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Analysis and discretization of an optimal control problem for the time-periodic MHD equations

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Abstract

We consider the mathematical formulation and analysis of an optimal control problem associated with the tracking of the velocity and the magnetic field of a viscous, incompressible, electrically conducting fluid in a bounded two-dimensional domain through the adjustment of distributed controls. Existence of optimal solutions is proved and first-order necessary conditions for optimality are used to derive an optimality system of partial differential equations whose solutions provide optimal states and controls. Semidiscrete-in-time approximations are defined and their convergence to the exact optimal solutions is shown.

1. Introduction

The need to control the flow of magnetically sensitive fluids arises in many applications, e.g., crystal growth processes, nuclear reactor cooling, fusion reactors, ship propulsion engines, etc. Although there have been extensive studies of the control of magnetically neutral flows (see, e.g., [3] and [4] and references cited therein), less attention has been paid to the control of MHD flows. In particular, the analysis of such problems and their

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discretization has not been considered in detail. In this paper, we consider a prototype MHD optimal control problem.

We consider the following optimal control problem: minimize

$$\int_Q \left(\frac{1}{2} \left| \nabla \big(u(x,t) - u^\bullet(x,t) \big) \right|^2 + \frac{1}{2} \left| \operatorname{curl} \big(B(x,t) - B^\bullet(x,t) \big) \right|^2 + \frac{\ell}{2} \big(\left| \psi_1(x,t) \right|^2 + \left| \psi_2(x,t) \right|^2 \big) \right) dx\,dt \tag{1.1}$$

over $\psi_1, \psi_2, u, B \in (L^2(Q))^2$ subject to the nondimensional magnetohydrodynamic equation (MHD equation) for a viscous incompressible resistive fluid (see [5–7])

$$\begin{aligned}
&\frac{\partial u}{\partial t} + (u \cdot \nabla) u - \frac{1}{\mathrm{Re}} \Delta u + \nabla p + S \nabla \left(\frac{1}{2} B^2 \right) - S (B \cdot \nabla) B = f_0 + \psi_1 \quad \text{in } \Omega \times R, \\
&\frac{\partial B}{\partial t} + (u \cdot \nabla) B - (B \cdot \nabla) u + \frac{1}{\mathrm{Rm}} \overline{\operatorname{curl}} (\operatorname{curl} B) = \psi_2 \quad \text{in } \Omega \times R, \\
&\operatorname{div} u = 0, \quad \operatorname{div} B = 0 \quad \text{in } \Omega \times R, \\
&u = 0 \quad \text{on } \partial\Omega \times R, \quad B \cdot n = 0 \quad \text{and} \quad \operatorname{curl} B = 0 \quad \text{on } \partial\Omega \times R, \\
&u(x,0) = u(x,T), \quad B(x,0) = B(x,T), \quad \forall (x,t) \in \Omega \times R.
\end{aligned} \tag{1.2}$$

Here $Q = \Omega \times (0,T)$, Ω is an open bounded simply-connected subset of R^2, with smooth boundary $\partial\Omega$, f_0 is a T-periodic (nondimensional) volume density force, $u = (u_1(x,t), u_2(x,t))$ is the velocity of the particle of fluid which is at point x at time t, $B = (B_1(x,t), B_2(x,t))$ is the magnetic field at point x at time t, $p = p(x,t)$ stands for the pressure of the fluid while $\psi_1, \psi_2 \in L^2_{\mathrm{loc}}(R; L^2(\Omega))$ are T-periodic inputs and $u^\bullet, B^\bullet \in L^2_{\mathrm{loc}}(R; L^2(\Omega))$ are a T-periodic reference velocity and magnetic field, respectively. The nondimensional quantities p, u, B correspond to the normalization by reference units denoted by $L_*, T_*, U_* = L_*/T_*, B_*$, for lengths, times, velocities, and magnetic fields, respectively. There are three nondimensional numbers in the equation which represent the Reynolds number $\mathrm{Re} = L_* u_* / \nu$ (where ν is the kinematic viscosity), the magnetic Reynolds number $\mathrm{Rm} = L_* u_* \sigma \mu$ (where μ is the magnetic permeability and σ the conductivity of the fluid, assumed to be constant), $S = M^2 / \mathrm{Re}\,\mathrm{Rm} = B_*^2 / \mu \rho_* u_*^2$ (where M is the Hartman number) and $\ell > 0$. We recall the definitions of the curl and $\overline{\operatorname{curl}}$ operators in two dimensions:

$$\begin{aligned}
&\operatorname{curl} u = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \quad \text{for every vector } u = (u_1, u_2), \\
&\overline{\operatorname{curl}}\, \phi = \left(\frac{\partial \phi}{\partial x_2}, -\frac{\partial \phi}{\partial x_1} \right) \quad \text{for every scalar function } \phi,
\end{aligned}$$

and the following formula:

$$\overline{\operatorname{curl}}\, u = \operatorname{grad} \operatorname{div} u - \Delta u.$$

2. Weak formulation and existence

Let us briefly recall the way we can represent the MHD equations (1.2) as an infinite-dimensional equation (see [7–9]). The spaces used are a combination of spaces for the Navier–Stokes equations (denoted with index 1) and spaces used in the theory of Maxwell equations (denoted with index 2). They are

$$
\begin{aligned}
&\mathcal{V}_1 = \left\{ v \in \left(\mathcal{C}_0^\infty(\Omega)\right)^2,\ \operatorname{div} v = 0 \right\},\\
&V_1 = \left\{ v \in \mathbb{H}_0^1(\Omega),\ \operatorname{div} v = 0 \right\} \quad \left(\text{the closure of } \mathcal{V}_1 \text{ in } \mathbb{H}_0^1(\Omega) = \left(H_0^1(\Omega)\right)^2\right),\\
&H_1 = \left\{ v \in \mathbb{L}^2(\Omega),\ \operatorname{div} v = 0 \text{ and } v \cdot n|_{\partial\Omega} = 0 \right\}\\
&\quad \left(\text{the closure of } \mathcal{V}_1 \text{ in } \mathbb{L}^2(\Omega) = \left(L^2(\Omega)\right)^2\right),\\
&\mathcal{V}_2 = \left\{ C \in \left(\mathcal{C}^\infty(\overline{\Omega})\right)^2,\ \operatorname{div} C = 0 \text{ and } C \cdot n|_{\partial\Omega} = 0 \right\},\\
&V_2 = \left\{ v \in \mathbb{H}^1(\Omega),\ \operatorname{div} C = 0 \text{ and } C \cdot n|_{\partial\Omega} = 0 \right\}\\
&\quad \left(\text{the closure of } \mathcal{V}_2 \text{ in } \mathbb{H}^1(\Omega) = \left(H^1(\Omega)\right)^2\right),\\
&H_2 = \left(\text{the closure of } \mathcal{V}_2 \text{ in } \mathbb{L}^2(\Omega)\right) = H_1.
\end{aligned}
$$

The space V_1 is endowed with the scalar product

$$
((u, v))_1 = \sum_{1 \leqslant i \leqslant 2} \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right) = \sum_{1 \leqslant i \leqslant 2} \int_\Omega \frac{\partial u}{\partial x_i} \cdot \frac{\partial v}{\partial x_i}\, dx,
$$

which is the scalar product on $\mathbb{H}_0^1(\Omega)$. The dual space of V_1 is characterized by (see [9])

$$
V_1' = \left\{ v \in \mathbb{H}^{-1}(\Omega),\ \operatorname{div} v = 0 \right\}.
$$

The space V_2 is endowed with the scalar product

$$
((u, v))_2 = (\operatorname{curl} u, \operatorname{curl} v)
$$

which is equivalent to the usual scalar product induced by $\mathbb{H}^1(\Omega)$ on V_2.

We set (see [9])

$$
V = V_1 \times V_2, \quad H = H_1 \times H_2, \quad V' \text{ the dual space of } V,
$$

and by identifying H with its own dual we have $V \subset H \subset V'$. The space H will be endowed with the following scalar products:

$$
\begin{aligned}
&(\Phi, \Psi) = (u, v) + (B, C) \quad \text{for all } \Phi = (u, B),\ \Psi = (v, C) \in H,\\
&[\Phi, \Psi] = (u, v) + S(B, C),
\end{aligned}
$$

and the induced (equivalent) norms

$$
|\Phi| = (\Phi, \Phi)^{1/2}, \qquad [\Phi] = [\Phi, \Phi]^{1/2}.
$$

The space V will also be endowed with three scalar products

$$
\begin{aligned}
&((\Phi, \Psi)) = \frac{1}{\mathrm{Re}}((u, v))_1 + \frac{1}{\mathrm{Rm}}((B, C))_2, \qquad [\![\Phi, \Psi]\!] = \frac{1}{\mathrm{Re}}((u, v))_1 + \frac{S}{\mathrm{Rm}}((B, C))_2,\\
&((\Phi, \Psi))_J = ((u, v))_1 + ((B, C))_2,
\end{aligned}
$$

and the equivalent norms

$$\|\Phi\| = ((\Phi,\Phi))^{1/2}, \quad [\![\Phi]\!] = [\![\Phi,\Phi]\!]^{1/2}, \quad \|\Phi\|_J = ((\Phi,\Phi))_J^{1/2}.$$

Let $\mathcal{A}_1 \in L(V_1, V_1')$, $\mathcal{A}_2 \in L(V_2, V_2')$, $\mathcal{A} \in L(V, V')$ be defined by

$$\langle \mathcal{A}_1 u, v\rangle = ((u,v))_1 \quad \text{for all } u, v \in V_1,$$
$$\langle \mathcal{A}_2 B, C\rangle = ((B,C))_2 \quad \text{for all } B, C \in V_2,$$
$$\langle \mathcal{A}\Phi, \Psi\rangle = ((\Phi,\Psi)), \quad \langle \mathcal{A}_J\Phi, \Psi\rangle = ((\Phi,\Psi))_J \quad \text{for all } \Phi, \Psi \in V.$$

As in [9] we consider $\mathcal{A}_1 \in \mathcal{L}(V_1, V_1')$, $\mathcal{A}_2 \in \mathcal{L}(V_2, V_2')$, $\mathcal{A} \in \mathcal{L}(V, V')$ as unbounded operators on H_1, H_2, H, for which the domains are

$$D(\mathcal{A}_1) = \{u \in V_1,\ \mathcal{A}_1 u \in H_1\} = \mathbb{H}^2(\Omega) \cap V_1,$$
$$D(\mathcal{A}_2) = \{B \in V_2,\ \mathcal{A}_2 B \in H_2\} = \mathbb{H}^2(\Omega) \cap V_2,$$
$$D(\mathcal{A}) = D(\mathcal{A}_2) \times D(\mathcal{A}_2) = \big(\mathbb{H}^2(\Omega)\big)^2 \cap V.$$

Let $b : \mathbb{L}^1(\Omega) \times \mathbb{W}^{1,1}(\Omega) \times \mathbb{L}^1(\Omega) \to R$ be defined by

$$b(u,v,w) = \sum_{1\leqslant i,j\leqslant 2} \int_\Omega u_i D_i v_j w_j \, dx$$

whenever the integrals make sense. We recall that, for $m_i \geqslant 0$ satisfying $m_1+m_2+m_3 > 1$ or $m_1+m_2+m_3 = 1$ where at least two m_i are nonzero, we have

$$\big|b(u,v,w)\big| \leqslant c_1 |u|_{H^{m_1}} |v|_{H^{m_2+1}} |w|_{H^{m_3}},$$
$$\forall (u,v,w) \in \mathbb{H}^{m_1}(\Omega) \times \mathbb{H}^{m_2+1}(\Omega) \times \mathbb{H}^{m_3}(\Omega). \tag{2.1}$$

For $m_1 = m_3 = 1$, $m_2 = 0$ we find that the trilinear form b is continuous on $(\mathbb{H}^1(\Omega))^3$ and satisfies

$$b(u,v,v) = 0, \quad \forall u \in V_\alpha\ (\alpha = 1,2),\ \forall v \in \mathbb{H}^1(\Omega),$$
$$b(u,v,w) = -b(u,w,v), \quad \forall u \in V_\alpha,\ \forall v, w \in \mathbb{H}^1(\Omega). \tag{2.2}$$

We also define the trilinear form $\mathcal{B}_0 : V \times V \times V \to R$ by setting

$$\mathcal{B}_0(\Phi_1,\Phi_2,\Phi_3) = b(u_1,u_2,u_3) - Sb(B_1,B_2,u_3) + b(u_1,B_2,B_3) - b(B_1,u_2,B_3),$$
$$\forall \Phi_i = (u_i, B_i) \in V,$$

and the bilinear continuous operator $\mathcal{B} : V \times V \to V'$,

$$\big\langle \mathcal{B}(\Phi_1,\Phi_2), \Phi_3\big\rangle = \mathcal{B}_0(\Phi_1,\Phi_2,\Phi_3), \quad \forall \Phi_i \in V.$$

From (2.1) we get

$$\big|\mathcal{B}_0(\Phi_1,\Phi_2,\Phi_3)\big| \leqslant c_2 \max(1,S) |\Phi_1|_{H^{m_1}} |\Phi_2|_{H^{m_2+1}} |\Phi_3|_{H^{m_3}},$$
$$\forall (\Phi_1,\Phi_2,\Phi_3) \in \mathbb{H}^{m_1}(\Omega) \times \mathbb{H}^{m_2+1}(\Omega) \times \mathbb{H}^{m_3}(\Omega). \tag{2.3}$$

This yields for $m_1 = m_2 = 1/2$, $m_3 = 0$ that

$$\left|\mathcal{B}_0(\Phi_1,\Phi_2,\Phi_3)\right| \leqslant c_3\big(|\Phi_1|\,\|\Phi_1\|\,\|\Phi_2\|\,|\mathcal{A}\Phi_2|\big)^{1/2}|\Phi_3|,$$
$$\forall(\Phi_1,\Phi_2,\Phi_3)\in V\times D(\mathcal{A})\times H, \tag{2.4}$$

where $c_3 = c_3(\Omega, S, \mathrm{Re}, \mathrm{Rm})$. Let us denote by $M \in M_4(R)$ the diagonal matrix

$$m_{ii} = 1 \quad \text{for } 1 \leqslant i \leqslant 2, \qquad m_{ii} = S \quad \text{for } 3 \leqslant i \leqslant 4. \tag{2.5}$$

From (2.2) and the identity

$$\mathcal{B}_0(\Phi_1,\Phi_2,M\Phi_2) = b(u_1,u_2,u_2) + Sb(u_1,B_2,B_2) - S\big(b(B_1,B_2,u_2) + b(B_1,u_2,B_2)\big)$$

we finally get

$$\mathcal{B}_0(\Phi_1,\Phi_2,M\Phi_2) = 0, \quad \forall\Phi_1,\Phi_2\in V,$$
$$\mathcal{B}_0(\Phi_1,\Phi_2,M\Phi_3) = -\mathcal{B}_0(\Phi_1,\Phi_3,M\Phi_2), \quad \forall\Phi_i\in V. \tag{2.6}$$

Let $f(t) = P(f_0(t),0)$, $\Psi(t) = P(\psi_1(t),\psi_2(t))$, where $P : (\mathbb{L}^2(\Omega))^2 \to H$ is the projection on H. Then we rewrite the state equation (1.2) as

$$\frac{d\Phi}{dt}(t) + \mathcal{A}\Phi(t) + \mathcal{B}\big(\Phi(t),\Phi(t)\big) = f(t) + \Psi(t), \quad t\in(0,T),$$
$$\Phi(0) = \Phi(T), \tag{2.7}$$

and confine to the *strong solutions* $\Phi \in L^2(0,T;D(\mathcal{A})) \cap W^{1,2}([0,T];H)$.

Assume that $\Phi^\bullet = (u^\bullet, B^\bullet) \in L^2(0,T;H)$. Then we may reformulate problem (1.1) as: minimize

$$J(\Phi,\Psi) = \int_0^T \left(\frac{1}{2}\big\|\Phi(t)-\Phi^\bullet(t)\big\|_J^2 + \frac{\ell}{2}\big|\Psi(t)\big|^2\right)dt \tag{P}$$

over $(\Phi,\Psi) \in (L^2(0,T;D(\mathcal{A})) \cap W^{1,2}([0,T];H)) \times L^2(0,T;H)$ subject to (2.7).

Theorem 2.1. *There is at least one solution (Φ^*,Ψ^*) to problem (P).*

Proof. Let $\{\Phi_n,\Psi_n\}$ be a minimizing sequence in problem (P), i.e.,

$$\inf(P) \leqslant J(\Phi_n,\Psi_n) \leqslant \inf(P) + \frac{1}{n}, \tag{2.8}$$

$$\Phi_n' + \mathcal{A}\Phi_n + \mathcal{B}(\Phi_n,\Phi_n) = f + \Psi_n, \quad \text{a.e. } t\in(0,T);$$
$$\Phi_n(0) = \Phi_n(T). \tag{2.9}$$

By (2.8) it follows that $\{\Phi_n\}$ is bounded in $L^2(0,T;V)$, $\{\Psi_n\}$ is bounded in $L^2(0,T;H)$ and therefore on a subsequence, again denoted n, we have

$$\Psi_n \to \Psi^* \quad \text{weakly in } L^2(0,T;H).$$

If we multiply (2.9) by $tM\Phi_n$, integrate on Ω we get by (2.6) that

$$\frac{1}{2}\frac{d}{dt}\big(t\big[\Phi_n(t)\big]^2\big) - \frac{1}{2}\big[\Phi_n(t)\big]^2 + t\,[\![\Phi_n(t)]\!]^2$$
$$= t\big[f(t) + \Psi_n(t), \Phi_n(t)\big], \qquad \text{a.e. } t \in (0,T). \tag{2.10}$$

This yields

$$t\big[\Phi_n(t)\big]^2 + \int_0^t s\,[\![\Phi_n(s)]\!]^2\,ds \leqslant C, \qquad \forall t \in [0,T],$$

and therefore

$$\big[\Phi_n(0)\big] = \big[\Phi_n(T)\big] \leqslant C, \qquad \forall n \in \mathbb{N}.$$

Here C denotes several positive constants independent of Φ and n. Next we multiply (2.9) by $t\mathcal{A}\phi_n$ and obtain after some calculus involving Young's inequality and (2.4) that

$$\frac{1}{2}\frac{d}{dt}\big(t\big\|\Phi_n(t)\big\|^2\big) - \frac{1}{2}\big\|\Phi_n(t)\big\|^2 + t\big|\mathcal{A}\Phi_n(t)\big|^2$$
$$= \big(f(t) + \Phi_n(t), t\mathcal{A}\Phi_n\big) - \mathcal{B}_0\big(\Phi_n(t), \Phi_n(t), t\mathcal{A}\Phi_n(t)\big)$$
$$\leqslant \frac{t}{4}\big|\mathcal{A}\Phi_n(t)\big|^2 + t\big|f(t) + \Phi_n(t)\big|^2 + tc_3\big|\Phi_n(t)\big|^{1/2}\big\|\Phi_n(t)\big\|\,|\mathcal{A}\Phi_n|^{3/2}$$
$$\leqslant \frac{t}{2}\big|\mathcal{A}\Phi_n(t)\big|^2 + tC\big|\Phi_n(t)\big|^2\big\|\Phi_n(t)\big\|^4 + t\big|f(t) + \Phi_n(t)\big|^2.$$

Now integrating on $(0,t)$ and using the above estimates we get

$$t\big\|\Phi_n(t)\big\|^2 + \int_0^t s\big|\mathcal{A}\Phi_n(s)\big|^2\,ds \leqslant C\int_0^t \big(1 + s\big\|\Phi_n(t)\big\|^4\big)\,ds$$

which by Grönwall's lemma gives

$$t\big\|\Phi_n(t)\big\|^2 \leqslant C, \qquad \forall t \in (0,T].$$

Since $\Phi_n(0) = \Phi_n(T)$ we infer that $\|\Phi_n(0)\| \leqslant C$. Finally, multiplying (2.9) by $\mathcal{A}\Phi_n$ and integrating on $\Omega \times (0,t)$ we obtain as above

$$\big\|\Phi_n(t)\big\|^2 + \int_0^t \big|\mathcal{A}\Phi_n(s)\big|^2\,ds \leqslant C\left(\big\|\Phi_n(0)\big\|^2 + \int_0^t \big\|\Phi_n(s)\big\|^4\,ds\right)$$

and therefore

$$\big\|\Phi_n(t)\big\|^2 + \int_0^t \big|\mathcal{A}\Phi_n(s)\big|^2\,ds \leqslant C, \qquad \forall t \in [0,T].$$

This yields

$$\|\Phi_n'\|_{L^2(0,T;H)} + \big\|\mathcal{B}(\Phi_n, \Phi_n)\big\|_{L^2(0,T;H)} \leqslant C.$$

Since $V \Subset H$ we infer that $\{\Phi_n\}$ is compact in $C([0,T];H) \cap L^2(0,T;V)$ and on subsequences we have

$$\begin{aligned}
&\Phi_n \to \Phi^* \quad \text{strongly in } L^2(0,T;V) \cap C\big([0,T];H\big),\\
&\mathcal{A}\Phi_n \to \mathcal{A}\Phi^* \quad \text{weakly in } L^2(0,T;H),\\
&\Phi_n' \to (\Phi^*)' \quad \text{weakly in } L^2(0,T;H).
\end{aligned}$$

By (2.4) we have

$$\begin{aligned}
&\big|\big(\mathcal{B}(\Phi_n,\Phi_n) - \mathcal{B}(\Phi^*,\Phi^*),\Phi\big)\big| \leqslant \big|\mathcal{B}_0(\Phi_n - \Phi^*,\Phi_n,\Phi)\big| + \big|\mathcal{B}_0(\Phi^*,\Phi_n - \Phi^*,\Phi)\big|\\
&\quad \leqslant C\big(|\Phi_n - \Phi^*|^{1/2}\|\Phi_n - \Phi^*\|^{1/2}\|\Phi_n\|^{1/2}|\mathcal{A}\Phi_n|^{1/2}\\
&\qquad + |\Phi^*|^{1/2}\|\Phi^*\|^{1/2}\|\Phi_n - \Phi^*\|^{1/2}\big|\mathcal{A}(\Phi_n - \Phi^*)\big|^{1/2}\big)|\Phi|,
\end{aligned} \tag{2.11}$$

for all $\Phi \in H$, and therefore

$$\mathcal{B}(\Phi_n,\Phi_n) \to \mathcal{B}(\Phi^*,\Phi^*) \quad \text{strongly in } L^2(0,T;H).$$

Letting n go to ∞ in (2.8), (2.9) we see that (Φ^*,Ψ^*) satisfies the system (2.7) and $J(\Phi^*,\Psi^*) = \inf(P)$. □

3. Optimality conditions

Let (Φ^*,Ψ^*) be an optimal pair in problem (P). For each $\varepsilon > 0$ consider the approximating problem: minimize

$$\int_0^T \left(\frac{1}{2}\|\Phi - \Phi^\bullet\|_J^2 + \frac{\ell}{2}|\Psi|^2 + \frac{1}{2\varepsilon}|\xi|^2\right) dt \tag{P_ε}$$

over $\Phi \in L^2(0,T;D(\mathcal{A})) \cap W^{1,2}([0,T];H)$, $\Psi,\xi \in L^2(0,T;H)$ subject to

$$\begin{aligned}
&\Phi'(t) + \mathcal{A}\Phi(t) + \mathcal{B}\big(\Phi(t),\Phi(t)\big) = f(t) + \Psi(t) + \xi(t), \quad t \in (0,T);\\
&\Phi(0) = \Phi(T).
\end{aligned} \tag{3.1}$$

By Theorem 2.1 for each $\varepsilon > 0$ problem (P_ε) has at least one solution $(\Phi_\varepsilon,\Psi_\varepsilon,\xi_\varepsilon)$.

Lemma 3.1. *For $\varepsilon \to 0$ we have*

$$\begin{aligned}
&\Phi_\varepsilon \to \Phi^* \quad \textit{strongly in } L^2(0,T;V) \cap C\big([0,T];H\big),\\
&\Phi_\varepsilon' \to (\Phi^*)', \quad \mathcal{A}\Phi_\varepsilon \to \mathcal{A}\Phi^* \quad \textit{weakly in } L^2(0,T;H),\\
&\Psi_\varepsilon \to \Psi^*, \quad \varepsilon^{-1/2}\xi_\varepsilon \to 0 \quad \textit{weakly in } L^2(0,T;H),\\
&\lim_{\varepsilon\to 0}\Big\{\inf_{\Phi_\varepsilon,\Psi_\varepsilon,\xi_\varepsilon}(P_\varepsilon)\Big\} = \inf_{\Phi,\Psi}(P).
\end{aligned} \tag{3.2}$$

Proof. By taking $(\Phi,\Psi,\xi) = (\Phi^*,\Psi^*,0)$ in (P_ε) we get

$$\inf_{\Phi,\Psi,\xi}(P_\varepsilon) \leqslant \int_0^T \left(\frac{1}{2}\|\Phi^* - \Phi^\bullet\|_J^2 + \frac{\ell}{2}|\Psi^*|^2\right) dt \equiv \inf_{\Phi,\Psi}(P).$$

If multiply (3.1) with $M\Phi_\varepsilon, tM\Phi_\varepsilon$ and integrate on $(0,T), (0,t)$, respectively, we get by (2.4), (2.6) that

$$t\big[\Phi_\varepsilon(t)\big]^2 + \int_0^T \big[\!\big[\Phi_\varepsilon(t)\big]\!\big]^2\,dt \leqslant C.$$

Now if we multiply (3.1) by $t\mathcal{A}\Phi_\varepsilon$, integrate on $(0,t)$, we see as above that

$$t\big\|\Phi_\varepsilon(t)\big\|^2 \leqslant C, \quad \forall \in (0,T],$$

and therefore

$$\big\|\Phi_\varepsilon(0)\big\| = \big\|\Phi_\varepsilon(T)\big\| \leqslant C, \quad \forall \varepsilon > 0.$$

When we multiply (3.1) by $\mathcal{A}\Phi_\varepsilon$ we obtain

$$\big\|\Phi_\varepsilon(t)\big\|^2 + \int_0^t \big|\mathcal{A}\Phi_\varepsilon(s)\big|^2\,ds \leqslant C, \quad \forall t \in [0,T],$$

and from (3.1) we have that

$$\|\Phi_\varepsilon'\|_{L^2(0,T;H)} + \big\|\mathcal{B}(\Phi_\varepsilon,\Phi_\varepsilon)\big\|_{L^2(0,T;H)} \leqslant C, \quad \forall \varepsilon > 0.$$

Hence on a subsequence we have

$$\begin{aligned}
&\Phi_\varepsilon \to \bar\Phi \quad \text{strongly in } C\big([0,T];H\big) \cap L^2(0,T;V),\\
&\Phi_\varepsilon' \to \bar\Phi', \quad \mathcal{A}\Phi_\varepsilon \to \mathcal{A}\bar\Phi \quad \text{weakly in } L^2(0,T;H),\\
&\Psi_\varepsilon \to \bar\Psi, \quad \xi_\varepsilon \to 0 \quad \text{weakly in } L^2(0,T;H).
\end{aligned}$$

On the other hand, by (2.4) we see that

$$\mathcal{B}(\Phi_\varepsilon,\Phi_\varepsilon) \to \mathcal{B}(\bar\Phi,\bar\Phi)$$

and therefore $(\bar\Phi,\bar\Psi)$ is a solution to the state system (2.7).

Finally, taking the limit in (P_ε), by the weak lower semicontinuity of the H-norm we obtain that

$$\inf_{\Phi,\Psi}(P) \leqslant \int_0^T \left(\frac{1}{2}\|\bar\Phi - \Phi^\bullet\|_J^2 + \frac{\ell}{2}|\bar\Psi|^2\right)dt \leqslant \lim_{\varepsilon\to 0}\Big\{\inf_{\Phi,\Psi,\xi}(P_\varepsilon)\Big\}$$

hence $\bar\Phi = \Phi^*$, $\bar\Psi = \Psi^*$ and the conclusions of Lemma 3.1 follow. □

In the space $L^2(0,T;H)$ we define the operators

$$\begin{aligned}
&\boldsymbol{\mathcal{A}}_\varepsilon\phi = \phi' + \mathcal{A}\phi + \mathcal{B}(\Phi_\varepsilon,\phi) + \mathcal{B}(\phi,\Phi_\varepsilon), \quad \forall \phi \in D(\boldsymbol{\mathcal{A}}_\varepsilon) = X,\\
&\boldsymbol{\mathcal{A}}_\varepsilon^*\phi = -\phi' + \mathcal{A}\phi + \mathcal{B}_0(\Phi_\varepsilon,\cdot,\phi) + \mathcal{B}_0(\cdot,\Phi_\varepsilon,\phi), \quad \forall \phi \in X,
\end{aligned} \tag{3.3}$$

where

$$X = \big\{\phi \in W^{1,2}\big([0,T];H\big) \cap L^2\big(0,T;D(\mathcal{A})\big),\ \phi(0) = \phi(T)\big\}.$$

It is easily seen that

$$\int_0^T (\mathcal{A}_\varepsilon^* \Upsilon, \phi)\,dt = \int_0^T (\mathcal{A}_\varepsilon \phi, \Upsilon)\,dt, \quad \forall \phi, \Upsilon \in D(\mathcal{A}_\varepsilon) = D(\mathcal{A}_\varepsilon^*) = X.$$

The operators $\mathcal{A}$ and $\mathcal{A}^*$ are defined by the same formulae (3.3) where $\Phi_\varepsilon = \Phi^*$.

Lemma 3.2. *The operators $\mathcal{A}_\varepsilon, \mathcal{A}_\varepsilon^*, \mathcal{A}, \mathcal{A}^*$ are closed, densely defined and have closed ranges in $L^2(0,T;H)$. Moreover,* $\dim N(\mathcal{A}_\varepsilon), \dim N(\mathcal{A}_\varepsilon^*) \leqslant n_0$, *independent of ε, $\mathcal{A}_\varepsilon^*$ is the adjoint of $\mathcal{A}_\varepsilon$ and the following estimates hold*:

$$\begin{aligned} &\left\| \mathcal{A}_\varepsilon^{-1} g \right\|_{L^2(0,T;D(\mathcal{A}))\cap W^{1,2}([0,T];H)} \leqslant C\|g\|_{L^2(0,T;H)}, \quad \forall g \in R(\mathcal{A}_\varepsilon), \\ &\left\| \left(\mathcal{A}_\varepsilon^*\right)^{-1} g \right\|_{L^2(0,T;D(\mathcal{A}))\cap W^{1,2}([0,T];H)} \leqslant C\|g\|_{L^2(0,T;H)}, \quad \forall g \in R(\mathcal{A}_\varepsilon^*). \end{aligned} \tag{3.4}$$

Similarly, the operators $\mathcal{A}^*, \mathcal{A}$ are mutually adjoint and the estimates (3.4) *remain true for $\mathcal{A}^*, \mathcal{A}$.*

We have used the symbols N and R to denote the null space and the range of the corresponding operators.

Proof. We shall use an argument similar to [1]. Let prove first that $\mathcal{A}_\varepsilon$ is closed and has closed range in $L^2(0,T;H)$. Let $(g,\phi_0) \in L^2(0,T;H) \times H$ be arbitrary, fixed. Taking $m_1 = m_3 = 1/2$, $m_2 = 0$ in (2.3), we get by interpolation inequalities that

$$\left|\mathcal{B}_0(\Phi_\varepsilon,\phi,\phi) + \mathcal{B}_0(\phi,\Phi_\varepsilon,\phi)\right| \leqslant C_1|\Phi_\varepsilon|^{1/2}\|\Phi_\varepsilon\|^{1/2}|\phi|\|\phi\| \leqslant C|\phi|\|\phi\|, \\ \forall \phi \in V,$$

since by Lemma 3.1 $\{\Phi_\varepsilon\}$ is bounded in $C([0,T];V)$. By standard existence result we know that the Cauchy problem

$$\phi' + \mathcal{A}\phi + \mathcal{B}(\Phi_\varepsilon,\phi) + \mathcal{B}(\phi,\Phi_\varepsilon) = g, \quad \text{a.e. } t \in (0,T); \qquad \phi(0) = \phi_0, \tag{3.5}$$

has a unique solution

$$\phi = \phi_\varepsilon(t,\phi_0,g) \in L^2(0,T,V) \cap W^{1,2}\big([0,T];V'\big) \subset C\big([0,T];H\big)$$

which satisfies

$$\left|\phi(t)\right|^2 + \int_0^T \left\|\phi(t)\right\|^2 dt \leqslant C\left(|\phi_0|^2 + \int_0^T \left|g(t)\right|^2 dt\right). \tag{3.6}$$

If $\phi_0 \in V$ we have a better regularity

$$\phi \in L^2\big(0,T;D(\mathcal{A})\big) \cap W^{1,2}\big([0,T];H\big) \subset C\big([0,T];V\big).$$

In this case, when multiply (3.5) by $t\mathcal{A}\phi(t)$ and integrate on $(0,t)$, by (2.4) and (3.6) we infer

$$
\begin{aligned}
&t\|\phi(t)\|^2+\int_0^t s|\mathcal{A}\phi(s)|^2\,ds\\
&\quad\leqslant C\Biggl(|\phi_0|^2+\int_0^T |g(s)|^2\,ds+\Biggl(\int_0^t s|\mathcal{A}\phi(s)|^2\,ds\Biggr)^{3/4}\Biggl(\int_0^t \|\phi(s)\|^2\,ds\Biggr)^{1/4}\\
&\qquad+\Biggl(\int_0^t s|\mathcal{A}\phi(s)|^2\,ds\Biggr)^{1/2}\Biggl(\int_0^t \|\phi(s)\|^2\,ds\Biggr)^{1/4}\Biggl(\int_0^T |\mathcal{A}\Phi_\varepsilon(s)|^2\,ds\Biggr)^{1/4}\Biggr).
\end{aligned}
$$

Since $\{\mathcal{A}\Phi_\varepsilon\}$ is bounded in $L^2(0,T;H)$ we conclude by (3.6) that

$$\|\phi(t)\|\leqslant t^{-1}\rho\bigl(|\phi_0|,\|g\|_{L^2(0,T;H)}\bigr),\quad \forall t>0.$$

The above estimate extends to all solutions ϕ to (3.5) where $\phi_0\in H$ and therefore satisfies

$$\phi_\varepsilon(T,\phi_0,g)\in V;\qquad \bigl\|\phi_\varepsilon(T,\phi_0,g)\bigr\|\leqslant\rho\bigl(|\phi_0|,\|g\|_{L^2(0,T;H)}\bigr),\quad\forall\varepsilon>0. \tag{3.7}$$

Let denote

$$\phi_\varepsilon(T,\phi_0,g)=\Gamma_\varepsilon\phi_0+E_\varepsilon g,$$

where $\Gamma_\varepsilon\phi_0=\phi_\varepsilon(T,\phi_0,0)$, $E_\varepsilon g=\phi_\varepsilon(T,0,g)$. By (3.7) we have $\Gamma_\varepsilon\in L(H,V)$, $E_\varepsilon\in L(L^2(0,T;H);V)$ and

$$\|E_\varepsilon\|_{L(L^2(0,T;H),V)}+\|\Gamma_\varepsilon\|_{L(H,V)}\leqslant C,\quad\forall\varepsilon>0. \tag{3.8}$$

Since the injection V into H is compact we infer that Γ_ε is completely continuous in H.

Now consider (Φ_ε,g) such that $\mathbf{\mathcal{A}}\Phi_\varepsilon=g$. Therefore $\Phi_\varepsilon(t)=\phi_\varepsilon(t,\phi_0,g)$, where $(I-\Gamma_\varepsilon)\phi_0=E_\varepsilon g$. The Fredholm–Riesz theory implies that $R(I-\Gamma_\varepsilon)$ is closed and $\dim N(I-\Gamma_\varepsilon)<\infty$. Hence $R(\mathbf{\mathcal{A}}_\varepsilon)$ is closed in $L^2(0,T;H)$ and $N(\mathbf{\mathcal{A}}_\varepsilon)$ are finite dimensional.

Consider (ϕ_n,g_n) such that $\mathcal{A}_\varepsilon\phi_n=g_n$ and

$$\phi_n\to\phi,\quad g_n\to g\quad\text{strongly in } L^2(0,T;H).$$

By the estimate (3.8) it follows that $\{\phi_n(0)\}$ is bounded in V and as seen earlier this implies that

$$
\begin{aligned}
&\{\phi_n\}\text{ is bounded in } L^2\bigl(0,T;D(\mathcal{A})\bigr)\cap W^{1,2}\bigl([0,T];H\bigr),\\
&\mathcal{B}(\Phi_\varepsilon,\phi_n)+\mathcal{B}(\phi_n,\Phi_\varepsilon)\to\mathcal{B}(\Phi_\varepsilon,\phi)+\mathcal{B}(\phi,\Phi_\varepsilon)\quad\text{weakly in } L^2(0,T;H).
\end{aligned}
$$

Hence $\mathbf{\mathcal{A}}\phi_\varepsilon=g$, i.e., $\mathbf{\mathcal{A}}_\varepsilon$ is closed.

Now let $\Gamma\in\mathcal{L}(H,H)$ be defined by $\Gamma\phi_0=\phi(T,\phi_0,0)$, where ϕ is the solution to

$$\phi'+\mathcal{A}\phi+\mathcal{B}(\Phi^*,\phi)+\mathcal{B}(\phi,\Phi^*)=g,\quad\text{a.e. } t\in(0,T);\qquad \phi(0)=\phi_0.$$

We have that $\Gamma\in\mathcal{L}(H,V)$ and so Γ is completely continuous from H to itself. Moreover, by Lemma 3.1 and (3.8) it follows that

$$\Gamma_\varepsilon\to\Gamma\quad\text{in } \mathcal{L}(H,H)$$

as $\varepsilon \to 0$. Since $\dim(I - \Gamma) < \infty$ the latter implies that there is n_0 such that $\dim N(I - \Gamma_\varepsilon) \leqslant n_0$, $\forall \varepsilon > 0$. Hence $\dim N(\mathcal{A}_\varepsilon) \leqslant n_0$, $\forall \varepsilon > 0$ as claimed. Moreover, we have the estimate

$$\left|(I - \Gamma_\varepsilon)^{-1} g_0\right| \leqslant C|g_0|, \quad \forall g_0 \in R(I - \Gamma_\varepsilon). \tag{3.9}$$

Indeed, otherwise there are $\phi_{0\varepsilon} \in R((I - \Gamma_\varepsilon)^*)$, $f_\varepsilon \in R(I - \Gamma_\varepsilon)$ such that

$$|f_\varepsilon| = 1, \quad (I - \Gamma_\varepsilon)\phi_{0\varepsilon} = f_\varepsilon, \quad |\phi_{0\varepsilon}| \to \infty$$

as $\varepsilon \to 0$. Then on a subsequence we have

$$\phi_{0\varepsilon}|\phi_{0\varepsilon}|^{-1} \to 0,$$

where $|\phi_0| = 1$, $\phi_0 \in R((I - \Gamma)^*)$, $(I - \Gamma)\phi_0 = 0$ which leads to a contradiction because $R((I - \Gamma)^*) \oplus N(I - \Gamma) = H$.

Finally, we recall that $\Phi = \phi_\varepsilon(t, \phi_0, g)$, where $(I - \Gamma_\varepsilon)\phi_0 = E_\varepsilon g$ is a solution to equation $\mathcal{A}_\varepsilon \Phi = g$ while by (3.8) and (3.9) we have

$$\left|\phi_\varepsilon(0)\right| \leqslant C\|g\|_{L^2(0,T;H)}, \quad \forall g \in R(\mathcal{A}_\varepsilon),$$

and so, by (3.7),

$$\left\|\phi_\varepsilon(T)\right\| \leqslant C\|g\|_{L^2(0,T;H)}, \quad \forall g \in R(\mathcal{A}_\varepsilon).$$

Then as seen above we have

$$\|\phi_\varepsilon\|_{W^{1,2}(0,T;H)} + \|\phi_\varepsilon\|_{L^2(0,T;D(\mathcal{A}))} \leqslant C\|g\|_{L^2(0,T;H)}, \quad \forall g \in R(\mathcal{A}_\varepsilon),$$

which implies (3.4).

The corresponding properties of the operator $\mathcal{A}_\varepsilon^*$ follow from the same arguments because in this case Eq. (3.5) is replaced by

$$\phi' + \mathcal{A}\phi + \mathcal{B}_0(\Phi_\varepsilon, \cdot, \phi) + \mathcal{B}_0(\cdot, \Phi_\varepsilon, \phi) = g, \quad \text{a.e. } t \in (0, T); \qquad \phi(0) = \phi_0,$$

and so the previous estimates remain valid. In particular, it follows that the operator $\mathcal{A}_\varepsilon^*$ is closed and its adjoint is precisely $\mathcal{A}_\varepsilon$. Also by Lemma 3.1 and the above estimates we have

$$\mathcal{A}_\varepsilon \Phi \to \mathcal{A}\Phi \quad \text{weakly in } L^2(0, T; H) \tag{3.10}$$

as $\varepsilon \to 0$ for each $\phi \in X$. $\square$

For $\lambda \in R$, $\Phi \in X$, $\Psi \in L^2(0, T; H)$ we set

$$\xi^\lambda = (\Phi_\varepsilon + \lambda\Phi)' + \mathcal{A}(\Phi_\varepsilon + \lambda\Phi) + \mathcal{B}(\Phi_\varepsilon + \lambda\Phi, \Phi_\varepsilon + \lambda\Phi) - (f + \Psi_\varepsilon + \lambda\Psi).$$

We may write ξ^λ as

$$\xi^\lambda = \xi_\varepsilon + \lambda\big(\Phi' + \mathcal{A}\Phi + \mathcal{B}(\Phi_\varepsilon, \Phi) + \mathcal{B}(\Phi, \Phi_\varepsilon) + \lambda\mathcal{B}(\Phi, \Phi) - \Psi\big)$$

and so by the optimality of $(\Phi_\varepsilon, \Psi_\varepsilon, \xi_\varepsilon)$ in (P_ε) we have

$$\int_0^T \bigg(((\Phi_\varepsilon - \Phi^\bullet, \Phi))_J + \ell(\Psi_\varepsilon, \Psi) + \frac{1}{\varepsilon}\big(\xi_\varepsilon, \Phi' + \mathcal{A}\Phi + \mathcal{B}(\Phi_\varepsilon, \Phi) + \mathcal{B}(\Phi, \Phi_\varepsilon) - \Psi\big)\bigg)\,dt \geqslant 0$$
$$\forall \phi \in X,\ \Psi \in L^2(0,T;H). \tag{3.11}$$

We set $q_\varepsilon = \frac{1}{\varepsilon}\xi_\varepsilon$. If we take $\Psi = 0$ we get

$$\int_0^T \big(((\Phi_\varepsilon - \Phi^\bullet, \Phi))_J + (q_\varepsilon, \mathscr{A}_\varepsilon \Phi)\big)\,dt = 0.$$

Hence $q_\varepsilon \in D(\mathscr{A}_\varepsilon^*)$ and

$$\mathscr{A}_\varepsilon^* q_\varepsilon = -\mathcal{A}_J(\Phi_\varepsilon - \Phi^\bullet), \tag{3.12}$$

and by (3.11) we obtain

$$\Psi_\varepsilon = \frac{1}{\ell} q_\varepsilon \quad \text{a.e. in } (0,T). \tag{3.13}$$

Then by Lemma 3.1 it follows that

$$\|q_\varepsilon\|_{L^2(0,T;H)} \leqslant C, \quad \forall \varepsilon > 0.$$

Now we may write q_ε as $q_\varepsilon^1 + q_\varepsilon^2$, where $q_\varepsilon^1 \in R(\mathscr{A}_\varepsilon)$, $q_\varepsilon^2 \in N(\mathscr{A}_\varepsilon^*)$. By Lemma 3.2 we know that

$$\|q_\varepsilon^1\|_{L^2(0,T;D(\mathcal{A}))\cap W^{1,2}([0,T];H)} \leqslant C, \quad \forall \varepsilon > 0,$$

hence on a subsequence, again denoted $\{\varepsilon\}$, we have

$$q_\varepsilon^1 \to q^1 \quad \text{weakly in } L^2\big(0,T;D(\mathcal{A})\big) \cap W^{1,2}\big([0,T];H\big),$$
$$q_\varepsilon^2 \to q^2 \quad \text{strongly in } L^2\big(0,T;D(\mathcal{A})\big) \cap W^{1,2}\big([0,T];H\big),$$

because $\{q_\varepsilon^2\} \subset N(\mathscr{A}_\varepsilon^*)$ and $\dim N(\mathscr{A}_\varepsilon^*) \leqslant n_0$. Now letting ε tend to 0 into (3.12), (3.13) it follows by Lemma 3.1 and (3.10) that

$$\mathscr{A}^*(q^1 + q^2) = -\mathcal{A}_J(\Phi^* - \Phi^\bullet); \qquad \Psi^* = \frac{1}{\ell}(q^1 + q^2) \quad \text{a.e. } t \in (0,T).$$

Let denote $q = q^1 + q^2$. We have established the following maximum principle result for problem (P).

Theorem 3.3. *If the pair (Φ^*, Ψ^*) is optimal in problem (P) then there is $q \in L^2(0,T; D(\mathcal{A})) \cap W^{1,2}([0,T];H)$ such that*

$$q'(t) - \mathcal{A}q - \mathcal{B}_0(\Phi^*, \cdot, q) - \mathcal{B}_0(\cdot, \Phi^*, q) = \mathcal{A}_J(\Phi^* - \Phi^\bullet),$$
$$\text{a.e. } t \in (0,T), \tag{3.14}$$
$$q(0) = q(T),$$
$$\Psi^*(t) = \frac{1}{\ell} q(t), \quad \text{a.e. } t \in (0,T). \tag{3.15}$$

If $q=(q_u,q_B)$ and $\Psi^*=(\psi_1^*,\psi_2^*)$ the adjoint system can be written as

$$\begin{aligned}
&\frac{\partial q_u}{\partial t}+\frac{1}{\mathrm{Re}}\Delta q_u+u^*\cdot\nabla q_u-B^*\cdot\nabla q_B+q_u\cdot\nabla u^*+q_B\cdot\nabla B^*+\nabla p_0\\
&\quad=-\Delta(u^*-u^\bullet)\quad\text{in } Q,\\
&\frac{\partial q_B}{\partial t}-\frac{1}{\mathrm{Rm}}\overline{\mathrm{curl}}(\mathrm{curl}\,q_B)-SB^*\cdot\nabla q_u+u^*\cdot\nabla q_B-Sq_u\cdot\nabla B^*-q_B\cdot\nabla u^*\\
&\quad=\overline{\mathrm{curl}}\big(\mathrm{curl}(B^*-B^\bullet)\big)\quad\text{in } Q,\\
&\operatorname{div} q_u=0,\quad \operatorname{div} q_B=0\quad\text{in } Q,\\
&q_u=0\quad\text{on }\Sigma,\quad q_B\cdot n=0\quad\text{and}\quad\mathrm{curl}\,q_B=0\quad\text{on }\Sigma,\\
&q_u(x,0)=q_u(x,T),\quad q_B(x,0)=q_B(x,T)\quad\text{in }\Omega,
\end{aligned}\tag{3.16}$$

and the optimality condition

$$\psi_1^*=\frac{1}{\ell}q_u,\quad \psi_2^*=\frac{1}{\ell}q_B\quad\text{in } Q.$$

4. Semidiscrete-in-time approximations

Let $\sigma_N=\{t_n\}_{n=0}^N$ be a partition of $[0,T]$ into equal intervals of duration $\Delta t=T/N$ with $t_0=0$ and $t_N=T$. We will denote by $\mathbf{v}$ the vector $(v^{(1)},v^{(2)},\dots,v^{(N)})$ of functions belonging to a space $\mathbf{Y}=Y^N$. We associate the following approximate function:

$$v^N(t,x)=v^{(n)}(x),\quad t\in(t_{n-1},t_n],\ n=1,2,\dots,N,$$

where $v^{(0)}=v^{(N)}$, and a continuous, piecewise (in time t) linear function $v_{\mathrm{pl}}^N=v_{\mathrm{pl}}^N(t,x)$ defined by the interpolating conditions

$$v_{\mathrm{pl}}^N(t_n,x)=v^{(n)}(x),\quad n=1,2,\dots,N.$$

On this partition we define the discrete target $\Phi^{\bullet(n)}(x)=\Phi^\bullet(t_n,x)$ for $n=0,1,\dots,N$. The state variables $\Phi^{(n)}\in D(\mathcal{A})$ are constrained to satisfy the semidiscrete MHD equation

$$\frac{1}{\Delta t}\big(\Phi^{(n)}-\Phi^{(n-1)}\big)+\mathcal{A}\Phi^{(n)}+\mathcal{B}\big(\Phi^{(n)},\Phi^{(n)}\big)=f^{(n)}+\Psi^{(n)},\tag{4.1}$$

obtained from (2.7) by a backward Euler discretization in time and the periodic condition

$$\Phi^{(0)}=\Phi^{(N)}.\tag{4.2}$$

Optimization is achieved by means of the minimization of the discretized-in-time functional

$$J^N(\mathbf{\Phi},\mathbf{\Psi})=\frac{1}{2}\Delta t\sum_{n=1}^N\big\|\Phi^{(n)}-\Phi^{\bullet(n)}\big\|_J^2+\frac{\ell}{2}\Delta t\sum_{n=1}^N\big|\Psi^{(n)}\big|^2.\tag{4.3}$$

This functional results from applying the right-point discretization rule in time to the continuous functional J. The discrete-in-time approximate optimal control problem is then given by given $\Delta t = T/N$, $\phi^\bullet \in L^2(0,T;H)$ find $(\mathbf{\Phi},\mathbf{\Psi})$ in $\mathbf{D}(\mathcal{A}) \times \mathbf{H}$ such that $(\mathbf{\Phi},\mathbf{\Psi})$ is the solution of (4.1) and the functional (4.3) is minimized. (P^N)

Theorem 4.1. *Given $T > 0$, $\Delta t = T/N$ there exists at least one optimal solution $(\mathbf{\Phi}^*,\mathbf{\Psi}^*) \in \mathbf{D}(\mathcal{A}) \times \mathbf{H}$ of the semidiscrete optimal control problem.*

Proof. Given N, let $\{(\mathbf{\Phi}_k,\mathbf{\Psi}_k)\}_{k=1}^{\infty}$ be a minimizing sequence. By (4.3) we have that

$$\left|\Phi_k^N\right|^2_{L^2(0,T;H)} + \left|\Psi_k^N\right|^2_{L^2(0,T;H)} = \Delta t\sum_{n=1}^N\left|\Phi_k^{(n)}\right|^2 + \Delta t\sum_{n=1}^N\left|\Psi_k^{(n)}\right|^2 \leqslant C,$$
$$\forall k = 1,\dots,\infty.$$

If we multiply (4.1) by $M\Phi_k^{(n)}$ and integrate over Ω we get

$$\begin{aligned}&\left[\Phi_k^{(n)}\right]^2 + \left[\Phi_k^{(n)} - \Phi_k^{(n-1)}\right]^2 + 2\Delta t \left[\!\left[\Phi_k^{(n)}\right]\!\right]^2\\ &\quad\leqslant \left[\Phi_k^{(n-1)}\right]^2 + \Delta t\left[\!\left[\Phi_k^{(n)}\right]\!\right]^2 + \frac{\Delta t}{\lambda_1}\left[f^{(n)} + \Psi_k^{(n)}\right]^2\end{aligned} \tag{4.4}$$

and when summating from $n = 1$ to N it yields

$$\begin{aligned}&\sum_{n=1}^N\left[\Phi_k^{(n)} - \Phi_k^{(n-1)}\right]^2 + \Delta t\sum_{n=1}^N\left[\!\left[\Phi_k^{(n)}\right]\!\right]^2\\ &\quad\leqslant C\left(|f^N|^2_{L^2(0,T;H)} + \left|\Psi_k^N\right|^2_{L^2(0,T;H)}\right).\end{aligned} \tag{4.5}$$

Now we multiply (4.1) by $n\Delta t M\Phi_k^{(n)}$ and get

$$n\Delta t\left[\Phi_k^{(n)}\right]^2 + n(\Delta t)^2\left[\!\left[\Phi_k^{(n)}\right]\!\right]^2 \leqslant n\Delta t\left[\Phi_k^{(n-1)}\right]^2 + \frac{n(\Delta t)^2}{\lambda_1}\left[f^{(n)} + \Psi_k^{(n)}\right]^2,$$
$$\text{for all } n = 1,\dots,N.$$

If we take the sum from $n = 1$ to N we obtain by the Poincare inequality and (4.5),

$$\begin{aligned}&N\Delta t\left[\Phi_k^{(N)}\right]^2 + \sum_{n=1}^N n(\Delta t)^2\left[\!\left[\Phi_k^{(n)}\right]\!\right]^2\\ &\quad\leqslant \Delta t\sum_{n=1}^N\left[\Phi_k^{(n)}\right]^2 + 2\sum_{n=1}^N\frac{n(\Delta t)^2}{\lambda_1}\left([f^{(n)}]^2 + \left[\Psi_k^{(n)}\right]^2\right) \leqslant C,\end{aligned}$$

for all $k = 1,\dots,\infty$. Using (4.2) we infer that $[\Phi_k^{(0)}] \leqslant C$, $\forall k$. Then by summation from $n = 1$ to r in (4.4) we get

$$\begin{aligned}&\left[\Phi_k^{(r)}\right]^2 + \Delta t\sum_{n=1}^r[\![\Phi^{(n)}]\!]^2 \leqslant C\left(1 + |f^N|^2_{L^2(0,T;H)} + \left|\Psi_k^N\right|^2_{L^2(0,T;H)}\right),\\ &\quad\forall r = 1,\dots,N,\ \forall k.\end{aligned} \tag{4.6}$$

Next we multiply (4.1) by $n\Delta t\mathcal{A}\Phi_k^{(n)}$ and integrate on Ω to get after some calculations involving (2.4) that

$$\begin{aligned} &n\Delta t\left\|\Phi_k^{(n)}\right\|^2+n(\Delta t)^2\left|\mathcal{A}\Phi_k^{(n)}\right|^2 \\ &\quad\leqslant n\Delta t\left\|\Phi_k^{(n-1)}\right\|^2+2n(\Delta t)^2\left|f^{(n)}+\Psi_k^{(n)}\right|^2+2Cn(\Delta t)^2\left|\Phi_k^{(n)}\right|^2\left\|\Phi_k^{(n)}\right\|^4\end{aligned}$$

for all $n=1,\dots,N$ and for all k. If we summate from $n=1$ to N, use the discrete Grönwall inequality and (4.6) we obtain that

$$\left\|\Phi_k^{(0)}\right\|\leqslant C,\quad \forall k.$$

Finally when we multiply (4.1) by $\mathcal{A}\Phi_k^{(n)}$ we have as above

$$\begin{aligned} &\left\|\Phi_k^{(n)}\right\|^2+\left\|\Phi_k^{(n)}-\Phi_k^{(n-1)}\right\|^2+\Delta t\left|\mathcal{A}\Phi_k^{(n)}\right|^2 \\ &\quad\leqslant\left\|\Phi_k^{(n-1)}\right\|^2+2\Delta t\left|f^{(n)}+\Psi_k^{(n)}\right|^2+C\Delta t\left|\Phi_k^{(n)}\right|^2\left\|\Phi_k^{(n)}\right\|^4\end{aligned}$$

and taking the sum from $n=1$ to r we infer by Grönwall inequality and (4.6) that

$$\left\|\Phi_k^{(r)}\right\|^2+\left\|\Phi_k^{(r)}-\Phi_k^{(r-1)}\right\|^2+\sum_{n=1}^{r}\Delta t\left|\mathcal{A}\Phi_k^{(n)}\right|^2\leqslant C \tag{4.7}$$

for all $r=1$ to N and for all k. Therefore we conclude that the sequences $(\mathbf{\Phi}_k,\mathbf{\Psi}_k)$ are uniformly bounded in $\mathbf{D}(\mathcal{A})\times\mathbf{H}$, and on subsequences we have

$$\begin{aligned} &\Psi_k^{(n)}\to\Psi^{*(n)}\quad \text{weakly in } H,\\ &\Phi_k^{(n)}\to\Phi^{*(n)}\quad \text{weakly in } D(\mathcal{A}),\ \text{strongly in } V,\end{aligned}$$

for $n=1,\dots,N$. Using the same argument as in (2.11) we get

$$\mathcal{B}\big(\Phi_k^{(n)},\Phi_k^{(n)}\big)\to\mathcal{B}(\Phi^{*(n)},\Phi^{*(n)})\quad \text{strongly in } H.$$

This allows us to pass to the limit in the semidiscrete equation and conclude the proof. □

Now we can prove the convergence of the semidiscrete optimal control problem.

Theorem 4.2. *For $\Delta t\to 0$ the solution $\{(\Phi^{*(n)},\Psi^{*(n)})\}_{n=1}^N$ of the semidiscrete-in-time optimal control problem tends to the solution (Φ^*,Ψ^*) of the corresponding continuous optimal control problem.*

Proof. Using the same computations as for the previous theorem we obtain easily that $\{(\Phi^{*N},\Psi^{*N})\}_{N=1}^\infty$ is uniformly bounded in $L^2(0,T;D(\mathcal{A}))\cap L^\infty(0,T;V)\times L^2(0,T;H)$ and $\{\frac{d}{dt}\Phi^{*N}_{\mathrm{pl}}\}$ is uniformly bounded in $L^2(0,T;V)$. Moreover, we have that

$$\left\|\Phi^{*N}-\Phi^{*N}_{\mathrm{pl}}\right\|^2_{L^2(0,T;V)}=\frac{\Delta t}{3}\sum_{n=1}^{N}\|\Phi^{*(n)}-\Phi^{*(n-1)}\|^2\to 0\quad \text{when } \Delta t\to 0.$$

Hence on subsequences we have that

$$\begin{aligned}&\Psi^{*N} \to \Psi^* \quad \text{weakly in } L^2(0,T;H),\\ &\Phi^{*N} \to \Phi^* \quad \text{strongly in } L^2(0,T;V),\ \text{weak-}* \text{ in } L^2\big(0,T;D(\mathcal{A})\big).\end{aligned} \tag{4.8}$$

Equation (4.1) can be interpreted as

$$\frac{d\Phi^{*N}_{\text{pl}}}{dt} + \mathcal{A}\Phi^{*N} + \mathcal{B}(\Phi^{*N}, \Phi^{*N}) = f^N + \Psi^{*N}$$

and as we pass $N \to \infty$ we find that the solution of the semidiscrete problem (P^N) converges to the corresponding solution of the continuous optimal control problem (P). □

Due to the lack of differentiability in the application $\mathbf{\Psi} \to \mathbf{\Phi}(\mathbf{\Psi})$ we will replace problem (P^N) by a sequence of approximating problems (P^N_ε), for which we can compute necessary conditions of optimality.

For each ε consider the following optimization problem: minimize

$$\begin{aligned}J^N_\varepsilon(\Phi,\Psi,\xi) = \frac{\Delta t}{2}\sum_{n=1}^N \|\Phi^{(n)} - \Phi^{\bullet(n)}\|_J^2 + \frac{\ell}{2}\Delta t \sum_{n=1}^N |\Psi^{(n)}|^2\\ + \frac{\Delta t}{2\varepsilon}\sum_{n=1}^N |\xi^{(n)}|^2\end{aligned} \tag{P^N_ε}$$

over $(\mathbf{\Phi}, \mathbf{\Psi}, \boldsymbol{\xi}) \in \mathbf{D}(\mathcal{A}) \times \mathbf{H} \times \mathbf{H}$ satisfying

$$\begin{aligned}&\frac{1}{\Delta t}(\Phi^{(n)} - \Phi^{(n-1)}) + \mathcal{A}\Phi^{(n)} + \mathcal{B}(\Phi^{(n)}, \Phi^{(n)}) = f^{(n)} + \Psi^{(n)} + \xi^{(n)},\\ &\Phi^{(0)} = \Phi^{(N)}.\end{aligned} \tag{4.9}$$

By Theorem 4.1, for each $\varepsilon > 0$ problem (P^N_ε) has at least one solution $(\mathbf{\Phi}_\varepsilon, \mathbf{\Psi}_\varepsilon, \boldsymbol{\xi}_\varepsilon)$.

Lemma 4.1. *For $\varepsilon \to 0$ we have*

$$\begin{aligned}&\Phi^{(n)}_\varepsilon \to \Phi^{*(n)} \quad \textit{weakly in } D(\mathcal{A}),\ \textit{strongly in } V,\\ &\Psi^{(n)}_\varepsilon \to \Psi^{*(n)} \quad \textit{weakly in } V,\ \textit{weakly in } H,\\ &\varepsilon^{-1/2}\xi^{(n)}_\varepsilon \to 0 \quad \textit{weakly in } H\end{aligned} \tag{4.10}$$

for all $n = 1, \ldots, N$ and

$$\lim_{\varepsilon\to 0}\Big\{\inf_{(\mathbf{\Phi},\Psi,\xi)}\big(P^N_\varepsilon\big)\Big\} = \inf_{(\mathbf{\Phi},\Psi)}(P^N).$$

Proof. By taking $(\mathbf{\Phi}, \mathbf{\Psi}, \boldsymbol{\xi}) = (\mathbf{\Phi}^*, \mathbf{\Psi}^*, 0)$ in (P^N_ε) we get

$$\inf_{(\mathbf{\Phi},\Psi,\xi)}\big(P^N_\varepsilon\big) \leqslant \frac{\Delta t}{2}\sum_{n=1}^N \|\Phi^{*(n)} - \Phi^{\bullet(n)}\|_J^2 + \frac{\ell}{2}\Delta t\sum_{n=1}^N |\Psi^{*(n)}|^2 = \inf_{(\mathbf{\Phi},\Psi)}(P^N).$$

Using a similar argument as in the proof of Theorem 4.1 we get that

$$\big\|\Phi^{(r)}_\varepsilon\big\|^2 + \big\|\Phi^{(r)}_\varepsilon - \Phi^{(r-1)}_\varepsilon\big\|^2 + \sum_{n=1}^r \Delta t\big|\mathcal{A}\Phi^{(n)}_\varepsilon\big|^2 \leqslant C$$

for all $r = 1, \dots, N$, where C does not depend on ε. On subsequences we have then

$$\begin{aligned} &\Phi_\varepsilon^{(n)} \to \bar{\Phi}^{(n)} \quad \text{weakly in } D(\mathcal{A}),\ \text{strongly in } V,\\ &\Psi_\varepsilon^{(n)} \to \bar{\Psi}^{(n)} \quad \text{weakly in } V,\ \text{weakly in } H,\\ &\xi_\varepsilon^{(n)} \to 0 \quad \text{weakly in } H \end{aligned}$$

for all $n = 1, \dots, N$, also $B(\Phi_\varepsilon^{(n)}, \Phi_\varepsilon^{(n)}) \to B(\bar{\Phi}^{(n)}, \bar{\Phi}^{(n)})$ and therefore $(\bar{\Phi}^{(n)}, \bar{\Psi}^{(n)})$ is a solution to the semidiscrete in time system (4.1)–(4.2). Now taking the limit in (P_ε^N) we obtain

$$\begin{aligned} \inf_{(\mathbf{\Phi},\Psi)} (P^N) &\leqslant \frac{\Delta t}{2} \sum_{n=1}^{N} \|\bar{\Phi}^{(n)} - \Phi^{\bullet(n)}\|_J^2 + \frac{\ell}{2}\Delta t \sum_{n=1}^{N} |\bar{\Psi}^{(n)}|^2 + \Delta t \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \sum_{n=1}^{N} \left|\xi_\varepsilon^{(n)}\right|^2 \\ &\leqslant \lim_{\varepsilon\to 0} \left\{ \inf_{(\mathbf{\Phi},\Psi,\xi)} \left(P_\varepsilon^N\right) \right\} \end{aligned}$$

hence $\bar{\mathbf{\Phi}} = \bar{\mathbf{\Phi}}^*$, $\bar{\mathbf{\Psi}} = \bar{\mathbf{\Psi}}^*$ and the conclusions of Lemma 4.1 hold. $\square$

In the space $\mathbf{D}(\mathcal{A})$ we define the operators

$$\begin{aligned} \boldsymbol{\mathcal{A}}_\varepsilon^N \begin{bmatrix} \Phi^{(1)} \\ \Phi^{(2)} \\ \vdots \\ \Phi^{(N-1)} \\ \Phi^{(N)} \end{bmatrix} &= \frac{1}{\Delta t} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & -1 \\ -1 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 1 \end{bmatrix} \begin{bmatrix} \Phi^{(1)} \\ \Phi^{(2)} \\ \vdots \\ \Phi^{(N-1)} \\ \Phi^{(N)} \end{bmatrix} + \begin{bmatrix} \mathcal{A}\Phi^{(1)} \\ \mathcal{A}\Phi^{(2)} \\ \vdots \\ \mathcal{A}\Phi^{(N-1)} \\ \mathcal{A}\Phi^{(N)} \end{bmatrix} \\ &\quad + \begin{bmatrix} \mathcal{B}(\Phi_\varepsilon^{(1)}, \Phi^{(1)}) \\ \mathcal{B}(\Phi_\varepsilon^{(2)}, \Phi^{(2)}) \\ \vdots \\ \mathcal{B}(\Phi_\varepsilon^{(N-1)}, \Phi^{(N-1)}) \\ \mathcal{B}(\Phi_\varepsilon^{(N)}, \Phi^{(N)}) \end{bmatrix} + \begin{bmatrix} \mathcal{B}(\Phi^{(1)}, \Phi_\varepsilon^{(1)}) \\ \mathcal{B}(\Phi^{(2)}, \Phi_\varepsilon^{(2)}) \\ \vdots \\ \mathcal{B}(\Phi^{(N-1)}, \Phi_\varepsilon^{(N-1)}) \\ \mathcal{B}(\Phi^{(N)}, \Phi_\varepsilon^{(N)}) \end{bmatrix} \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} \boldsymbol{\mathcal{A}}_\varepsilon^{N*} \begin{bmatrix} \Phi^{(1)} \\ \Phi^{(2)} \\ \vdots \\ \Phi^{(N-1)} \\ \Phi^{(N)} \end{bmatrix} &= \frac{1}{\Delta t} \begin{bmatrix} 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1 \\ -1 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} \Phi^{(1)} \\ \Phi^{(2)} \\ \vdots \\ \Phi^{(N-1)} \\ \Phi^{(N)} \end{bmatrix} \\ &\quad + \begin{bmatrix} \mathcal{A}\Phi^{(1)} \\ \mathcal{A}\Phi^{(2)} \\ \vdots \\ \mathcal{A}\Phi^{(N-1)} \\ \mathcal{A}\Phi^{(N)} \end{bmatrix} + \begin{bmatrix} \mathcal{B}_0(\Phi_\varepsilon^{(1)}, \cdot, \Phi^{(1)}) \\ \mathcal{B}_0(\Phi_\varepsilon^{(2)}, \cdot, \Phi^{(2)}) \\ \vdots \\ \mathcal{B}_0(\Phi_\varepsilon^{(N-1)}, \cdot, \Phi^{(N-1)}) \\ \mathcal{B}_0(\Phi_\varepsilon^{(N)}, \cdot, \Phi^{(N)}) \end{bmatrix} \end{aligned}$$

$$+\begin{bmatrix} \mathcal{B}_0(\Phi^{(1)}, \cdot, \Phi_\varepsilon^{(1)}) \\ \mathcal{B}_0(\Phi^{(2)}, \cdot, \Phi_\varepsilon^{(2)}) \\ \vdots \\ \mathcal{B}_0(\Phi^{(N-1)}, \cdot, \Phi_\varepsilon^{(N-1)}) \\ \mathcal{B}_0(\Phi^{(N)}, \cdot, \Phi_\varepsilon^{(N)}) \end{bmatrix}.$$

It is easily seen that

$$\left(\boldsymbol{\mathcal{A}}_\varepsilon^N \Upsilon, \Theta\right)_{\mathbf{H}} = \left(\Theta, \boldsymbol{\mathcal{A}}_\varepsilon^{N*} \Upsilon\right)_{\mathbf{H}}, \quad \forall \Upsilon, \Theta \in \mathbf{D}(\mathcal{A}).$$

The operators $\boldsymbol{\mathcal{A}}^N$ and $\mathbf{A}^{N*}$ are defined by the same formulae (4.11) where $\Phi_\varepsilon^{(n)} = \Phi^{(n)*}$.

Lemma 4.2. *The operators $\boldsymbol{\mathcal{A}}_\varepsilon^N$, $\boldsymbol{\mathcal{A}}_\varepsilon^{N*}$, $\boldsymbol{\mathcal{A}}^N$, $\boldsymbol{\mathcal{A}}^{N*}$ are closed, densely defined and have closed ranges in* $\mathbf{H}$. *Moreover,* $\dim N(\boldsymbol{\mathcal{A}}_\varepsilon^N)$, $\dim N(\boldsymbol{\mathcal{A}}_\varepsilon^{N*}) \leqslant k_0$, *independent of ε and the following estimates hold*:

$$\begin{aligned} &\left\|\boldsymbol{\mathcal{A}}_\varepsilon^{N^{-1}} \mathbf{g}\right\|_{\mathbf{D}(\mathcal{A})} \leqslant C\|\mathbf{g}\|_{\mathbf{H}}, \quad \forall \mathbf{g} \in R\left(\boldsymbol{\mathcal{A}}_\varepsilon^N\right), \\ &\left\|\boldsymbol{\mathcal{A}}_\varepsilon^{N*^{-1}} \mathbf{g}\right\|_{\mathbf{D}(\mathcal{A})} \leqslant C\|\mathbf{g}\|_{\mathbf{H}}, \quad \forall \mathbf{g} \in R\left(\boldsymbol{\mathcal{A}}_\varepsilon^{N*}\right). \end{aligned} \tag{4.12}$$

Similarly, the operators $\boldsymbol{\mathcal{A}}^N$, $\boldsymbol{\mathcal{A}}^{N*}$ are mutually adjoint and estimates (4.12) *remain true for $\boldsymbol{\mathcal{A}}^N$, $\boldsymbol{\mathcal{A}}^{N*}$.*

Proof. Using the Galerkin method it can be easily proved that the semidiscrete-in-time Cauchy problem

$$\begin{aligned} &\frac{1}{\Delta t}(\Upsilon^{(n)} - \Upsilon^{(n-1)}) + \mathcal{A}\Upsilon^{(n)} + \mathcal{B}\left(\Phi_\varepsilon^{(n)}, \Upsilon^{(n)}\right) + \mathcal{B}\left(\Upsilon^{(n)}, \Phi_\varepsilon^{(n)}\right) = g^{(n)}, \\ &\Upsilon^{(0)} = \xi_0 \end{aligned} \tag{4.13}$$

has a unique solution $\{\Upsilon_\varepsilon^{(n)}(\xi_0, \mathbf{g})\}_{n=1}^N \subset \mathbf{D}(\mathcal{A})$ for $\{(g^{(n)}, \xi_0)\}_{n=1}^N \subset \mathbf{H} \times V$, which by (2.3), (4.10) satisfies

$$[\Upsilon^{(n)}]^2 + \Delta t \sum_{r=1}^n [\![\Upsilon^{(r)}]\!]^2 \leqslant C\left([\xi_0]^2 + \sum_{n=1}^N [g^{(n)}]^2\right), \quad \forall n = 1, \ldots, N.$$

Moreover, when multiply (4.13) by $n\Delta t\mathcal{A}\Phi^{(n)}$ and summate from 1 to N we get from (4.10) that

$$\begin{aligned} &n\|\Upsilon^{(n)}\|^2 + \sum_{r=1}^n r\Delta t|\mathcal{A}\Upsilon^{(r)}|^2 \\ &\quad \leqslant C\Delta t\left(|\xi_0|^2 + \sum_{r=1}^n |g^{(r)}|^2 + \left(\sum_{r=1}^n r|\mathcal{A}\Upsilon^{(r)}|^2\right)^{3/4} \left(\sum_{r=1}^n \|\Upsilon^{(r)}\|^2\right)^{1/4} \right. \\ &\quad\quad \left. + \left(\sum_{r=1}^n r|\mathcal{A}\Upsilon^{(r)}|^2\right)^{1/2} \left(\sum_{r=1}^n \|\Upsilon^{(r)}\|^2\right)^{1/4} \left(\sum_{r=1}^n \left|\mathcal{A}\Phi_\varepsilon^{(r)}\right|^2\right)^{1/4}\right), \\ &\quad \forall n = 1, \ldots, N, \end{aligned}$$

and consequently

$$\|\Upsilon^{(n)}\| \leqslant \rho\left(|\xi_0|, \sum_{r=1}^{n} |g^{(r)}|^2\right)\frac{1}{n}, \quad \forall n = 1, \dots, N.$$

This estimation holds for all solutions $\{\Upsilon^{(n)}\}_{n=1}^{N}$ of (4.13) where $\xi_0 \in H$ and we have

$$\Upsilon_\varepsilon^{(N)}(\xi_0, \mathbf{g}) \in V; \qquad \|\Upsilon_\varepsilon^{(N)}\| \leqslant \rho(|\xi_0|, |\mathbf{g}|_{\mathbf{H}}), \quad \forall \varepsilon > 0. \tag{4.14}$$

We set

$$\Upsilon_\varepsilon^{(N)}(\xi_0, \mathbf{g}) = \Gamma_\varepsilon \xi_0 + E_\varepsilon \mathbf{g},$$

where $\Gamma_\varepsilon \xi_0 = \Upsilon_\varepsilon^{(N)}(\xi_0, \mathbf{0})$, $E_\varepsilon \mathbf{g} = \Upsilon_\varepsilon^{(N)}(0, \mathbf{g})$. Clearly $\Gamma_\varepsilon \in \mathcal{L}(H, V)$, $E_\varepsilon \in \mathcal{L}(\mathbf{H}, V)$ and by estimate (4.14) we have

$$\|E_\varepsilon\|_{L(\mathbf{H},V)} + \|\Gamma_\varepsilon\|_{L(H,V)} \leqslant C, \quad \forall \varepsilon > 0. \tag{4.15}$$

Since injection $V \subset H$ is compact we infer that Γ_ε is completely continuous in H. Now let $(\mathbf{\Phi}, \mathbf{g}) \in \mathcal{A}_\varepsilon^N$, i.e., $\mathcal{A}_\varepsilon^N \mathbf{\Phi} = \mathbf{g}$. We have therefore $\Phi^{(n)} = \Upsilon_\varepsilon^{(n)}(\xi_0, \mathbf{g})$, where $(I - \Gamma_\varepsilon)\xi_0 = E_\varepsilon \mathbf{g}$. By Fredholm–Riesz theory we know that $R(I - \Gamma_\varepsilon)$ is closed and $\dim N(I - \Gamma_\varepsilon) < \infty$. Hence $R(\mathcal{A}_\varepsilon^N)$ is closed in $\mathbf{H}$ and $N(\mathcal{A}_\varepsilon^N)$ is finite dimensional. Moreover, if $(\mathbf{\Upsilon}_m, \mathbf{g}_m) \in \mathcal{A}_\varepsilon^N$ and

$$\mathbf{\Upsilon}_m \to \mathbf{\Upsilon}, \quad \mathbf{g}_m \to \mathbf{g} \quad \text{strongly in } \mathbf{H}$$

then by estimate (4.14) it follows that $\{\mathbf{\Upsilon}_m(0)\}$ is bounded in V and as seen earlier we have

$$\mathbf{\Upsilon}_m \text{ is bounded in } \mathbf{D}(\mathcal{A}),$$
$$\mathcal{B}(\Phi_\varepsilon^{(n)}, \Upsilon_m^{(n)}) + \mathcal{B}(\Upsilon_m^{(n)}, \Phi_\varepsilon^{(n)}) \to \mathcal{B}(\Phi_\varepsilon^{(n)}, \Upsilon^{(n)}) + \mathcal{B}(\Upsilon^{(n)}, \Phi_\varepsilon^{(n)}) \quad \text{weakly in } H.$$

Hence $(\mathbf{\Upsilon}, \mathbf{g}) \in \mathcal{A}_\varepsilon^N$, i.e., $\mathcal{A}_\varepsilon^N$ is closed.

Now let $\Gamma \in \mathcal{L}(H, H)$ be defined by $\Gamma \xi_0 = \Upsilon^{(N)}(\xi_0, \mathbf{0})$, where Υ is the solution

$$\frac{1}{\Delta t}(\Upsilon^{(n)} - \Upsilon^{(n-1)}) + \mathcal{A}\Upsilon^{(n)} + \mathcal{B}(\Phi^{*(n)}, \Upsilon^{(n)}) + \mathcal{B}(\Upsilon^{(n)}, \Phi^{*(n)}) = g^{(n)},$$
$$\Upsilon^{(0)} = \xi_0. \tag{4.16}$$

As seen earlier $\Gamma \in \mathcal{L}(H, V)$ and so is completely continuous from H to itself. Moreover, by Lemma 4.1 and estimate (4.15) it follows that

$$\Gamma_\varepsilon \to \Gamma \quad \text{in } \mathcal{L}(H, H)$$

as $\varepsilon \to 0$. Since $\dim(I - \Gamma) \leqslant \infty$ the latter implies that there is k_0 such that $\dim(I - \Gamma_\varepsilon) \leqslant k_0$, $\forall \varepsilon > 0$. Hence $\dim N(\mathcal{A}_\varepsilon^N) \leqslant k_0$, $\forall \varepsilon > 0$ as claimed. Moreover we have the estimate

$$|(I - \Gamma_\varepsilon)^{-1} g_0| \leqslant C|g_0|, \quad \forall g_0 \in R(I - \Gamma_\varepsilon). \tag{4.17}$$

Indeed, otherwise there are $\xi_\varepsilon \in R(I - \Gamma_\varepsilon)^*$, $f_\varepsilon \in R(I - \Gamma_\varepsilon)$ such that

$$|f_\varepsilon| = 1, \quad (I - \Gamma_\varepsilon)\xi_\varepsilon = f_\varepsilon, \quad |\xi_\varepsilon| \to 0$$

as $\varepsilon \to 0$. Then on a subsequence we have

$$\xi_\varepsilon|\xi_\varepsilon|^{-1} \to \xi_0,$$

where $|\xi_0| = 1$, $\xi_0 \in R((I-\Gamma)^*) \cap N(I-\Gamma)$ which leads to a contradiction.

Now we recall that $\Phi^{(n)} = \Upsilon_\varepsilon^{(n)}(\xi_0, \mathbf{g})$, $\forall n = 1, \ldots, N$, where $(I-\Gamma_\varepsilon)\xi_0 = E_\varepsilon \mathbf{g}$ is a solution to equation $\mathcal{A}_\varepsilon^N \mathbf{\Phi} = \mathbf{g}$ while by (4.15) and (4.17) we have

$$\left|\Upsilon_\varepsilon^{(0)}(\xi_0, \mathbf{g})\right| \leqslant C|\mathbf{g}|, \quad \forall \mathbf{g} \in R\big(\mathcal{A}_\varepsilon^N\big),$$

and so, by (4.14),

$$\left\|\Upsilon_\varepsilon^{(N)}(\xi_0, \mathbf{g})\right\| \leqslant C|\mathbf{g}|, \quad \forall \mathbf{g} \in R\big(\mathcal{A}_\varepsilon^N\big).$$

Then as seen above we have

$$\sum_{n=1}^{N} \Delta t \left|\mathcal{A}\Upsilon_\varepsilon^{(n)}\right|^2 \leqslant C|\mathbf{g}|, \quad \forall \mathbf{g} \in R\big(\mathcal{A}_\varepsilon^N\big),$$

which implies (4.12).

The corresponding properties of the operator $\mathcal{A}_\varepsilon^{N*}$ follows similarly from the previous arguments because in this case Eq. (4.13) is replaced by

$$\frac{1}{\Delta t}\big(\Upsilon^{(n)} - \Upsilon^{(n-1)}\big) + \mathcal{A}\Upsilon^{(n)} + \mathcal{B}_0\big(\Phi_\varepsilon^{(n)}, \cdot, \Upsilon^{(n)}\big) + \mathcal{B}_0\big(\Upsilon^{(n)}, \cdot, \Phi_\varepsilon^{(n)}\big) = g^{(n)},$$
$$\Upsilon^{(0)} = \xi_0,$$

and so the previous estimates remain valid. □

For $\lambda \in R$, $\Phi^{(n)} \in D(\mathcal{A})$, $\forall n = 0, \ldots, N$ with $\Phi^{(0)} = \Phi^{(N)}$ we set

$$\begin{aligned}\zeta_\lambda^{(n)} = {} & \frac{(\Phi_\varepsilon^{(n)} + \lambda\Phi^{(n)}) - (\Phi_\varepsilon^{(n-1)} + \lambda\Phi^{(n-1)})}{\Delta t} + \mathcal{A}\big(\Phi_\varepsilon^{(n)} + \lambda\Phi^{(n)}\big) \\ & + \mathcal{B}\big(\Phi_\varepsilon^{(n)} + \lambda\Phi^{(n)}, \Phi_\varepsilon^{(n)} + \lambda\Phi^{(n)}\big) - f^{(n)} - \big(\Psi_\varepsilon^{(n)} + \lambda\Psi^{(n)}\big).\end{aligned} \tag{4.18}$$

We may write $\zeta_\lambda^{(n)}$ as

$$\begin{aligned}\zeta_\lambda^{(n)} = \zeta_\varepsilon^{(n)} + \lambda\bigg(& \frac{\Phi^{(n)} - \Phi^{(n-1)}}{\Delta t} + \mathcal{A}\Phi^{(n)} + \mathcal{B}\big(\Phi_\varepsilon^{(n)}, \Phi^{(n)}\big) + \mathcal{B}\big(\Phi^{(n)}, \Phi_\varepsilon^{(n)}\big) \\ & + \lambda\mathcal{B}(\Phi^{(n)}, \Phi^{(n)}) - \Psi^{(n)}\bigg)\end{aligned}$$

and by the optimality of $(\Phi_\varepsilon^{(n)}, \Psi_\varepsilon^{(n)}, \zeta_\varepsilon^{(n)})$ we get

$$\begin{aligned}& \Delta t \sum \big(\big(\Phi_\varepsilon^{(n)} - \Phi^{\bullet(n)}, \Phi^{(n)}\big)\big)_J + \ell \Delta t \sum \big(\Psi_\varepsilon^{(n)}, \Psi^{(n)}\big) \\ & \quad + \frac{\Delta t}{\varepsilon} \sum_{n=1}^{N} \bigg(\zeta_\varepsilon^{(n)}, \frac{\Phi^{(n)} - \Phi^{(n-1)}}{\Delta t} + \mathcal{A}\Phi^{(n)} + \mathcal{B}\big(\Phi_\varepsilon^{(n)}, \Phi^{(n)}\big) + \mathcal{B}\big(\Phi^{(n)}, \Phi_\varepsilon^{(n)}\big) \\ & \quad + \lambda\mathcal{B}(\Phi^{(n)}, \Phi^{(n)}) - \Psi^{(n)}\bigg) \geqslant 0\end{aligned} \tag{4.19}$$

for all $\{\Phi^{(n)}\}_{n=0}^{N} \subset D(\mathcal{A})^{N+1}$ with $\Phi^{(0)} = \Phi^{(N)}$, and for all $\Psi^{(n)} \in H$, $n = 1, \ldots, N$.

We set $q_\varepsilon^{(n)} = \frac{1}{\varepsilon}\zeta_\varepsilon^{(n)}$. By taking $\Psi^{(n)} = 0$, $\forall n = 1, \dots, N$, we get

$$\Delta t \sum_{n=1}^{N} \big(\big(\Phi_\varepsilon^{(n)} - \Phi^{\bullet(n)}, \Phi^{(n)}\big)\big)_J + \Delta t \big(\boldsymbol{\mathcal{A}}_\varepsilon^N \boldsymbol{\Phi}, \mathbf{q}_\varepsilon\big) = 0. \tag{4.20}$$

Hence $\mathbf{q}_\varepsilon \in D(\boldsymbol{\mathcal{A}}_\varepsilon^N)$ and

$$\boldsymbol{\mathcal{A}}_\varepsilon^{N*}\mathbf{q}_\varepsilon = -\begin{pmatrix} \mathcal{A}_J(\Phi_\varepsilon^{(1)} - \Phi^{\bullet(1)}) \\ \vdots \\ \mathcal{A}_J(\Phi_\varepsilon^{(N)} - \Phi^{\bullet(N)}) \end{pmatrix}. \tag{4.21}$$

Using once again (4.19) we see that

$$\ell \Psi_\varepsilon^{(n)} = q_\varepsilon^{(n)}, \quad \forall n = 1, \dots, N. \tag{4.22}$$

Then by Lemma 4.1 it follows that

$$\big|q_\varepsilon^{(n)}\big| \leqslant C, \quad \forall n = 1, \dots, N,\ \forall \varepsilon > 0.$$

Now we may write $\mathbf{q}_\varepsilon = \mathbf{q}_\varepsilon^1 + \mathbf{q}_\varepsilon^2$ with $\mathbf{q}_\varepsilon^1 \in R(\boldsymbol{\mathcal{A}}_\varepsilon^N)$, $\mathbf{q}_\varepsilon^2 \in N(\boldsymbol{\mathcal{A}}_\varepsilon^{N*})$. Then by Lemma 4.1 it follows that

$$\big\|\mathbf{q}_\varepsilon^2\big\|_{\mathbf{D}(\mathcal{A})} \leqslant C, \quad \forall \varepsilon > 0,$$

while by Lemma 4.2 we know that

$$\big\|\mathbf{q}_\varepsilon^1\big\|_{\mathbf{D}(\mathcal{A})} \leqslant C, \quad \forall \varepsilon > 0.$$

Hence on a subsequence, again denoted $\{\varepsilon\}$, we have

$$\mathbf{q}_\varepsilon^1 \to \mathbf{q}^1 \quad \text{weakly in } \mathbf{D}(\mathcal{A}), \qquad \mathbf{q}_\varepsilon^2 \to \mathbf{q}^2 \quad \text{strongly in } \mathbf{D}(\mathcal{A})$$

because $\mathbf{q}_\varepsilon^2 \in N(\boldsymbol{\mathcal{A}}_\varepsilon^{N*})$ and $\dim N(\boldsymbol{\mathcal{A}}_\varepsilon^{N*}) \leqslant k_0$. Now letting $\varepsilon \to 0$ into (4.21) and (4.22) it follows that

$$\boldsymbol{\mathcal{A}}^{N*}(\mathbf{q}^1 + \mathbf{q}^2) = -\begin{pmatrix} \mathcal{A}_J(\Phi^{*(1)} - \Phi^{\bullet(1)}) \\ \vdots \\ \mathcal{A}_J(\Phi^{*(N)} - \Phi^{\bullet(N)}) \end{pmatrix}, \qquad \boldsymbol{\Psi}^* = \frac{1}{\ell}\mathbf{q}.$$

Hence we have established the following maximum principle for the semidiscrete-in-time optimal control problem (P^N).

Theorem 4.3. *If the pair $(\boldsymbol{\Phi}^*, \boldsymbol{\Psi}^*)$ is optimal in problem (P^N) then there is $\mathbf{q} \in \mathbf{D}(\mathcal{A})$ such that*

$$\frac{1}{\Delta t}(q^{(n)} - q^{(n-1)}) - \mathcal{A}q^{(n-1)} - \mathcal{B}_0\big(\Phi_\varepsilon^{(n-1)}, \cdot, q^{(n-1)}\big) - \mathcal{B}_0(\cdot, \Phi^{(n-1)}, q^{(n-1)})$$
$$= \mathcal{A}_J(\Phi^{*(n-1)} - \Phi^{\bullet(n-1)}), \tag{4.23}$$
$$\Psi^{*(n)} = \frac{1}{\ell}q^{(n)} \tag{4.24}$$

for all $n = 1, \dots, N$, with $q^{(0)} = q^{(N)}$.

5. Error estimates

The error estimates we shall derive make use of the results of [2] concerning the approximation of a class of nonlinear problems. For the sake of completeness we will state the relevant result, specialized to our needs. The nonlinear problems to be considered are of the type

$$F(\lambda,\varphi) \equiv \varphi + TG(\lambda,\varphi) = 0, \tag{5.1}$$

where $T \in \mathcal{L}(Y,X)$, G is a C^2 mapping from $\Lambda \times X$ into Y, X and Y are Banach spaces, and Λ is a compact interval of R. We say that $\{(\lambda,\varphi(\lambda))\colon\ \lambda \in \Lambda\}$ is a branch of solutions of (5.1) if $\lambda \to \varphi(\lambda)$ is a continuous function from Λ into X such that $F(\lambda,\varphi(\lambda)) = 0$. The branch is called a *nonsingular branch* if we also have that $D_\varphi F(\lambda,\varphi(\lambda))$ is an isomorphism from X into X for all $\lambda \in \Lambda$. Here D_φ denotes the Fréchet derivative with respect to φ.

Approximations are defined by introducing an approximating operator $T^N \in \mathcal{L}(Y,X)$. Then we seek $\varphi^N \in X$ such that

$$F^N(\lambda,\varphi^N) \equiv \varphi^N + T^N G(\lambda,\varphi^N). \tag{5.2}$$

Suppose that (5.1) has a branch of nonsingular solutions $\{(\lambda,\varphi(\lambda))\colon\ \lambda \in \Lambda\}$. We make the following assumptions. First, there is another Banach space Z contained in Y, with continuous imbedding, such that

$$D_\varphi G(\lambda,\varphi) \in \mathcal{L}(X,Z), \quad \forall\lambda \in \Lambda,\ \forall\varphi \in X. \tag{5.3}$$

Concerning the operator T^N we assume that

$$\lim_{N\to\infty} \left\| (T^N - T)g \right\|_X = 0, \quad \forall g \in Y, \tag{5.4}$$

and

$$\lim_{N\to\infty} \left\| (T^N - T) \right\|_{\mathcal{L}(Z,X)} = 0. \tag{5.5}$$

We state now the result (see [2, Theorem 3.3]) that will be used in the sequel. In the statement of the theorem D^2G represents any and all second Fréchet derivatives of G.

Theorem 5.1. *Let X and Y be Banach spaces and Λ a compact set of R. Assume that G is a C^2 mapping from $\Lambda \times X$ into Y and that D^2G is bounded on all bounded subsets of $\Lambda \times X$. Assume that* (5.3)–(5.5) *hold and that $\{(\lambda,\varphi(\lambda))\colon\ \lambda \in \Lambda\}$ is a branch of nonsingular solutions of* (5.1). *Then there exists a neighborhood $\mathcal{O}$ of the origin in X and for $N \geqslant N_0$ big enough a unique C^2 function $\lambda \to \varphi^N(\lambda) \in X$ such that*

$$\bigl\{\bigl(\lambda,\varphi^N(\lambda)\bigr)\colon\ \lambda \in \Lambda\bigr\} \text{ is a branch of nonsingular solutions of (5.2)}, \tag{5.6}$$

$$\varphi^N(\lambda) - \varphi(\lambda) \in \mathcal{O} \quad \text{for all } \lambda \in \Lambda. \tag{5.7}$$

Moreover, there exists a constant C independent of N and λ such that

$$\left\| \varphi^N(\lambda) - \varphi(\lambda) \right\|_X \leqslant C \left\| (T^N - T)G\bigl(\lambda,\varphi(\lambda)\bigr) \right\|_X, \quad \forall\lambda \in \Lambda. \tag{5.8}$$

Now we recast the optimality system (1.2), (3.16) and its discretization (4.1), (4.23), (4.24) into a form that fits the above framework. We will use a vector $\lambda \in \Lambda$ (a compact set in R^2), and note that the theorem holds without major modification.

Theorem 5.2. *Let* $\{(\lambda = (\mathrm{Re}, \mathrm{Rm}),\ \varphi(\lambda) = (\Phi(\lambda), \lambda\Phi'(\lambda), \mathrm{Re}\, p(\lambda), q(\lambda), \lambda q'(\lambda), \mathrm{Re}\, p_0(\lambda)));\ \lambda \in \Lambda\}$ *be a nonsingular branch of solutions of* (1.1), (3.14), *and* (3.15). *Then there exists a neighborhood* $\mathcal{O}$ *of the origin in* X *and for* $N \geqslant N_0$ *big enough a unique* $\mathcal{C}^\infty$ *branch* $\{(\lambda, \varphi(\lambda) = (\Phi^N(\lambda), \lambda\Phi_{\mathrm{pl}}^{N\prime}(\lambda), \mathrm{Re}\, p^N(\lambda), q^N(\lambda), \lambda q_{\mathrm{pl}}^{N\prime}(\lambda), \mathrm{Re}\, p_0^N(\lambda)));\ \lambda \in \Lambda\}$ *of solutions of* (4.1), (4.23), *and* (4.24) *such that*

$$\varphi^N(\lambda) \in \varphi(\lambda) + \mathcal{O}, \quad \forall \lambda \in \Lambda.$$

Moreover, if

$$f \in C^1\big([0,T]; H^2(\Omega) \cap V\big) \cap C\big([0,T]; H^4(\Omega) \cap V\big), \text{ and there exists } t_0 \in [0,T] \tag{5.9a}$$

$$\text{such that } \Phi(t_0), q(t_0) \in H^4(\Omega) \cap V,\ \Phi'(t_0), q'(t_0) \in H^2(\Omega) \cap V, \tag{5.9b}$$

we have the estimate

$$\begin{aligned}
&\big\|\Phi^N(\lambda) - \Phi(\lambda)\big\|_{L^\infty(0,T;D(\mathcal{A}))} + \big\|\Phi_{\mathrm{pl}}^{N\prime}(\lambda) - \Phi'(\lambda)\big\|_{L^\infty(0,T;H)} \\
&\quad + \big\|p^N(\lambda) - p(\lambda)\big\|_{L^\infty(0,T;H^1(\Omega))} + \big\|q^N(\lambda) - q(\lambda)\big\|_{L^\infty(0,T;D(\mathcal{A}))} \\
&\quad + \big\|q_{\mathrm{pl}}^{N\prime}(\lambda) - q'(\lambda)\big\|_{L^\infty(0,T;H)} + \big\|p_0^N(\lambda) - p_0(\lambda)\big\|_{L^\infty(0,T;H^1(\Omega))} \leqslant C\Delta t,
\end{aligned}$$

where the constant C *is independent of* Δt.

Proof. We define the spaces

$$\begin{aligned}
X := X_N = {}& \big[X_N^\Phi \times X_N^{\Phi'} \times X_N^p\big]^2 \\
= {}& \big[\big\{\Phi \in L^\infty\big(0,T; \mathbb{H}^2(\Omega) \cap V\big):\ \Phi_{|(t_{n-1},t_n]} \in C^0\big\} \\
& \times \big\{\Phi \in L^\infty(0,T;H):\ \Phi_{|(t_{n-1},t_n]} \in C^0\big\} \\
& \times \big\{p \in L^\infty\big(0,T;H^1(\Omega)\big):\ p_{|(t_{n-1},t_n]} \in C^0\big\}\big]^2, \\
Y \equiv Z := Y_N = {}& \big[W^{1,2}\big([0,T];V'\big) \cap \big\{\Phi \in L^\infty(0,T;H): \\
& \Phi_{|(t_{n-1},t_n]} \in C^0,\ \Phi(0) = \Phi(T)\big\}\big]^2.
\end{aligned}$$

Let the operator $T \in \mathcal{L}(Y_N, X_N)$ be defined in the following manner: $T(f_u, f_B, g_u, g_B) = ((u,B), (u',B'), p, (q_u,q_B), (q_u',q_B'), p_0)$ for $(f_u, f_B, g_u, g_B) \in Y_N$ and $((u,B), (u',B'), p, (q_u,q_B), (q_u',q_B'), p_0) \in X_N$, if and only if

$$-\Delta u + \nabla p + u' = f_u, \quad \overline{\mathrm{curl}}(\mathrm{curl}\, B) + B' = f_B, \quad \text{in } Q, \tag{5.10a}$$

$$\Delta q_u + \nabla p_0 + q_u' = g_u, \quad -\overline{\mathrm{curl}}(\mathrm{curl}\, q_B) + q_B' = g_B, \quad \text{in } Q, \tag{5.10b}$$

$$\operatorname{div} u = \operatorname{div} B = \operatorname{div} q_u = \operatorname{div} q_B = 0, \quad \text{in } Q,$$

$$u = q_u = 0, \quad B \cdot n = q_B \cdot n = 0, \quad \mathrm{curl}\, B = \mathrm{curl}\, q_B = 0, \quad \text{on } \partial\Omega \times (0,T),$$

$$u(\cdot,0) = u(\cdot,T), \quad B(\cdot,0) = B(\cdot,T), \quad q_u(\cdot,0) = q_u(\cdot,T),$$

$$q_B(\cdot,0) = q_B(\cdot,T) \quad \text{in } \Omega.$$

Analogously, the operator $T^N \in \mathcal{L}(Y_N, X_N)$ is defined as follows: $T^N(f_u, f_B, g_u, g_B) = ((u^N, B^N), (u_{\mathrm{pl}}^{N\prime}, B_{\mathrm{pl}}^{N\prime}), p^N, (q_u^N, q_B^N), ({q_u}_{\mathrm{pl}}^{N\prime}, {q_B}_{\mathrm{pl}}^{N\prime}), p_0^N)$ for $(f_u, f_B, g_u, g_B) \in Y_N$ and $(u^N, B^N, u_{\mathrm{pl}}^{N\prime}, B_{\mathrm{pl}}^{N\prime}, p^N, q_u^N, q_B^N, {q_u}_{\mathrm{pl}}^{N\prime}, {q_B}_{\mathrm{pl}}^{N\prime}, p_0^N) \in X_N$, if and only if

$$-\Delta u^N + \nabla p^N + u_{\mathrm{pl}}^{N\prime} = f_u, \quad \overline{\mathrm{curl}}(\mathrm{curl}\, B^N) + B_{\mathrm{pl}}^{N\prime} = f_B, \quad \text{in } Q, \tag{5.11a}$$

$$\Delta q_u^N + \nabla p_0^N + {q_u}_{\mathrm{pl}}^{N\prime} = g_u, \quad -\overline{\mathrm{curl}}\big(\mathrm{curl}\, q_B^N\big) + {q_B}_{\mathrm{pl}}^{N\prime} = g_B, \quad \text{in } Q, \tag{5.11b}$$

$$\begin{aligned}
&\mathrm{div}\, u^N = \mathrm{div}\, B^N = \mathrm{div}\, q_u^N = \mathrm{div}\, q_B^N = 0, \quad \text{in } Q,\\
&u^N = q_u^N = 0, \quad B^N \cdot n = q_B^N \cdot n = 0, \quad \mathrm{curl}\, B^N = \mathrm{curl}\, q_B^N = 0,\\
&\quad \text{on } \partial\Omega \times (0,T),\\
&u^N(\cdot,0) = u^N(\cdot,T), \quad B^N(\cdot,0) = B^N(\cdot,T), \quad q_u^N(\cdot,0) = q_u(\cdot,T),\\
&q_B^N(\cdot,0) = q_B^N(\cdot,T) \quad \text{in } \Omega.
\end{aligned}$$

We note that the operators T, T^N are well defined in this framework (see Theorem I.6.1 in [10] for the case of Navier–Stokes equations).

Next we define the nonlinear mapping $G : \Lambda \times X_N \to Y_N$ as follows:

$$G\big(\lambda, (u, B, u', B', p, q_u, q_B, q_u', q_B', p_0)\big) = (f_u, f_B, g_u, g_B) \tag{5.12}$$

for $\lambda \in \Lambda$, $(u, B, u', B', p, q_u, q_B, q_u', q_B', p_0) \in X_N$, $(f_u, f_B, g_u, g_B) \in Y_N$ if and only if

$$\begin{aligned}
f_u &= \mathrm{Re}\bigg(-f_0 - \Psi_1 + (u\cdot\nabla)u - S\nabla\bigg(\frac{1}{2}B^2\bigg) + S(B\cdot\nabla)B\bigg),\\
f_B &= \mathrm{Rm}\big(\Psi_2 - (u\cdot\nabla)B + (B\cdot\nabla)u\big),\\
g_u &= \mathrm{Re}(-u\cdot\nabla q_u + B\cdot\nabla q_B - q_u\cdot\nabla u - q_B\cdot\nabla B),\\
g_B &= \mathrm{Rm}(S\,B\cdot\nabla q_u - u\cdot\nabla q_B + S q_u\cdot\nabla B + q_b\cdot\nabla u).
\end{aligned}$$

Clearly the solution to the optimality system (1.1), (3.14), and (3.15) is equivalent to

$$\begin{aligned}
&(u, B, \mathrm{Re}\, u', \mathrm{Rm}\, B', \mathrm{Re}\, p, q_u, q_B, \mathrm{Re}\, q_u', \mathrm{Rm}\, q_B', \mathrm{Re}\, p_0)\\
&\quad + TG\big(\lambda, (u, B, \mathrm{Re}\, u', \mathrm{Rm}\, B', \mathrm{Re}\, p, q_u, q_B, \mathrm{Re}\, q_u', \mathrm{Rm}\, q_B', \mathrm{Re}\, p_0)\big) = 0
\end{aligned}$$

and the discrete optimality system (4.1), (4.23), (4.24) is equivalent to

$$\begin{aligned}
&\big(u^N, B^N, \mathrm{Re}\, u_{\mathrm{pl}}^{N\prime}, \mathrm{Rm}\, B_{\mathrm{pl}}^{N\prime}, \mathrm{Re}\, p^N, q_u^N, q_B^N, \mathrm{Re}\, {q_u}_{\mathrm{pl}}^{N\prime}, \mathrm{Rm}\, {q_B}_{\mathrm{pl}}^{N\prime}, \mathrm{Re}\, p_0\big)\\
&\qquad + T^N G\big(\lambda, \big(u^N, B^N, \mathrm{Re}\, u_{\mathrm{pl}}^{N\prime}, \mathrm{Rm}\, B_{\mathrm{pl}}^{N\prime}, \mathrm{Re}\, p^N, q_u^N, q_B^N, \mathrm{Re}\, {q_u}_{\mathrm{pl}}^{N\prime},\\
&\qquad\qquad \mathrm{Rm}\, {q_B}_{\mathrm{pl}}^{N\prime}, \mathrm{Re}\, p_0\big)\big)\\
&\quad = 0.
\end{aligned}$$

Using an argument similar to the Navier–Stokes case (Theorem I.6.1 in [10]), one can conclude that $\Phi = (u, B)$, $q = (q_u, q_B) \in C([t_{n-1}, t_n), \mathbb{H}^2(\Omega) \cap V)$, for all $n = 1, \ldots, N$, and $\Phi'', q'' \in L^2(0, T; H)$. Therefore

$$
\begin{aligned}
&\|\Phi - \Phi^N\|_{X_N^\Phi} + \|q - q^N\|_{X_N^\Phi} \\
&\quad \leqslant \max_{n=1,\dots,N} \sup_{(t_{n-1},t_n]} \left(\left\|\Phi(t) - \Phi(t_n)\right\|_{H^2} + \left\|q(t) - q(t_n)\right\|_{H^2}\right) \\
&\qquad + \max_{n=1,\dots,N} \left(\left\|\Phi(t_n) - \Phi^n\right\|_{H^2} + \left\|q(t_n) - q^n\right\|_{H^2}\right). \qquad (5.13)
\end{aligned}
$$

Under the assumptions above, we first prove the following estimation:

$$
\left\|\Phi(t_m) - \Phi^m\right\| + \Delta t \sum_{n=1}^{m} \left|A\Phi(t_n) - A\Phi^n\right|^2 \leqslant C(\Delta t)^2, \quad \forall m = 1, \dots, N. \qquad (5.14)
$$

Let denote $d^n = \Phi(t_n) - \Phi^n$. By subtracting (5.11a) from (5.10a) at $t = t_n$ we get

$$
\frac{d_n - d_{n-1}}{\Delta t} + \mathcal{A} d_n = -\frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} (t - t_{n-1}) \Phi''(t)\, dt. \qquad (5.15)
$$

By multiplying (5.15) with d_n and integrating over Ω we get

$$
2|d_n|^2 + 2|d_n - d_{n-1}|^2 + \Delta t \|d_n\|^2 \leqslant 2|d_{n-1}|^2 + C(\Delta t)^2 \int_{t_{n-1}}^{t_n} \left|\Phi''(t)\right|^2 dt.
$$

Taking the summation for $n = 1$ to N we obtain

$$
\sum_{n=1}^{N} \|d_n\|^2 \leqslant C \Delta t \int_0^T \left|\Phi''(t)\right|^2 dt. \qquad (5.16)
$$

Now we multiply (5.15) by nAd_n, integrate over Ω, and summate from 1 to N to derive

$$
\begin{aligned}
&2N\|d_N\|^2 + 2\sum_{n=1}^{N} n\|d_n - d_{n-1}\|^2 + \Delta t \sum_{n=1}^{N} n|Ad_n|^2 \\
&\quad \leqslant 2\sum_{n=1}^{N} \|d_n\|^2 + C(\Delta t)^2 \int_0^T \left|\Phi''(t)\right|^2 dt.
\end{aligned}
$$

Since $N = T/\Delta t$, and $d_0 = d_N$, by using (5.16) we obtain

$$
\|d_0\|^2 \leqslant C(\Delta t)^2 \int_0^T \left|\Phi(t)''\right|^2 dt. \qquad (5.17)
$$

Finally multiplying (5.15) by Ad_n, integrating over Ω and summating from 1 to m, we obtain, via (5.17),

$$
\|d_m\|^2 + \Delta t \sum_{n=1}^{m} |Ad_n|^2 \leqslant C(\Delta t)^2 \int_0^T \left|\Phi(t)''\right|^2 dt, \quad \forall m = 1, \dots, N,
$$

which is the desired result (5.14).

For the backward in time equations (5.10b) and (5.11b), we obtain similarly that

$$\left\|q(t_m)-q^m\right\|^2+\sum_{n=m}^{N}\left|Aq(t_m)-Aq^m\right|^2\leqslant C(\Delta t)^2\int_0^T\left|\Phi(t)''\right|^2dt,$$
$$\forall m=1,\dots,N. \tag{5.18}$$

Since Φ, q are piecewise continuous in time with values in $\mathbb{H}^2(\Omega)$, we derive from (5.13), (5.14) and (5.18) that

$$\lim_{N\to\infty}\|T^N-T\|_{\mathcal{L}(Y,X)}=0.$$

Now by the definition (5.12) it follows that D^2G is independent of u, B, thus is bounded on all bounded subsets of $\Lambda\times X$ in virtue of the Sobolev imbeddings. Moreover, G is $\mathcal{C}^\infty$ and D^pG is zero for all $p\geqslant 2$. Since $\{(\lambda,\varphi(\lambda)=(\Phi(\lambda),\lambda\Phi'(\lambda),\operatorname{Re}p(\lambda),q(\lambda),\lambda q'(\lambda),\operatorname{Re}p_0(\lambda)));\ \lambda\in\Lambda\}$ is a nonsingular branch of solutions of (1.1), (3.14), and (3.15) by Theorem 5.1, we deduce that for N_0 big enough there exists a real $a>0$ and a unique branch $\{(\lambda,\varphi^N(\lambda)=(\Phi^N(\lambda),\lambda\Phi_{\mathrm{pl}}^{N}{}'(\lambda),\operatorname{Re}p^N(\lambda),q^N(\lambda),\lambda q_{\mathrm{pl}}^{N}{}'(\lambda),\operatorname{Re}p_0^N(\lambda)));\ \lambda\in\Lambda\}$ of solutions of (4.1), (4.23), (4.24) such that

$$\left\|\varphi^N(\lambda)-\varphi(\lambda)\right\|_X\leqslant a.$$

The mapping $\lambda\to\varphi^N(\lambda)$ is $\mathcal{C}^\infty$ due to the $\mathcal{C}^\infty$-regularity of G and the boundedness of D^pG on all bounded subsets of $\Lambda\times X$ (see Remark IV.3.6 in [2]).

Under the additional assumptions (5.9a)–(5.9b) we have that $\Phi, q\in C^1([0,T];H^2(\Omega)\cap V)$ and by (5.13)–(5.14) we have

$$\|T^N-T\|_{\mathcal{L}(Y,X)}\leqslant C\Delta t.$$

Hence

$$\begin{aligned}
&\left\|\varphi^N(\lambda)-\varphi(\lambda)\right\|_X\\
&\quad=\left\|\Phi^N(\lambda)-\Phi(\lambda)\right\|_{L^\infty(0,T;D(\mathcal{A}))}+\left\|\Phi_{\mathrm{pl}}^{N}{}'(\lambda)-\Phi'(\lambda)\right\|_{L^\infty(0,T;H)}\\
&\qquad+\left\|p^N(\lambda)-p(\lambda)\right\|_{L^\infty(0,T;H^1(\Omega))}+\left\|q^N(\lambda)-q(\lambda)\right\|_{L^\infty(0,T;D(\mathcal{A}))}\\
&\qquad+\left\|q_{\mathrm{pl}}^{N}{}'(\lambda)-q'(\lambda)\right\|_{L^\infty(0,T;H)}+\left\|p_0^N(\lambda)-p_0(\lambda)\right\|_{L^\infty(0,T;H^1(\Omega))}\\
&\quad\leqslant C\left\|(T^N-T)G\big(\lambda,\varphi(\lambda)\big)\right\|_X\leqslant C\left\|(T^N-T)\right\|_{\mathcal{L}(Y,X)}\left\|G\big(\lambda,\varphi(\lambda)\big)\right\|_X\leqslant C\Delta t,
\end{aligned}$$

which completes the proof. □

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