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Optimal Control of the Periodic String Equation with Internal Control

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Abstract. This paper is concerned with the existence and the maximum principle for optimal control problems governed by the periodic vibrating string equation on $(0, \pi) \times (0, T)$ with Dirichlet boundary conditions. The case of internal controllers supported on $\omega \subset (0, \pi)$ is examined.

Key Words. Convex functions, weak solutions, self-adjoint operators, periodic conditions, null spaces, wave operators, internal controls.

1. Introduction

We are concerned with the following optimal control problem:

$$\min \int_Q (g(y(x,t)+y_0(x,t))+h(u(x,t)+u_0(x,t)))\,dx\,dt, \tag{1}$$

s.t. $u \in L^2(Q)$, $Q=(0,\pi)\times(0,T)$, and

$$y_{tt}(x,t)-y_{xx}(x,t)=m(x)u(x,t), \quad x\in(0,\pi),\ t\in\mathbb{R}, \tag{2a}$$

$$y(0,t)=y(\pi,t)=0, \qquad t\in\mathbb{R}, \tag{2b}$$

$$y(x,t+T)=y(x,t), \qquad x\in(0,\pi),\ t\in\mathbb{R}, \tag{2c}$$

where m is the characteristic function of ω, i.e.,

$$m(x)=\begin{cases}1, & \text{for } x\in\omega,\\ 0, & \text{for } x\bar{\in}\omega,\end{cases} \tag{2d}$$

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and ω is an open subset of $(0, \pi)$. Here, $g: \mathbb{R} \to \bar{\mathbb{R}} =]-\infty, +\infty]$ and $h: \mathbb{R} \to \bar{\mathbb{R}}$, are lower semicontinuous, convex functions, and

$$y_0 \in L^\infty(Q), \qquad u_0 \in L^\infty(Q),$$

$$u_0(x, t) = u_0(x, t+T), y_0(x, t) = y_0(x, t+T), \qquad \text{a.e. in } Q.$$

By solution to (2) we mean a weak solution; i.e., $y \in L^2(Q)$ and

$$\int_Q y(x, t)(\varphi_{tt}(x, t) - \varphi_{xx}(x, t))\, dx\, dt$$
$$= \int_Q m(x)u(x, t)\varphi(x, t)\, dx\, dt, \qquad \forall \varphi \in X, \tag{3}$$

where

$$X = \{\varphi \in C^2([0, \pi] \times [0, T)];$$
$$\varphi(0, t) = \varphi(\pi, t) = 0, \varphi(x, 0) = \varphi(x, T) = 0,$$
$$\varphi_t(x, 0) = \varphi_t(x, T), \forall (x, t) \in [0, \pi] \times [0, T]\}.$$

Let $A: L^2(Q) \to L^2(Q)$ be the wave operator (2), i.e.,

$$Ay = f, \qquad \text{for } [y, f] \in D(A) \times R(A), \text{ if and only if} \tag{4}$$

$$\int_Q y(\varphi_{tt} - \varphi_{xx})\, dx\, dt = \int_Q f\varphi\, dx\, dt, \qquad \forall \varphi \in X. \tag{5}$$

In terms of A, the weak solution of (2) is the solution to operator equation $Ay = mu$. In Proposition 1.1 below, we recall for later use some properties of A (see Refs. 1 and 2). Let $R(A)$ denote the range of A.

Proposition 1.1. Assume that T/π is rational. Then, A is self-adjoint, $R(A)$ is closed, and $A^{-1} \in L(R(A), R(A))$. Moreover, A^{-1} is compact on $R(A)$, and, in addition,

$$\|A^{-1}f\|_{L^\infty(Q)} \leq C\|f\|_{L^1(Q)}, \qquad \forall f \in R(A), \tag{6}$$

$$\|A^{-1}f\|_{H^1(Q)} \leq C\|f\|_{L^2(Q)}, \qquad \forall f \in R(A). \tag{7}$$

In particular, it follows from Proposition 1.1 that the weak solution y of (2), if any, is unique modulo the null space

$$N(A) = \{y \in L^2(Q); Ay = 0\}.$$

Note that the space $L^2(Q)$ admits the orthogonal decomposition $L^2(Q) = R(A) \oplus N(A)$ and that A^{-1} is continuous from $R(A)$ into itself. Here, we are concerned with two distinct questions: maximum principle and existence of optimal controllers for Problem (1). These two problems were approached in Ref. 3 in the case where $m \equiv 1$. Here, we may take

$$m \in L^\infty(0, \pi), \qquad m(x) \geq \rho > 0, \text{ a.e. in } \omega, \qquad m(x) = 0, \text{ else.}$$

This corresponds to the situation of the string equation with internal controller u supported on $\omega \subset (0, \pi)$. The treatment is similar to that used in Ref. 3, but with some important differences. In Ref. 4, this problem was treated in dimension 2, with h having a quadratic growth. For other related works, see Refs. 5 and 6.

The conditions that we impose on g and h allow state and control constraints into Problem (1). A typical example is the situation where

$$g(y) = (1/2)|y|^2, \qquad h(u) = \alpha |u|^2 + I_{[a,b]}(u), \qquad \alpha > 0,$$

with $I_{[a,b]}$ the indicator function of $[a, b]$.

2. Maximum Principle

Here, we assume that:

(A1) the function $g: \mathbb{R} \to \mathbb{R}$ is continuous and convex;

(A2) the function $h: \mathbb{R} \to \bar{\mathbb{R}}$ is convex, lower semicontinuous and $u_0(x, t) \in K \subset \operatorname{int} D(h)$, a.e. $(x, t) \in Q$, where K is a compact subset.

Theorem 2.1. Assume that T is a rational multiple of π and that Assumptions (A1) and (A2) hold. Then, the pair $(y^*, u^*) \in L^\infty(Q) \times L^\infty(Q)$ is optimal for Problem (1) if, and only if, there are $p \in L^\infty(Q)$ and $w \in L^\infty(Q)$ such that

$$p_{tt} - p_{xx} = -w, \qquad \text{in } Q = (0, \pi) \times (0, T), \tag{8a}$$

$$p(0, t) = p(\pi, t) = 0, \qquad \forall t \in (0, T), \tag{8b}$$

$$p(x, 0) = p(x, T), \qquad p_t(x, 0) = p_t(x, T), \qquad \forall x \in (0, \pi), \tag{8c}$$

$$w(x, t) \in \partial g(y^*(x, t) + y_0(x, t)), \qquad \text{a.e. } (x, t) \in Q, \tag{9}$$

$$u^*(x, t) \in \partial h^*(m(x) p(x, t)) - u_0(x, t), \qquad \text{a.e. } (x, t) \in Q. \tag{10}$$

Here, h^* is the conjugate function of h, i.e.,

$$h^*(p) = \sup\{pu - h(u); u \in \mathbb{R}\}, \qquad \forall p \in \mathbb{R},$$

and $\partial g: \mathbb{R} \to 2^{\mathbb{R}}$ and $\partial h^*: \mathbb{R} \to 2^{\mathbb{R}}$ are the subdifferentials of g and h^*, respectively. Throughout what follows, we denote by $(\cdot, \cdot)$ the usual scalar product in $L^2(Q)$. The solution $p \in L^\infty(Q)$ to (8) should, of course, be considered in the weak sense, i.e.,

$$Ap = -w. \tag{8d}$$

Proof. It is readily seen that Eqs. (8–10) are sufficient for optimality. To prove necessity, we fix an optimal pair (y^*, u^*) and consider the approximating control problem

$$\min\left\{\int_Q (g_\epsilon(y+y_0) + (1/2)|y-y^*|^2)\, dx\, dt + \int_Q (h_\epsilon(u+u_0) + (1/2)|u-u^*|^2)\, dx\, dt; \quad (y, u) \in L^2(Q) \times L^2(Q),\ Ay = mu\right\}, \tag{11}$$

where $g_\epsilon \in C^1(R)$ is the convex regularization of g, i.e.,

$$g_\epsilon(r) = \inf\{|r-s|^2/2\epsilon + g(s);\, s \in \mathbb{R}\}, \qquad \forall r \in \mathbb{R};$$

h_ϵ is defined similarly.

Since by Proposition 1.1 the affine manifold $\{(y, u) \in L^2(Q) \times L^2(Q);\ Ay = mu\}$ is closed and the cost functional is strictly convex and coercive, Problem (11) has a unique solution $(y_\epsilon, u_\epsilon) \in L^2(Q) \times L^2(Q)$. We have

$$\begin{aligned} &\int_Q (g_\epsilon(y_\epsilon + y_0) + (1/2)|y_\epsilon - y^*|^2)\, dx\, dt \\ &\quad + \int_Q (h_\epsilon(u_\epsilon + u_0) + (1/2)|u_\epsilon - u^*|^2)\, dx\, dt \\ &\le \int_Q (g_\epsilon(y^* + y_0) + h(u^* + u_0))\, dx\, dt \\ &\le \inf(1). \end{aligned}$$

Since the function

$$(y, u) \to \int_Q g(y+y_0)\, dx\, dt + \int_Q h(u+u_0)\, dx\, dt$$

is weakly lower semicontinuous on $L^2(Q) \times L^2(Q)$ and $g_\epsilon \to g$ as $\epsilon \to 0$, the latter yields

$$\lim_{\epsilon \to 0} \int_Q (|u_\epsilon - u^*|^2 + |y_\epsilon - y^*|^2)\, dx\, dt = 0. \tag{12}$$

Next, we have

$$\begin{aligned}
&\int_Q (g_\epsilon(y_\epsilon + y_0) + (1/2)|y_\epsilon - y^*|^2)\, dx\, dt \\
&\quad + \int_Q (h_\epsilon(u_\epsilon + u_0) + (1/2)|u_\epsilon - u^*|^2)\, dx\, dt \\
&\leq \int_Q (g_\epsilon(y_\epsilon + \lambda z + y_0) + (1/2)|y_\epsilon + \lambda z - y^*|^2)\, dx\, dt \\
&\quad + \int_Q (h_\epsilon(u_\epsilon + \lambda u + u_0) + (1/2)|u_\epsilon + \lambda u - u^*|^2)\, dx\, dt,
\end{aligned}$$

for all $\lambda > 0$, $(z, u) \in L^2(Q) \times L^2(Q)$, $Az = mu$. This yields

$$\begin{aligned}
&\int_Q (g'_\epsilon(y_\epsilon + y_0)z + (y_\epsilon - y^*)z)\, dx\, dt \\
&\quad + \int_Q (h'_\epsilon(u_\epsilon + u_0)u + (u_\epsilon - u^*)u)\, dx\, dt = 0,
\end{aligned} \tag{13}$$

for all $(z, u) \in L^2(Q) \times L^2(Q)$ such that $Az = mu$. In particular, for $u = 0$, Equation (13) yields

$$g'_\epsilon(y_\epsilon + y_0) + y_\epsilon - y^* \in N(A)^\perp = R(A). \tag{14}$$

Hence, there is $p_\epsilon \in L^2(Q)$ such that

$$Ap_\epsilon = -g'_\epsilon(y_\epsilon + y_0) - y_\epsilon + y^*. \tag{15}$$

Substituting the latter in (13) yields

$$(Ap_\epsilon, z) - \int_Q (h'_\epsilon(u_\epsilon + u_0)u + (u_\epsilon - u^*)u)\, dx\, dt = 0.$$

Therefore, we have

$$\int_Q (mp_\epsilon + u^* - u_\epsilon - h'_\epsilon(u_\epsilon + u_0))u\, dx\, dt = 0, \qquad \forall u \in Y, \tag{16}$$

where

$$Y = \{u \in L^2(Q), mu \in R(A)\}.$$

Note that the orthogonal complement $Y^\perp$ of Y in $L^2(Q)$ is precisely the space $\{mv;\, v \in N(A)\}$. Indeed, $\forall u \in N(A)$, $\forall v \in Y$, i.e., $mv \in R(A)$, we have

$$0 = (mv, u) = (v, mu),$$

which yields $mN(A) \subset Y^\perp$. Conversely, $\forall v \in (mN(A))^\perp$, $\forall u \in N(A)$, since $(v, mu) = 0 = (mv, u)$, we get $v \in Y$ and equivalently $Y^\perp \subset mN(A)$. Hence,

$$mp_\epsilon + u^* - u_\epsilon - h'_\epsilon(u_\epsilon) \in Y^\perp = mN(A).$$

Let $\eta_\epsilon \in N(A)$. If we denote again by p_ϵ the function $p_\epsilon - \eta_\epsilon$, we get

$$mp_\epsilon + u^* - u_\epsilon = h'_\epsilon(u_\epsilon + u_0), \qquad \text{a.e. in } Q, \tag{17a}$$

$$u_\epsilon = (1 + h'_\epsilon)^{-1}(mp_\epsilon + u^* + u_0) - u_0, \qquad \text{a.e. } (x, t) \in Q. \tag{17b}$$

We write $p_\epsilon = p^1_\epsilon + p^2_\epsilon$ and $y_\epsilon = y^1_\epsilon + y^2_\epsilon$, where p^1_ϵ, $y^1_\epsilon \in R(A)$ and p^2_ϵ, $y^2_\epsilon \in N(A)$. It is readily seen that (see Ref. 1)

$$N(A) = \left\{ y \in L^2(Q);\, y(x, t) = q(t + x) - q(t - x),\, q \text{ is } \tau\text{-periodic}, \right.$$

$$\left. \int_0^\tau q(s)\, ds = 0,\, \tau = 2\pi/n = T/m \right\}, \tag{18}$$

and $\eta \in L^2(Q)$ belongs to $R(A) = N(A)^\perp$ if and only if

$$\int_0^\pi (\eta(x, t - x) - \eta(x, t + x))\, dx = 0, \qquad \text{a.e. } t \in (0, T). \tag{19}$$

Since $\{u_\epsilon\}$ is bounded in $L^2(Q)$, we have by Proposition 1.1 and (2) that

$$\|y^1_\epsilon\|_{L^\infty(Q)} \le C, \qquad \forall \epsilon > 0. \tag{20a}$$

Concerning y^2_ϵ, we have the following lemma.

Lemma 2.1. For each $\epsilon > 0$, $y^2_\epsilon \in L^\infty(Q)$ and there is $C > 0$ such that

$$\|y^2_\epsilon\|_{L^\infty(Q)} \le C, \qquad \forall \epsilon > 0. \tag{20b}$$

Proof. The proof is as in Ref. 3, but it will be sketched for the reader's convenience. Since $y^2_\epsilon \in N(A)$, we may write

$$y^2_\epsilon(x, t) = q_\epsilon(t - x) - q_\epsilon(t + x), \qquad \forall (x, t) \in Q, \tag{21}$$

where q_ϵ is τ-periodic. This yields

$$q_\epsilon(t)=(1/2)\int_0^\pi (y_\epsilon^2(x,t-x)-y_\epsilon^2(x,t+x))\,dx, \quad \forall t\in(0,\tau). \tag{22}$$

By (12) and (20a), we infer that

$$\|q_\epsilon\|_{L^2(0,\tau)}\leq C, \qquad \forall\epsilon>0. \tag{23}$$

Then, by (15), we have

$$\int_0^\pi (G_\epsilon(y_\epsilon^1(x,t-x)+y_\epsilon^2(x,t-x)+y_0(x,t-x))$$
$$-G_\epsilon(y_\epsilon^1(x,t+x)+y_\epsilon^2(x,t+x)+y_0(x,t-x)))\,dx$$
$$=-\int_0^\pi (y^*(x,t-x)+y_0(x,t-x)-y^*(x,t+x)$$
$$-y_0(x,t+x))\,dx, \qquad \text{a.e. } t\in(0,T),$$

where

$$G_\epsilon(r)=r+g'_\epsilon(r), \qquad \forall r\in\mathbb{R}.$$

We note that the weak solution to (2) is also a mild solution (see Refs. 4 and 7); i.e., $y\in C([0,T], L^2(0,\pi))$ and the above relation is true for all $t\in(0,T)$. Since G_ϵ is nondecreasing and $y^*\in L^\infty(Q)$, by (20a) and (21) we have

$$(1/2\pi)\int_0^\pi (G_\epsilon(-C+q_\epsilon(t)-q_\epsilon(t-2x))-G_\epsilon(C-q_\epsilon(t)+q_\epsilon(t+2x)))\,dx$$
$$=(1/2\pi)\int_0^\pi (G_\epsilon(-C+y_\epsilon^2(x,t-x))-G_\epsilon(C+y_\epsilon^2(x,t+x)))\,dx$$
$$\leq\|y^*+y_0\|_{L^\infty(Q)}, \qquad \forall t\in(0,T).$$

Finally, since q_ϵ is τ-periodic, we see that

$$(1/4\pi)\int_0^{2\pi} (G_\epsilon(-C+q_\epsilon(t)-q_\epsilon(s))-G_\epsilon(C-q_\epsilon(t)+q_\epsilon(s)))\,ds$$
$$\leq\|y^*+y_0\|_{L^\infty(Q)}, \qquad \forall t\in(0,T). \tag{24}$$

Let

$$M_\epsilon = \text{ess sup}\{q_\epsilon(t); \forall t \in (0, T)\},$$
$$E_\epsilon = \{s \in (0, 2\pi); q_\epsilon(s) > 2^{-1}M_\epsilon\}.$$

By (23), we have

$$m(E_\epsilon) \leq CM_\epsilon^{-2}, \qquad \forall \epsilon > 0, \tag{25}$$

and so (23) yields

$$(4\pi)^{-1}(2\pi - m(E_\epsilon))(G_\epsilon(-C + 2^{-1}M_\epsilon) - G_\epsilon(C - 2^{-1}M_\epsilon))$$
$$\leq m(E_\epsilon)(G_\epsilon(C) - G_\epsilon(-C)) + \|y^* + y_0\|_{L^\infty(Q)}. \tag{26}$$

In particular, it follows that

$$M_\epsilon < +\infty, \qquad \forall \epsilon > 0.$$

Since

$$G_\epsilon(C) = C + g'_\epsilon(C) \leq C + \inf\{|\theta|; \theta \in \partial g(C)\},$$

we infer that

$$\sup\{M_\epsilon; \epsilon > 0\} < \infty.$$

Similarly, it follows that

$$\text{ess inf}\{q_\epsilon(t); T \in (0, T)\} > C, \qquad \forall \epsilon > 0.$$

Hence,

$$\|q_\epsilon\|_{L^\infty(0,T)} \leq C,$$

and by (21) we get (20b) as desired. This completes the proof of Lemma 2.1. □

Since ∂g is locally bounded in $\mathbb{R} \times \mathbb{R}$, we have that

$$\|g'_\epsilon(y_\epsilon + y_0)\|_{L^\infty(Q)} \leq C, \qquad \forall \epsilon > 0, \tag{27}$$

and so by (15) we deduce via Proposition 1.1 that

$$\|p_\epsilon^1\|_{L^\infty(Q)} \leq C, \qquad \forall \epsilon > 0. \tag{28}$$

In order to prove the boundedness of $\{p_\epsilon^2\}$ in $L^\infty(Q)$, we need the boundedness of $\{mp_\epsilon\}$ in $L^1(Q)$. Indeed, by (15) and $Ay_\epsilon = mu_\epsilon$, we have

$$(mu_\epsilon, p_\epsilon) = (Ap_\epsilon, y_\epsilon)$$
$$\leq -\int_Q (g_\epsilon(y_\epsilon + y_0) - g_\epsilon(y_0))\, dx\, dt + (y^* - y_\epsilon, y_\epsilon).$$

Since (17) is equivalent to

$$h_\epsilon(u_\epsilon+u_0)+h_\epsilon^*(mp_\epsilon+u^*-u_\epsilon)$$
$$=(u_\epsilon+u_0)(mp_\epsilon+u^*-u_\epsilon), \qquad \text{a.e. in } Q,$$

the latter yields

$$\int_Q (h_\epsilon^*(mp_\epsilon+u^*-u_\epsilon)+h_\epsilon(u_\epsilon+u_0)+(u_\epsilon+u_0)(u_\epsilon-u^*)-u_0 mp_\epsilon)\,dx\,dt$$
$$\leq -\int_Q (g_\epsilon(y_\epsilon+y_0)-g_\epsilon(y_0))\,dx\,dt+(y^*-y_\epsilon, y_\epsilon). \tag{29}$$

By Assumption (A2), it follows that

$$\rho|mp_\epsilon| \leq h_\epsilon^*(mp_\epsilon+u^*-u_\epsilon)+h_\epsilon(u_0+\rho(mp_\epsilon/|mp_\epsilon|))$$
$$-u_0 mp_\epsilon+\rho|u^*-u_\epsilon|-u_0(u^*-u_\epsilon)$$
$$\leq h_\epsilon^*(mp_\epsilon+u^*-u_\epsilon)-u_0 mp_\epsilon+\rho|u^*-u_\epsilon|$$
$$-u_0(u^*-u_\epsilon)+C, \qquad \text{a.e. in } Q, \tag{30}$$

for $\rho>0$ and sufficiently small. Then, (29) and (30) imply that

$$\|mp_\epsilon\|_{L^1(Q)} \leq C, \qquad \forall \epsilon>0, \tag{31}$$

and so by (28) we have that $\{mp_\epsilon^2\}$ is bounded in $L^1(Q)$.

Lemma 2.2. For each $\epsilon>0$, $p_\epsilon^2 \in L^\infty(Q)$ and

$$\|p_\epsilon^2\|_{L^\infty(Q)} \leq C, \qquad \forall \epsilon>0. \tag{32}$$

Proof. We argue as in the proof of Lemma 2.1. Since $p_\epsilon^2 \in N(A)$, we have that

$$p_\epsilon^2(x,t)=q_\epsilon(t+x)-q_\epsilon(t-x), \qquad \forall (x,t)\in Q,$$
$$\text{where } q_\epsilon \text{ is } \tau\text{-periodic}, \int_0^\tau q_\epsilon(s)\,ds=0,\ \tau=2\pi/m=T/n. \tag{33}$$

We set

$$M_\epsilon = \operatorname{ess\,sup}\{q_\epsilon(t);\, t\in(0,T)\},$$
$$E_\epsilon=\{x\in(0,\pi);\, q_\epsilon(t+2x)>M_\epsilon/2\}$$
$$=\{\sigma\in(t,t+2\pi);\, q_\epsilon(\sigma)>M_\epsilon/2\},$$
$$\tilde{E}_\epsilon=\{x\in(0,\pi);\, q_\epsilon(t-2x)>M_\epsilon/2\}$$
$$=\{\sigma\in(t-2\pi,t);\, q_\epsilon(\sigma)>M_\epsilon/2\}=E_\epsilon-2\pi.$$

By (17b), we see that

$$-mu_0+m(1+\partial h_\epsilon)^{-1}(mp_\epsilon+u^*)\in R(A)=N(A)^\perp,$$

so by (19), we have

$$\int_0^\pi m(x)(u_0(x,t-x)-u_0(x,t+x))\,dx$$
$$=\int_0^\pi (m(x)H_\epsilon(m(x)p_\epsilon^1(x,t-x)+m(x)p_\epsilon^2(x,t-x)+u^*(x,t-x))$$
$$-m(x)H_\epsilon(m(x)p_\epsilon^1(x,t+x)+m(x)p_\epsilon^2(x,t+x)+u^*(x,t+x)))\,dx,\qquad \text{a.e. } t\in(0,T),$$

where

$$H_\epsilon=(1+\partial h_\epsilon)^{-1}.$$

Then, by (33), we get

$$\int_0^\pi m(x)(u_0(x,t-x)-u_0(x,t+x))\,dx$$
$$\geq\int_0^\pi (m(x)H_\epsilon(-C+m(x)q_\epsilon(t)-m(x)q_\epsilon(t-2x))$$
$$-m(x)H_\epsilon(C+m(x)q_\epsilon(t+2x)-m(x)q_\epsilon(t)))\,dx,\qquad \text{a.e. } t\in(0,T),$$

where

$$C=\|mp_\epsilon^1\|_{L^\infty(Q)}+\|u^*\|_{L^\infty(Q)}.$$

We have

$$\int_0^\pi m(x)(u_0(x,t-x)-u_0(x,t+x))\,dx$$
$$\geq(1/2)\int_{(t-2\pi,t)\cap\tilde{E}_\epsilon} m((t-s)/2)$$
$$\times H_\epsilon(-C+m((t-s)/2)q_\epsilon(t)-m((t-s)/2)q_\epsilon(s))\,ds$$
$$+(1/2)\int_{(t-2\pi,t)\setminus\tilde{E}_\epsilon} m((t-s)/2)$$
$$\times H_\epsilon(-C+m((t-s)/2)q_\epsilon(t)-m((t-s)/2)q_\epsilon(s))\,ds$$

$$-(1/2)\int_{(t,t+2\pi)\cap E_\epsilon} m((s-t)/2)$$
$$\times H_\epsilon(C+m((s-t)/2)q_\epsilon(s)-m((s-t)/2)q_\epsilon(t))\,ds$$
$$-(1/2)\int_{(t,t+2\pi)\setminus E_\epsilon} m((s-t)/2)$$
$$\times H_\epsilon(C+m((s-t)/2)q_\epsilon(s)-m((s-t)/2)q_\epsilon(t))\,ds$$
$$\geq (1/2)\int_{(t-2\pi,t)\cap \tilde{E}_\epsilon} m((t-s)/2)H_\epsilon(-C+m((t-s)/2)(q_\epsilon(t)-M_\epsilon))\,ds$$
$$+(1/2)\int_{(t-2\pi,t)\setminus \tilde{E}_\epsilon} m((t-s)/2)H_\epsilon(-C+m((t-s)/2)(q_\epsilon(t)-M_\epsilon/2))\,ds$$
$$-(1/2)\int_{(t,t+2\pi)\cap E_\epsilon} m((s-t)/2)H_\epsilon(C+m((s-t)/2)(M_\epsilon-q_\epsilon(t)))\,ds$$
$$-(1/2)\int_{(t,t+2\pi)\setminus E_\epsilon} m((s-t)/2)H_\epsilon(C+m((s-t)/2)(M_\epsilon/2-q_\epsilon(t)))\,ds,$$
$$\forall t\in(0,T).$$

Therefore,

$$\int_0^\pi m(x)(u_0(x,t-x)-u_0(x,t+x))\,dx$$
$$\geq (1/2)\int_{(t-2\pi,t)\cap \tilde{E}_\epsilon} m((t-s)/2)H_\epsilon(-C)\,ds$$
$$+(1/2)\int_{(t-2\pi,t)\setminus \tilde{E}_\epsilon} m((t-s)/2)H_\epsilon(-C+m((t-s)/2)M_\epsilon/2)\,ds$$
$$-(1/2)\int_{(t,t+2\pi)\cap E_\epsilon} m((s-t)/2)H_\epsilon(C)\,ds$$
$$-(1/2)\int_{(t,t+2\pi)\setminus E_\epsilon} m((s-t)/2)H_\epsilon(C-(M_\epsilon/2)m((s-t)/2))\,ds, \qquad \forall t\in\Sigma,$$

where

$$\Sigma=\{t\in(0,T);\, q_\epsilon(t)=\|q_\epsilon\|_{C([0,T])}\}.$$

Finally,

$$\begin{aligned}
&\int_0^\pi m(x)(u_0(x, t-x) - u_0(x, t+x))\, dx \\
&\geq \int_{(0,\pi)\cap \tilde{E}_\epsilon} m(x) H_\epsilon(-C)\, dx - \int_{(0,\pi)\cap E_\epsilon} m(x) H_\epsilon(C)\, dx \\
&\quad + \int_{(0,\pi)\setminus \tilde{E}_\epsilon} m(x) H_\epsilon(-C + m(x)(M_\epsilon/2))\, dx \\
&\quad - \int_{(0,\pi)\setminus E_\epsilon} m(x) H_\epsilon(C - m(x)(M_\epsilon/2))\, dx.
\end{aligned} \tag{34}$$

On the other hand,

$$\nu(E_\epsilon) \leq C M_\epsilon^{-1}, \qquad \forall \epsilon > 0, \tag{35}$$

where $\nu(E_\epsilon)$ is the Lebesgue measure of E_ϵ. Indeed, by (33), we have

$$q_\epsilon(t) = q_\epsilon(t-2x) + p_\epsilon^2(x, t-x), \qquad \text{a.e. } (x, t) \in Q.$$

Since $\nu(E_\epsilon)$ does not depend on t, multiplying by $m(x)$ and integrating on $E_\epsilon \times (0, T)$, it follows that

$$\begin{aligned}
&\|m\|_{L^\infty(0,\pi)} \nu(E_\epsilon) \int_0^T q_\epsilon(t)\, dt \\
&\geq \int_0^T \int_{E_\epsilon} m(x) p_\epsilon(x, t-x)\, dx\, dt + (M_\epsilon/2) T \int_{E_\epsilon} m(x)\, dx.
\end{aligned}$$

By (31) and (33), the latter implies (35). Hence, by (34) and (35), we see that

$$\limsup_{\epsilon \to 0} M_\epsilon < \infty.$$

Similarly, it follows that $\{\operatorname{ess\,inf}\{q_\epsilon(t); t \in (0, T)\}\}$ is bounded from below, and so,

$$\|q_\epsilon\|_{L^\infty(0,T)} \leq C, \qquad \forall \epsilon > 0.$$

Then, by (33), we find (32) as desired. This completes the proof of Lemma 2.2. □

On a subsequence, again denoted ϵ, we have

$$p_\epsilon \to p \text{ weak star in } L^\infty(Q),$$

$$g'_\epsilon(y_\epsilon + y_0) \to w \text{ weak star in } L^\infty(Q).$$

Then, letting ϵ tend to zero in (15), (17), by (12) we see that w and p satisfy (8–10). This completes the proof of Theorem 2.1. □

Theorem 2.1 remains true for the nonhomogeneous equation

$$Ay = mu + f, \qquad f \in L^\infty(Q), \tag{36}$$

in place of (2), if we assume, in addition to (A1) and (A2), that

$$(1-m)f \in R(A), \qquad u_0 + f \in \operatorname{int} D(h), \qquad \text{a.e. in } Q.$$

Indeed, if we set $\bar{y}$ the solution to $A\bar{y} = (1-m)f$, then Problem (1) reduces to

$$\inf \int_Q (g(y+\bar{y}+y_0) + h(u+f+u_0))\, dx\, dt. \tag{37}$$

The proof of Theorem 2.1 applies with

$$g_1(y+y_0) = g(y+\bar{y}+y_0), \qquad h_1(u+u_0) = h(u+f+u_0).$$

3. Existence of Optimal Controllers

In this section, we deal with the existence in Problem (1). The main result is the following theorem.

Theorem 3.1. Assume that T/π is a rational number, there is at least one admissible pair (y, u) in Problem (1), and in addition to Assumptions (A1) and (A2) of Theorem 2.1, we have

$$g(r) \geq \rho|r| + \beta, \qquad \forall r \in \mathbb{R}, \tag{38}$$

$$h(r) \geq \omega r^2 + \gamma, \qquad \forall r \in \mathbb{R}, \tag{39}$$

for some $\rho, \omega > 0$ and $\beta, \gamma \in \mathbb{R}$. Then, Problem (1) has at least one solution $(y^*, u^*) \in L^\infty(Q) \times L^\infty(Q)$.

Proof. Consider the optimization problem

$$\min\left\{\int_Q (g_\epsilon(y+y_0) + h(u+u_0))\, dx\, dt + (\epsilon/2)\int_Q y^2\, dx\, dt;\right.$$

$$\left.(y, u) \in L^2(Q) \times L^2(Q),\ Ay = mu\right\}. \tag{40}$$

Since the cost functional is coercive and weakly lower semicontinuous in $L^2(Q) \times L^2(Q)$, for each $\epsilon > 0$, Problem (40) has one optimal pair $(y_\epsilon, u_\epsilon) \in L^2(Q) \times L^2(Q)$. Since (40) is a smooth optimal control problem, as in the previous proof it follows that there is $p_\epsilon \in L^2(Q)$ such that

$$Ap_\epsilon = -g'_\epsilon(y_\epsilon + y_0) - \epsilon y_\epsilon, \tag{41}$$

$$mp_\epsilon \in \partial h(u_\epsilon + u_0), \qquad \text{a.e. in } Q. \tag{42}$$

By assumptions (38) and(39), it follows that

$$\|y_\epsilon\|_{L^1(Q)} + \|u_\epsilon\|_{L^2(Q)} + \epsilon^{1/2}\|y_\epsilon\|_{L^2(Q)} \leq C, \qquad \forall \epsilon > 0. \tag{43}$$

Let

$$y_\epsilon = y_\epsilon^1 + y_\epsilon^2,$$

where $y_\epsilon^1 \in R(A)$ and $y_\epsilon^2 \in N(A)$. Then, by Proposition 1.1, we have

$$\|y_\epsilon^1\|_{L^\infty(Q)} \leq C, \qquad \forall \epsilon > 0, \tag{44}$$

while (43) yields

$$\|y_\epsilon^2\|_{L^1(Q)} \leq C, \qquad \forall \epsilon > 0. \tag{45}$$

By (41), we see that

$$\epsilon y_\epsilon + g'_\epsilon(y_\epsilon + y_0) \in R(A),$$

and so, by (18) and (19), we have

$$y_\epsilon^2(x, t) = q_\epsilon(t + x) - q_\epsilon(t - x), \qquad \text{a.e. } (x, t) \in Q,$$

where q_ϵ is τ-periodic, $\tau = 2\pi/n = T/m$, and

$$\begin{aligned}\int_0^\pi &(g'_\epsilon(y_\epsilon^1(x, t - x) + y_\epsilon^2(x, t - x) + y_0(x, t - x)) \\ &\quad - g'_\epsilon(y_\epsilon^1(x, t + x) + y_\epsilon^2(x, t + x) + y_0(x, t + x)))\, dx \\ &= -\epsilon \int_0^\pi (y_\epsilon(x, t - x) - y_\epsilon(x, t + x))\, dx.\end{aligned}$$

This yields

$$\begin{aligned}&(1/4\pi) \int_0^{2\pi} (g'_\epsilon(C - q_\epsilon(t) + q_\epsilon(s)) - g'_\epsilon(C - q_\epsilon(t) + q_\epsilon(s)))\, ds \\ &\leq -\epsilon q(t) + C\epsilon.\end{aligned} \tag{46}$$

Let

$$M_\epsilon = \text{ess sup}\{q_\epsilon(t);\, t\in(0,T)\}, \qquad E_\epsilon = \{s;\, q_\epsilon(s) > M_\epsilon/2\}.$$

By (45), we know that

$$m(E_\epsilon) \le CM_\epsilon^{-1},$$

and then, by (46), we have

$$\begin{aligned}&(1/4\pi)(2\pi - CM_\epsilon^{-1})(g'_\epsilon(-C+(1/2)M_\epsilon) - g'_\epsilon(C-(1/2)M_\epsilon))\\ &\le CM_\epsilon^{-1}(g'_\epsilon(C) - g'_\epsilon(-C)) + C\epsilon \le C(\epsilon + M_\epsilon^{-1}).\end{aligned} \tag{47}$$

If

$$\lim_{\epsilon\to 0} M_\epsilon = +\infty,$$

then

$$(1+\epsilon\partial g)^{-1}(-C+(1/2)M_\epsilon) \to +\infty,$$

and so, by (47), we see that there are $\xi \in \partial g(-\infty)$, $\eta \in \partial g(+\infty)$ such that $\xi \ge \eta$. Hence,

$$\partial g(r) = \xi = \eta, \qquad \forall r \in \mathbb{R},$$

which clearly contradicts assumption (38). Hence,

$$M_\epsilon \le C, \qquad \forall \epsilon > 0,$$

and by a similar argument it follows from (46) that ess $\inf\{q_\epsilon(t);\, t\in(0,T)\}$ is bounded from below. Therefore, we have shown that $\{y_\epsilon^2\}$ is bounded in $L^\infty(Q)$. Then, on a subsequence, we have

$$u_\epsilon \to u^*, \qquad \text{weakly in } L^2(Q), \tag{48}$$

$$y_\epsilon \to y^*, \qquad \text{weakly star in } L^\infty(Q), \tag{49}$$

and letting ϵ tend to zero in (40), we see that (y^*, u^*) is an optimal pair in Problem (1). To conclude the proof, it remains to be shown that $u^* \in L^\infty(Q)$. Indeed, since $\{y_\epsilon\}$ is bounded in $L^\infty(Q)$ and ∂g is locally bounded in $\mathbb{R}$, we have that

$$\|g'_\epsilon(y_\epsilon + y_0)\|_{L^\infty(Q)} \le C, \qquad \forall \epsilon > 0. \tag{50}$$

On the other hand, by (42), we have

$$u_\epsilon = \partial h^*(mp_\epsilon) - u_0, \qquad \text{a.e. in } Q, \tag{51}$$

and so

$$m\partial h^*(mp_\epsilon) - mu_0 \in R(A), \qquad \forall \epsilon > 0.$$

Then, arguing as in proof of Theorem 2.1, it follows that

$$\int_0^\pi (m(x)\partial h^*(m(x)p_\epsilon^1(x,t-x)+m(x)p_\epsilon^2(x,t-x))$$
$$-m(x)\partial h^*(m(x)p_\epsilon^1(x,t+x)+m(x)p_\epsilon^2(x,t+x)))\,dx$$
$$=\int_0^\pi m(x)(u_0(x,t-x)-u_0(x,t+x))\,dx,$$

where

$$p_\epsilon=p_\epsilon^1+p_\epsilon^2, \qquad p_\epsilon^1\in R(A), \qquad p_\epsilon^2\in N(A).$$

Since by (41) and (50), $\{p_\epsilon^1\}$ is bounded in $L^\infty(Q)$, we have

$$\int_{t-2b}^{t-2a} m((t-s)/2)\partial h^*(-C+m((t-s)/2)q_\epsilon(t)-m((t-s)/2)q_\epsilon(s))\,ds$$
$$-\int_{t+2a}^{t+2b} m((s-t)/2)\partial h^*(C+m((s-t)/2)q_\epsilon(s)-m((s-t)/2)q_\epsilon(t))\,ds\le C,$$
$$\forall t\in(0,t),$$

where q_ϵ is as in (33). This implies as above that $\{p_\epsilon^2\}$ is bounded in $L^\infty(Q)$; and since, by virtue of assumption (39), ∂h^* is locally bounded, we see by (51) that $\{u_\epsilon\}$ is bounded in $L^\infty(Q)$. Hence, $u^*\in L^\infty(Q)$, and the proof of Theorem 3.1 is complete. □

4. State Constraint Problem

Here, we study Problem (1) in the more general case where Assumption (A1) is replaced by

(A1)′ the function $g:\mathbb{R}\to\bar{\mathbb{R}}=]-\infty,+\infty]$ is convex, lower semicontinuous, and $y_0(x,t)\in\mathcal{K}\subset\operatorname{int} D(g)$, a.e. $(x,t)\in Q$, where $\mathcal{K}$ is a compact subset.

Theorem 4.1. Assume that T is a rational multiple of π and that Assumptions (A1)′ and (A2) hold. Then, the pair $(y^*,u^*)\in L^\infty(Q)\times L^\infty(Q)$ is optimal in Problem (1) if, and only if, there are $p\in L^\infty(Q)$ and $\mu\in(L^\infty(Q))^*$

such that

$$p_{tt}-p_{xx}=-\mu, \qquad \text{in } Q=(0,\pi)\times(0,T), \tag{52a}$$

$$p(0,t)=p(\pi,t)=0, \qquad \forall t\in(0,T), \tag{52b}$$

$$p(x,0)=p(x,T), \qquad p_t(x,0)=p_t(x,T), \qquad \forall x\in(0,\pi), \tag{52c}$$

$$\mu(y^*-y)\geq\int_Q (g(y^*+y_0)-g(y+y_0))\,dx\,dt, \quad \forall y\in L^\infty(Q), \tag{53}$$

$$u^*(x,t)\in\partial h^*(m(m)p(x,t))-u_0(x,t), \qquad \text{a.e. } (x,t)\in Q. \tag{54}$$

Proof. The proof is essentially the same as that of Theorem 2.1, so it will only be sketched. If we denote by $(y_\epsilon, u_\epsilon)\in L^\infty(Q)\times L^\infty(Q)$ the solution to Problem (11) and by p_ϵ the function satisfying (15) and (17), we know from the proof of Theorem 2.1 that [see (31)]

$$\|y_\epsilon\|_{L^\infty(Q)}+\|mp_\epsilon\|_{L^1(Q)}\leq C, \qquad \forall\epsilon>0, \tag{55}$$

$$y_\epsilon\to y^*, \qquad u_\epsilon\to u^*, \qquad \text{strongly in } L^2(Q), \text{ as } \epsilon\to 0. \tag{56}$$

Note that, in this case, ∂g is no longer locally bounded. However, we prove that $\{g'_\epsilon(y_\epsilon+y_0)\}$ is bounded in $L^1(Q)$. Indeed, we have

$$g'_\epsilon(y_\epsilon+y_0)(y_\epsilon-\rho w)\geq g_\epsilon(y_\epsilon+y_0)-g_\epsilon(y_0+\rho w), \qquad \text{a.e. in } Q,$$

for $|w|=1$ and $\rho>0$. Then, by Assumption (A1)′, we obtain

$$\rho|g'_\epsilon(y_\epsilon+y_0)|\leq y_\epsilon g'_\epsilon(y_\epsilon+y_0)+C, \qquad \text{a.e. in } Q, \tag{57}$$

because

$$g_\epsilon(r)\geq C_1 r+C_2, \qquad \forall r\in\mathbb{R}, \qquad \epsilon>0.$$

By (15), it follows that

$$\begin{aligned}(g'_\epsilon(y_\epsilon+y_0), y_\epsilon)&=(y^*-y_\epsilon-Ap_\epsilon, y_\epsilon)\\ &=(y^*-y_\epsilon, y_\epsilon)-(u_\epsilon, mp_\epsilon)\leq C, \qquad \forall\epsilon>0.\end{aligned}$$

Then, by (57), we have

$$\|g'_\epsilon(y_\epsilon+y_0)\|_{L^1(Q)}\leq C, \qquad \forall\epsilon>0, \tag{58}$$

and so, by (15) and Proposition 1.1,

$$\|p^1_\epsilon\|_{L^\infty(Q)}\leq C, \qquad \forall\epsilon>0, \tag{59}$$

where

$$p_\epsilon=p^1_\epsilon+p^2_\epsilon, \qquad p^1_\epsilon\in R(A), \qquad p^2_\epsilon\in N(A).$$

Now, arguing as in the proof of Theorem 2.1, we find that $\{p_\epsilon^2\}$ is bounded in $L^\infty(Q)$. Then, on a generalized sequence, we have

$$p_\epsilon \to p, \qquad \text{weak star in } L^\infty(Q), \tag{60}$$

$$g'_\epsilon(y_\epsilon + y_0) \to \mu, \qquad \text{weak star in } (L^\infty(Q))^*. \tag{61}$$

Letting ϵ tend to zero in (17), we see that p satisfies (54). On the other hand, we have

$$\int_Q g'_\epsilon(y_\epsilon + y_0)(y_\epsilon - y)\,dx\,dt$$

$$\geq \int_Q (g_\epsilon(y_\epsilon + y_0) - g(y + y_0))\,dx\,dt, \qquad \forall y \in L^\infty(Q),$$

$$\liminf_{\epsilon \to 0} \int_Q g_\epsilon(y_\epsilon + y_0)\,dx\,dt \geq \int_Q g(y^* + y_0)\,dx\,dt.$$

This implies that μ satisfies (53). It is readily seen that (52–54) are sufficient for optimality. The proof of Theorem 4.1 is complete. □

Concerning existence under the weaker Assumption (A1)′, we have the following theorem.

Theorem 4.2. Let T/π be a rational number. Then, under Assumptions (A1)′ and (A2), (38), and (39), Problem (1) has at least one solution $(y^*, u^*) \in L^\infty(Q) \times L^\infty(Q)$.

Proof. Let $(y_\epsilon, u_\epsilon, p_\epsilon)$ be as in the proof of Theorem 3.1. As seen there, we have

$$\|mp_\epsilon\|_{L^1(Q)} + \|y_\epsilon\|_{L^\infty(Q)} + \|u_\epsilon\|_{L^2(Q)} + \epsilon^{1/2}\|y_\epsilon\|_{L^2(Q)} \leq C.$$

Concerning the estimate (50), arguing as in the proof of Theorem 4.1, by (41), we find that [see (58)]

$$\rho \|g'_\epsilon(y_\epsilon + y_0)\|_{L^1(Q)} \leq \int_Q y_\epsilon\, g'_\epsilon(y_\epsilon + y_0)\,dx\,dt + C \leq C.$$

From here on, the proof is identical with that of Theorem 3.1. □

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