$$

We recall that $\partial I_K^\epsilon(q) = (\partial I_K)_\epsilon(q) \xrightarrow{\epsilon\to 0} (\partial I_K)^0(q) = 0$ on $D(\partial I_K)$, where $(\partial I_K)_\epsilon$ denote the Yosida approximation of the maximal monotone operator ∂I_K and $(\partial I_K)^0(q)$ is the element of minimum norm in $(\partial I_K)(q)$. Therefore letting $\epsilon \to 0$ in (7.32) we obtain the second adjoint state equation from (5.1) and the proof of Theorem 5.1 is complete. □

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