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Optimal control of the time-periodic MHD equations

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Abstract

We consider the mathematical formulation and analysis of an optimal control problem associated with the tracking of the velocity and the magnetic field of a viscous, incompressible, electrically conducting fluid in a bounded two-dimensional domain through the adjustment of distributed controls.

Keywords: Magnetohydrodynamic equations; Optimal control

1. Introduction

The paper studies the optimal control problem: minimize

$$\frac{1}{2}\int_Q (|\nabla(u(x,t)-u^\bullet(x,t))|^2 + |\mathrm{curl}(B(x,t)-B^\bullet(x,t))|^2 + \ell(|\psi_1(x,t)|^2 + |\psi_2(x,t)|^2))\,\mathrm{d}t\,\mathrm{d}x \tag{1.1}$$

over $\psi_1, \psi_2, u, B \in (L^2(Q))^2$ subject to the nondimensional magnetohydrodynamic equations (MHD equations) for a viscous incompressible resistive fluid (see [5,6])

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u - \frac{1}{Re}\Delta u + \nabla p + S\nabla(\tfrac{1}{2}B^2) - S(B\cdot\nabla)B = f_0 + \psi_1 \quad \text{in } \Omega\times R,$$

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$$\frac{\partial B}{\partial t} + (u\cdot\nabla)B - (B\cdot\nabla)u + \frac{1}{Rm}\,\bar{\text{curl}}(\text{curl}\,B) = \psi_2 \quad \text{in } \Omega\times R, \tag{1.2}$$

$$\text{div}\ u = 0,\ \ \text{div}\,B = 0 \text{ in } \Omega\times R,$$

$$u = 0 \text{ on } \partial\Omega\times R,\ \ B\cdot n = 0 \text{ and curl}\,B = 0 \text{ on } \partial\Omega\times R,$$

$$u(x,0) = u(x,T),\ \ B(x,0) = B(x,T) \quad \forall (x,t)\in\Omega\times R.$$

Here $Q = \Omega\times(0,T)$, Ω is an open bounded simply connected subset of R^2 with smooth boundary $\partial\Omega$, f_0 is a T-periodic (nondimensional) volume density force, $u = (u_1(x,t), u_2(x,t))$ is the velocity of the particle of fluid which is at point x at time t, $B = (B_1(x,t), B_2(x,t))$ is the magnetic field at point x at time t, $p = p(x,t)$ stands for the pressure of the fluid while $\psi_1, \psi_2 \in L^2_{\text{loc}}(R; L^2(\Omega))$ are divergence free T-periodic inputs, and $u^\bullet, B^\bullet \in L^2_{\text{loc}}(R; H^1(\Omega))$ are the T-periodic reference velocity and magnetic field, respectively. The nondimensional quantities p, u, B correspond to the normalization by reference units denoted by $L_*, T_*, U_* = L_*/T_*, B_*$, for lengths, times, velocities, and magnetic fields. There are three nondimensional numbers in the equation which represent the Reynolds number $Re = L_* u_*/v$ (where v is the kinematic viscosity), the magnetic Reynolds number $Rm = L_* u_* \sigma\mu$ (where μ is the magnetic permeability and σ the conductivity of the fluid, assumed to be constant), $S = M^2/ReRm = B_*^2/\mu\rho_* u_*^2$ (where M is the Hartman number) and $\ell > 0$. We recall the definitions of the curl and $\bar{\text{curl}}$ operators in 2-dimensions

$$\text{curl}\ u = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \text{ for every vector } u = (u_1, u_2),$$

$$\bar{\text{curl}}\ \phi = \left(\frac{\partial\phi}{\partial x_2}, -\frac{\partial\phi}{\partial x_1}\right) \text{ for every scalar function } \phi$$

and the following formula:

$$\bar{\text{curl}}\ \text{curl}\ u = \text{grad}\,\text{div}\ u - \Delta u. \tag{1.3}$$

2. Weak formulation and existence

Let us briefly recall the way we can represent the MHD equations (1.2) as an infinite-dimensional equation (see [6,8,2]). The spaces used are a combination of spaces for the Navier–Stokes equations (denoted with subscript 1) and spaces used in the theory of Maxwell equations (denoted with subscript 2). They are

$$\mathscr{V}_1 = \{v \in (\mathscr{C}_0^\infty(\Omega))^2, \text{div}\ v = 0\},$$

$$V_1 = \{v \in \mathbb{H}_0^1(\Omega),\ \text{div}\ v = 0\} \text{ (the closure of } \mathscr{V}_1 \text{ in } \mathbb{H}_0^1(\Omega) = (H_0^1(\Omega))^2),$$

$$H_1 = \{v \in \mathbb{L}^2(\Omega),\ \text{div}\ v = 0 \text{ and } v\cdot n|_{\partial\Omega} = 0\} \text{ (the closure of } \mathscr{V}_1 \text{ in } \mathbb{L}^2(\Omega) = (L^2(\Omega))^2),$$

$$\mathscr{V}_2 = \{C \in (\mathscr{C}^\infty(\overline{\Omega}))^2, \text{div}\ C = 0 \text{ and } C\cdot n|_{\partial\Omega} = 0\},$$

$$V_2 = \{v \in \mathbb{H}^1(\Omega),\ \text{div}\, C = 0 \text{ and } C \cdot n|_{\partial\Omega} = 0\} \text{ (the closure of } \mathscr{V}_2 \text{ in } \mathbb{H}^1(\Omega) = (H^1(\Omega))^2),$$

$$H_2 = (\text{the closure of } \mathscr{V}_2 \text{ in } \mathbb{L}^2(\Omega)) = H_1.$$

The space V_1 is endowed with the scalar product

$$((u, v))_1 = \sum_{1 \leqslant i \leqslant 2} \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right) = \sum_{1 \leqslant i \leqslant 2} \int_\Omega \frac{\partial u}{\partial x_i} \cdot \frac{\partial v}{\partial x_i}\, \mathrm{d}x,$$

which is the scalar product on $\mathbb{H}^1_0(\Omega)$. The dual space of V_1 is characterized by (see [8])

$$V_1' = \{v \in \mathbb{H}^{-1}(\Omega),\ \text{div}\, v = 0\}.$$

The space V_2 is endowed with the scalar product

$$((u, v))_2 = (\text{curl}\, u, \text{curl}\, v)$$

which is equivalent to the usual scalar product induced by $\mathbb{H}^1(\Omega)$ on V_2. We set (see [8])

$$V = V_1 \times V_2, \quad H = H_1 \times H_2, \ \ V' \text{ the dual space of } V,$$

and by identifying H with its own dual we have $V \subset H \subset V'$. The space H will be endowed with the following scalar products:

$$(\Phi, \Psi) = (u, v) + (B, C) \ \text{ for all } \Phi = (u, B),\ \ \Psi = (v, C) \in H,$$

$$[\Phi, \Psi] = (u, v) + S(B, C)$$

and the induced (equivalent) norms

$$|\Phi| = (\Phi, \Phi)^{1/2}, \quad [\Phi] = [\Phi, \Phi]^{1/2}.$$

The space V will be endowed with three scalar products

$$((\Phi, \Psi)) = \frac{1}{Re}((u, v))_1 + \frac{1}{Rm}((B, C))_2,$$

$$[\![\Phi, \Psi]\!] = \frac{1}{Re}((u, v))_1 + \frac{S}{Rm}((B, C))_2,$$

$$((\Phi, \Phi))_J = ((u, v))_1 + ((B, C))_2$$

and the equivalent norms

$$\|\Phi\| = ((\Phi, \Phi))^{1/2}, \quad [\![\Phi]\!] = [\![\Phi, \Phi]\!]^{1/2}, \quad \|\Phi\|_J = ((\Phi, \Phi))_J^{1/2}.$$

Let $\mathscr{A}_1 \in L(V_1, V_1')$, $\mathscr{A}_2 \in L(V_2, V_2')$, $\mathscr{A} \in L(V, V')$, $\mathscr{A}_J \in L(V, V')$ be defined by

$$\langle \mathscr{A}_1 u, v \rangle = ((u, v))_1 \ \text{ for all } u, v \in V_1,$$

$$\langle \mathscr{A}_2 B, C\rangle = ((B, C))_2 \quad \text{for all } B, C \in V_2,$$

$$\langle \mathscr{A}\Phi, \Psi\rangle = ((\Phi, \Psi)),\ \ \langle \mathscr{A}_J\Phi, \Psi\rangle = ((\Phi, \Psi))_J \quad \text{for all } \Phi, \Psi \in V.$$

As in [8] we consider $\mathscr{A}_1 \in L(V_1, V_1')$, $\mathscr{A}_2 \in L(V_2, V_2')$, $\mathscr{A} \in L(V, V')$ as unbounded operators on H_1, H_2, H, for which the domains are

$$D(\mathscr{A}_1) = \{u \in V_1, \mathscr{A}_1 u \in H_1\} = \mathbb{H}^2(\Omega) \cap V_1,$$

$$D(\mathscr{A}_2) = \{B \in V_2, \mathscr{A}_2 B \in H_2\} = \mathbb{H}^2(\Omega) \cap V_2,$$

$$D(\mathscr{A}) = D(\mathscr{A}_2) \times D(\mathscr{A}_2) = (\mathbb{H}^2(\Omega))^2 \cap V.$$

Let $b : \mathbb{L}^1(\Omega) \times \mathbf{W}^{1,1}(\Omega) \times \mathbb{L}^1(\Omega) \to R$ be defined by

$$b(u, v, w) = \sum_{1\leqslant i,j\leqslant 2} \int_\Omega u_i D_i v_j\, w_j$$

whenever the integrals make sense. We recall that, for $m_i \geqslant 0$ satisfying $m_1 + m_2 + m_3 > 1$ or $m_1 + m_2 + m_3 = 1$ where at least two m_i are nonzero, we have

$$|b(u, v, w)| \leqslant c_1 |u|_{H^{m_1}} |v|_{H^{m_2+1}} |w|_{H^{m_3}},$$
$$\forall (u, v, w) \in \mathbb{H}^{m_1}(\Omega) \times \mathbb{H}^{m_2+1}(\Omega) \times \mathbb{H}^{m_3}(\Omega). \tag{2.1}$$

For $m_1 = m_3 = 1$, $m_2 = 0$ we find that the trilinear form b is continuous on $(\mathbb{H}^1(\Omega))^3$ and satisfies

$$b(u, v, v) = 0, \quad \forall u \in V_\alpha\ (\alpha = 1, 2), \quad \forall v \in \mathbb{H}^1(\Omega),$$

$$b(u, v, w) = -b(u, w, v), \quad \forall u \in V_\alpha, \quad \forall v, w \in \mathbb{H}^1(\Omega). \tag{2.2}$$

We also define the trilinear form $\mathscr{B}_0 : V \times V \times V \to R$ by setting

$$\mathscr{B}_0(\Phi_1, \Phi_2, \Phi_3) = b(u_1, u_2, u_3) - Sb(B_1, B_2, u_3) + b(u_1, B_2, B_3) - b(B_1, u_2, B_3)$$

for all $\Phi_i = (u_i, B_i) \in V$, and the bilinear continuous operator $\mathscr{B} : V \times V \to V'$

$$\langle \mathscr{B}(\Phi_1, \Phi_2), \Phi_3\rangle = \mathscr{B}_0(\Phi_1, \Phi_2, \Phi_3) \quad \forall \Phi_i \in V.$$

From (2.1) we get

$$|\mathscr{B}_0(\Phi_1, \Phi_2, \Phi_3)| \leqslant c_2 \max(1, S) |\Phi_1|_{H^{m_1}} |\Phi_2|_{H^{m_2+1}} |\Phi_3|_{H^{m_3}},$$
$$\forall (\Phi_1, \Phi_2, \Phi_3) \in \mathbb{H}^{m_1}(\Omega) \times \mathbb{H}^{m_2+1}(\Omega) \times \mathbb{H}^{m_3}(\Omega). \tag{2.3}$$

This yields for $m_1 = m_2 = 1/2$, $m_3 = 0$ that

$$|\mathscr{B}_0(\Phi_1, \Phi_2, \Phi_3)| \leqslant c_3 (|\Phi_1|\, \|\Phi_1\|\, \|\Phi_2\|\, |\mathscr{A}\Phi_2|)^{1/2} |\Phi_3|,$$
$$\forall (\Phi_1, \Phi_2, \Phi_3) \in V \times D(\mathscr{A}) \times H, \tag{2.4}$$

where $c_3 = c_3(\Omega, S, Re, Rm)$. Let $M \in M_4(R)$ denote the diagonal matrix

$$m_{ii} = 1 \text{ for } 1 \leqslant i \leqslant 2, \quad m_{ii} = S \text{ for } 3 \leqslant i \leqslant 4. \tag{2.5}$$

From (2.1) and the identity

$$\mathscr{B}_0(\Phi_1, \Phi_2, M\Phi_2) = b(u_1, u_2, u_2) + Sb(u_1, B_2, B_2) - S(b(B_1, B_2, u_2) + b(B_1, u_2, B_2))$$

we finally get

$$\mathscr{B}_0(\Phi_1, \Phi_2, M\Phi_2) = 0 \quad \forall \Phi_1, \Phi_2 \in V,$$
$$\mathscr{B}_0(\Phi_1, \Phi_2, M\Phi_3) = -\mathscr{B}_0(\Phi_1, \Phi_3, M\Phi_2) \quad \forall \Phi_i \in V. \tag{2.6}$$

Let $f(t) = P(f_0(t), 0)$, $\Psi(t) = P(\psi_1(t), \psi_2(t))$ where $P : (\mathbb{L}^2(\Omega))^2 \to H$ is the projection on H. Then we rewrite the state equation (1.2) as

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t}(t) + \mathscr{A}\Phi(t) + \mathscr{B}(\Phi(t), \Phi(t)) = f(t) + \Psi(t), \quad t \in (0, T),$$
$$\Phi(0) = \Phi(T) \tag{2.7}$$

and confine to the *strong solutions* $\Phi \in L^2(0, T; D(\mathscr{A})) \cap W^{1,2}([0, T]; H)$.

Assume that $\Phi^\bullet = (u^\bullet, B^\bullet) \in L^2(0, T; V)$. Then we may reformulate problem (1.1) as

$$\text{Minimize } J(\Phi, \Psi) = \int_0^T \left(\frac{1}{2}\|\Phi(t) - \Phi^\bullet(t)\|_J^2 + \frac{\ell}{2}|\Psi(t)|^2 \right) \mathrm{d}t \, (P)$$

over $(\Phi, \Psi) \in (L^2(0, T; D(\mathscr{A})) \cap W^{1,2}([0, T]; H)) \times L^2(0, T; H)$ subject to (2.7).

Theorem 2.1. *There is at least one solution* (Φ^*, Ψ^*) *to problem* (P).

Proof. Let $\{\Phi_n, \Psi_n\}$ be a minimizing sequence in problem (P), i.e.,

$$\inf (P) \leqslant J(\Phi_n, \Psi_n) \leqslant \inf (P) + \frac{1}{n}, \tag{2.8}$$

$$\Phi_n' + \mathscr{A}\Phi_n + \mathscr{B}(\Phi_n, \Phi_n) = f + \Psi_n, \quad \text{a.e. } t \in (0, T); \; \Phi_n(0) = \Phi_n(T). \tag{2.9}$$

By (2.8) it follows that $\{\Phi_n\}$ is bounded in $L^2(0, T; V)$, $\{\Psi_n\}$ is bounded in $L^2(0, T; H)$ and therefore on a subsequence, again denoted by n, we have

$$\Psi_n \to \Psi^* \text{ weakly in } L^2(0, T; H).$$

If we multiply (2.9) by $tM\Phi_n$, integrate on Ω we get by (2.6) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(t[\Phi_n(t)]^2) - \frac{1}{2}[\Phi_n(t)]^2 + t[\![\Phi_n(t)]\!]^2 = t[f(t) + \Psi_n(t), \Phi_n(t)], \text{ a.e. } t \in (0, T). \tag{2.10}$$

This yields

$$t[\Phi_n(t)]^2 + \int_0^t \tau[\![\Phi_n(s)]\!]^2 \, \mathrm{d}s \leqslant C, \quad \forall t \in [0, T]$$

and therefore

$$[\Phi_n(0)] = [\Phi_n(T)] \leqslant C, \quad \forall n \in \mathbb{N}.$$

Here C denotes several positive constants independent of Φ and n. Next we multiply (2.9) by $t\mathscr{A}\phi_n$ and obtain after some calculus involving Young's inequality and (2.4) that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(t\|\Phi_n(t)\|^2\right) - \frac{1}{2}\|\Phi_n(t)\|^2 + t|\mathscr{A}\Phi_n(t)|^2 \\
&\quad = (f(t) + \Phi_n(t), t\mathscr{A}\Phi_n) - \mathscr{B}_0(\Phi_n(t), \Phi_n(t), t\mathscr{A}\Phi_n(t)) \\
&\quad \leqslant \frac{t}{4}|\mathscr{A}\Phi_n(t)|^2 + t|f(t) + \Phi_n(t)|^2 + tc_3|\Phi_n(t)|^{1/2}\|\Phi_n(t)\|\,|\mathscr{A}\Phi_n|^{3/2} \\
&\quad \leqslant \frac{t}{2}|\mathscr{A}\Phi_n(t)|^2 + tC|\Phi_n(t)|^2\|\Phi_n(t)\|^4 + t|f(t) + \Phi_n(t)|^2.
\end{aligned}$$

Now integrating on $(0, t)$ and using the above estimates we get

$$t\|\Phi_n(t)\|^2 + \int_0^t s|\mathscr{A}\Phi_n(s)|^2\,\mathrm{d}s \leqslant C\int_0^t (1 + s\|\Phi_n(t)\|^4)\,\mathrm{d}s$$

which by Grönwall's lemma gives

$$t\|\Phi_n(t)\|^2 \leqslant C, \quad \forall t \in (0, T].$$

Since $\Phi_n(0) = \Phi_n(T)$ we infer that $\|\Phi_n(0)\| \leqslant C$. Finally, multiplying (2.9) by $\mathscr{A}\Phi_n$ and integrating on $\Omega \times (0, t)$ we obtain as above

$$\|\Phi_n(t)\|^2 + \int_0^t |\mathscr{A}\Phi_n(s)|^2\,\mathrm{d}s \leqslant C\left(\|\Phi_n(0)\|^2 + \int_0^t \|\Phi_n(s)\|^4\,\mathrm{d}s\right)$$

and therefore

$$\|\Phi_n(t)\|^2 + \int_0^t |\mathscr{A}\Phi_n(s)|^2\,\mathrm{d}s \leqslant C, \quad \forall t \in [0, T].$$

This yields

$$\|\Phi_n'\|_{L^2(0,T;H)} + \|\mathscr{B}(\Phi_n, \Phi_n)\|_{L^2(0,T;H)} \leqslant C.$$

Since $V \subset\subset H$ we infer that $\{\Phi_n\}$ is compact in $C([0, T]; H) \cap L^2(0, T; V)$ and on subsequences we have

$$\begin{aligned}
&\Phi_n \to \Phi^* \text{ strongly in } L^2(0, T; V) \cap C([0, T]; H), \\
&\mathscr{A}\Phi_n \to \mathscr{A}\Phi^* \text{ weakly in } L^2(0, T; H), \\
&\Phi_n' \to (\Phi^*)' \text{ weakly in } L^2(0, T; H).
\end{aligned}$$

By (2.4) we have

$$\begin{aligned}
&|(\mathscr{B}(\Phi_n, \Phi_n) - \mathscr{B}(\Phi^*, \Phi^*), \Phi)| \\
&\quad \leqslant |\mathscr{B}_0(\Phi_n - \Phi^*, \Phi_n, \Phi)| + \mathscr{B}_0(\Phi^*, \Phi_n - \Phi^*, \Phi)| \\
&\quad \leqslant C\|\Phi_n - \Phi^*\|^{1/2}(|\Phi_n - \Phi^*|^{1/2}\|\Phi_n\|^{1/2}|\mathscr{A}\Phi_n|^{1/2} \\
&\qquad + |\Phi^*|^{1/2}\|\Phi^*\|^{1/2}|\mathscr{A}(\Phi_n - \Phi^*)|^{1/2})|\Phi| \qquad (2.11)
\end{aligned}$$

for all $\Phi \in H$, and therefore

$$\mathscr{B}(\Phi_n, \Phi_n) \to \mathscr{B}(\Phi^*, \Phi^*) \text{ strongly in } L^2(0, T; H).$$

Letting n go to ∞ in (2.8), (2.9) we see that (Φ^*, Ψ^*) satisfies system (2.7) and $J(\Phi^*, \Psi^*) = \inf(P)$.

3. Optimality conditions

Let (Φ^*, Ψ^*) be an optimal pair in problem (P). For each $\varepsilon > 0$ consider the approximating problem: minimize

$$\int_0^T \left(\frac{1}{2}\|\Phi - \Phi^\bullet\|_J^2 + \frac{\ell}{2}|\Psi|^2 + \frac{1}{2\varepsilon}|\xi|^2\right) \mathrm{d}t \, (P_\varepsilon)$$

over $\Phi \in L^2(0, T; D(\mathscr{A})) \cap W^{1,2}([0, T]; H)$, $\Psi, \xi \in L^2(0, T; H)$ subject to

$$\begin{aligned} &\Phi'(t) + \mathscr{A}\Phi(t) + \mathscr{B}(\Phi(t), \Phi(t)) = f(t) + \Psi(t) + \xi(t), \quad t \in (0, T); \\ &\Phi(0) = \Phi(T). \end{aligned} \tag{3.1}$$

By Theorem 2.1 for each $\varepsilon > 0$ problem (P_ε) has at least one solution $(\Phi_\varepsilon, \Psi_\varepsilon, \xi_\varepsilon)$.

Lemma 3.1. *For $\varepsilon \to 0$ we have*

$$\begin{aligned} &\Phi_\varepsilon \to \Phi^* \quad \textit{strongly in } L^2(0, T; V) \cap C([0, T]; H), \\ &\Phi'_\varepsilon \to (\Phi^*)', \ \mathscr{A}\Phi_\varepsilon \to \mathscr{A}\Phi^* \quad \textit{weakly in } L^2(0, T; H), \\ &\Psi_\varepsilon \to \Psi^*, \ \varepsilon^{-1/2}\xi_\varepsilon \to 0 \quad \textit{weakly in } L^2(0, T; H), \\ &\lim_{\varepsilon \to 0} \left\{ \inf_{\Phi_\varepsilon, \Psi_\varepsilon, \xi_\varepsilon} (P_\varepsilon) \right\} = \inf_{\Phi, \Psi} (P). \end{aligned} \tag{3.2}$$

Proof. By taking $(\Phi, \Psi, \xi) = (\Phi^*, \Psi^*, 0)$ in (P_ε) we get

$$\inf_{\Phi, \Psi, \xi} (P_\varepsilon) \leqslant \int_0^T \left(\frac{1}{2}\|\Phi^* - \Phi^\bullet\|_J^2 + \frac{\ell}{2}|\Psi^*|^2\right) \mathrm{d}t \equiv \inf_{\Phi, \Psi} (P).$$

If multiply (3.1) with $M\Phi_\varepsilon$, $tM\Phi_\varepsilon$ and integrate on $(0, T)$, $(0, t)$ respectively, we get by (2.4), (2.6) that

$$t[\Phi_\varepsilon(t)]^2 + \int_0^T [\![\Phi_\varepsilon(t)]\!]^2 \, \mathrm{d}t \leqslant C.$$

Now if we multiply (3.1) by $t\mathscr{A}\Phi_\varepsilon$, integrate on $(0, t)$, we see as above that

$$t\|\Phi_\varepsilon(t)\|^2 \leqslant C, \quad \forall \in (0, T]$$

and therefore

$$\|\Phi_\varepsilon(0)\| = \|\Phi_\varepsilon(T)\| \leqslant C, \quad \forall \varepsilon > 0.$$

When we multiply (3.1) by $\mathscr{A}\Phi_\varepsilon$ we obtain

$$\|\Phi_\varepsilon(t)\|^2 + \int_0^t |\mathscr{A}\Phi_\varepsilon(s)|^2\,\mathrm{d}s \leqslant C, \quad \forall t \in [0, T]$$

and from (3.1) we have that

$$\|\Phi'_\varepsilon\|_{L^2(0,T;H)} + \|\mathscr{B}(\Phi_\varepsilon, \Phi_\varepsilon)\|_{L^2(0,T;H)} \leqslant C, \quad \forall \varepsilon > 0.$$

Hence on a subsequence we have

$$\Phi_\varepsilon \to \bar{\Phi} \quad \text{strongly in } C([0,T]; H) \cap L^2(0,T; V)$$

$$\Phi'_\varepsilon \to \bar{\Phi}', \ \mathscr{A}\Phi_\varepsilon \to \mathscr{A}\bar{\Phi} \quad \text{weakly in } L^2(0,T; H)$$

$$\Psi_\varepsilon \to \bar{\Psi}, \ \xi_\varepsilon \to 0 \quad \text{weakly in } L^2(0,T; H).$$

On the other hand, by (2.11) we see that

$$\mathscr{B}(\Phi_\varepsilon, \Phi_\varepsilon) \to \mathscr{B}(\bar{\Phi}, \bar{\Phi}) \text{ strongly in } L^2(0,T; H)$$

and therefore $(\bar{\Phi}, \bar{\Psi})$ is a solution to the state system (2.7).

Finally, taking the limit in (P_ε), by the weak lower semicontinuity of the H-norm we obtain that

$$\inf_{\Phi,\Psi}(P) \leqslant \int_0^T \left(\frac{1}{2}\|\bar{\Phi} - \Phi^\bullet\|_J^2 + \frac{\ell}{2}|\bar{\Psi}|^2\right)\mathrm{d}t \leqslant \lim_{\varepsilon\to 0}\left\{\inf_{\Phi,\Psi,\xi}(P_\varepsilon)\right\}$$

hence $\bar{\Phi} = \Phi^*$, $\bar{\Psi} = \Psi^*$ and the conclusions of Lemma 3.1 follow. □

In the space $L^2(0,T; H)$ we define the operators (see [1])

$$\mathfrak{A}_\varepsilon\phi = \phi' + \mathscr{A}\phi + \mathscr{B}(\Phi_\varepsilon, \phi) + \mathscr{B}(\phi, \Phi_\varepsilon), \quad \forall \phi \in D(\mathfrak{A}_\varepsilon) = X,$$

$$\mathfrak{A}^*_\varepsilon\phi = -\phi' + \mathscr{A}\phi + \mathscr{B}_0(\Phi_\varepsilon, \cdot, \phi) + \mathscr{B}_0(\cdot, \Phi_\varepsilon, \phi), \quad \forall \phi \in X, \tag{3.3}$$

where

$$X = \{\phi \in W^{1,2}([0,T]; H) \cap L^2(0,T; D(\mathscr{A})), \phi(0) = \phi(T)\}.$$

It is easily seen that

$$\int_0^T (\mathfrak{A}^*_\varepsilon \Upsilon, \phi)\,\mathrm{d}t = \int_0^T (\mathfrak{A}_\varepsilon\phi, \Upsilon)\,\mathrm{d}t, \quad \forall \phi, \Upsilon \in D(\mathfrak{A}_\varepsilon) = D(\mathfrak{A}^*_\varepsilon) = X.$$

The operators $\mathfrak{A}$ and $\mathfrak{A}^*$ are defined by the same formulae (3.3) where $\Phi_\varepsilon = \Phi^*$.

Lemma 3.2. *The operators* $\mathfrak{A}_\varepsilon, \mathfrak{A}^*_\varepsilon, \mathfrak{A}, \mathfrak{A}^*$ *are closed, densely defined and have closed ranges in* $L^2(0,T;H)$. *Moreover*, $\dim N(\mathfrak{A}_\varepsilon), \dim N(\mathfrak{A}^*_\varepsilon) \leqslant n_0$, *independent of* ε, $\mathfrak{A}^*_\varepsilon$ *is the adjoint of* $\mathfrak{A}_\varepsilon$ *and the following estimates hold*:

$$\|\mathfrak{A}_\varepsilon^{-1} g\|_{L^2(0,T;D(\mathscr{A}))\cap W^{1,2}([0,T];H)} \leqslant C\|g\|_{L^2(0,T;H)}, \quad \forall g \in R(\mathfrak{A}_\varepsilon),$$

$$\|(\mathfrak{A}^*_\varepsilon)^{-1} g\|_{L^2(0,T;D(\mathscr{A}))\cap W^{1,2}([0,T];H)} \leqslant C\|g\|_{L^2(0,T;H)}, \quad \forall g \in R(\mathfrak{A}^*_\varepsilon). \tag{3.4}$$

Similarly, the operators $\mathfrak{A}^*$, $\mathfrak{A}$ *are mutually adjoint and estimates* (3.4) *remain true for* $\mathfrak{A}^*$, $\mathfrak{A}$.

We have used the symbols N and R to denote the null space and the range of the corresponding operators. (For hints on the proof see the case of Navier–Stokes equations in [1]).

For $\lambda \in R$, $\Phi \in X$, $\Psi \in L^2(0,T;H)$ we set

$$\xi^\lambda = (\Phi_\varepsilon + \lambda\Phi)' + \mathscr{A}(\Phi_\varepsilon + \lambda\Phi) + \mathscr{B}(\Phi_\varepsilon + \lambda\Phi, \Phi_\varepsilon + \lambda\Phi) - (f + \Psi_\varepsilon + \lambda\Psi).$$

We may write ξ^λ as

$$\xi^\lambda = \xi_\varepsilon + \lambda(\Phi' + \mathscr{A}\Phi + \mathscr{B}(\Phi_\varepsilon, \Phi) + \mathscr{B}(\Phi, \Phi_\varepsilon) + \lambda\mathscr{B}(\Phi, \Phi) - \Psi)$$

and so by the optimality of $(\Phi_\varepsilon, \Psi_\varepsilon, \xi_\varepsilon)$ in (P_ε) we have

$$\int_0^T \Bigg(((\Phi_\varepsilon - \Phi^\bullet, \Phi))_J + \ell(\Psi_\varepsilon, \Psi) + \frac{1}{\varepsilon}(\xi_\varepsilon, \Phi' + \mathscr{A}\Phi + \mathscr{B}(\Phi_\varepsilon, \Phi) + \mathscr{B}(\Phi, \Phi_\varepsilon) - \Psi) \Bigg) \,\mathrm{d}t \geqslant 0 \tag{3.5}$$

for all $\phi \in X$, $\Psi \in L^2(0,T;H)$. We set $q_\varepsilon = \frac{1}{\varepsilon}\xi_\varepsilon$, and for $\Psi = 0$ we get from above

$$\int_0^T \big(((\Phi_\varepsilon - \Phi^\bullet, \Phi))_J + (q_\varepsilon, \mathfrak{A}_\varepsilon\Phi) \big) \,\mathrm{d}t = 0.$$

Hence $q_\varepsilon \in D(\mathfrak{A}^*_\varepsilon)$ and

$$\mathfrak{A}^*_\varepsilon q_\varepsilon = -\mathscr{A}_J(\Phi_\varepsilon - \Phi^\bullet). \tag{3.6}$$

Therefore by (3.5) we obtain

$$\Psi_\varepsilon = \frac{1}{\ell} q_\varepsilon \ \text{ a.e. in } (0,T). \tag{3.7}$$

Then by Lemma 3.1 it follows that

$$\|q_\varepsilon\|_{L^2(0,T;H)} \leqslant C, \quad \forall \varepsilon > 0.$$

Now we may write q_ε as $q^1_\varepsilon + q^2_\varepsilon$ where $q^1_\varepsilon \in R(\mathfrak{A}_\varepsilon)$, $q^2_\varepsilon \in N(\mathfrak{A}^*_\varepsilon)$. By Lemma 3.2 we know that

$$\|q^1_\varepsilon\|_{L^2(0,T;D(\mathscr{A}))\cap W^{1,2}([0,T];H)} \leqslant C, \quad \forall \varepsilon > 0$$

hence on a subsequence, again denoted $\{\varepsilon\}$, we have

$$q^1_\varepsilon \to q^1 \text{ weakly in } L^2(0,T;D(\mathscr{A}))\cap W^{1,2}([0,T];H),$$

$$q^2_\varepsilon \to q^2 \text{ strongly in } L^2(0,T;D(\mathscr{A}))\cap W^{1,2}([0,T];H)$$

because $\{q^2_\varepsilon\} \subset N(\mathfrak{A}^*_\varepsilon)$ and $\dim N(\mathfrak{A}^*_\varepsilon)\leqslant n_0$. Now letting ε tend to 0 into (3.6), (3.7) it follows by Lemma 3.1 that

$$\mathfrak{A}^*(q^1+q^2) = -\mathscr{A}_J(\Phi^*-\Phi^\bullet);\quad \Psi^* = \frac{1}{\ell}(q^1+q^2) \text{ a.e. } t\in(0,T).$$

Let denote $q=q^1+q^2$. We have established the following maximum principle result for problem (P):

Theorem 3.1. *If the pair (Φ^*,Ψ^*) is optimal in problem (P) then there is $q\in L^2(0,T;D(\mathscr{A}))\cap W^{1,2}([0,T];H)$ such that*

$$q'(t) - \mathscr{A}q - \mathscr{B}_0(\Phi^*,\cdot,q) - \mathscr{B}_0(\cdot,\Phi^*,q) = \mathscr{A}_J(\Phi^*-\Phi^\bullet),\quad a.e.\ t\in(0,T),$$

$$q(0)=q(T), \tag{3.8}$$

$$\Psi^*(t) = \frac{1}{\ell}q(t),\quad a.e.\ t\in(0,T). \tag{3.9}$$

If $q=(q_u,q_B)$ and $\Psi^*=(\psi_1^*,\psi_2^*)$ the adjoint system can be written as

$$\frac{\partial q_u}{\partial t} + \frac{1}{Re}\Delta q_u + u^*\cdot\nabla q_u - B^*\nabla q_B + q_u\cdot\nabla u^* + q_B\cdot\nabla B^* + \nabla p_0$$
$$= -\Delta(u^*-u^\bullet) \quad \text{in } Q,$$

$$\frac{\partial q_B}{\partial t} - \frac{1}{Rm}\,\bar{\text{curl}}(\text{curl}\, q_B) - SB^*\cdot\nabla q_u + u^*\cdot\nabla q_B - Sq_u\cdot\nabla B^* - q_B\cdot\nabla u^*$$
$$= \bar{\text{curl}}(\text{curl}(B^*-B^\bullet)),$$

$$\text{div}\, q_u = 0,\ \text{div}\, q_B = 0 \quad \text{in } Q,$$

$$q_u = 0 \text{ on } \Sigma,\ q_B\cdot n = 0 \quad \text{and} \quad \text{curl}\, q_B = 0 \quad \text{on } \Sigma,$$

$$q_u(x,0) = q_u(x,T),\ q_B(x,0) = q_B(x,T) \quad \text{in } Q$$

and the optimality condition

$$\psi_1^* = \frac{1}{\ell}q_u,\quad \psi_2^* = \frac{1}{\ell}q_B \quad \text{in } Q.$$

4. Semidiscrete-in-time approximations

Let $\sigma_N = \{t_n\}_{n=0}^N$ be a partition of $[0,T]$ into equal intervals of duration $\Delta t = T/N$ with $t_0=0$ and $t_N=T$. We will denote by $\mathbf{v}$ the vector $(v^{(1)}, v^{(2)},\ldots,v^{(N)})$ of functions belonging to a space $\mathbf{Y} = Y^N$. We associate the following approximate function:

$$v^N(t,x) = v^{(n)}(x), \;\; t \in [t_{n-1}, t_n], \;\; n = 1, 2, \ldots, N,$$

where $v^{(0)} = v^{(N)}$, and a continuous, piecewise (in time t) linear function $v_{\mathrm{pl}}^N = v_{\mathrm{pl}}^N(t,x)$ defined by the interpolating conditions

$$v_{\mathrm{pl}}^N(t_n, x) = v^{(n)}(x), \;\; n = 1, 2, \ldots, N.$$

On this partition we define the discrete target $\Phi^{\bullet(n)}(x) = \Phi^{\bullet}(t_n, x)$ for $n = 0, 1, \ldots, N$. The state variables $\Phi^{(n)} \in D(\mathscr{A})$ are constrained to satisfy the semidiscrete MHD equation

$$\frac{1}{\Delta t}(\Phi^{(n)} - \Phi^{(n-1)}) + \mathscr{A}\Phi^{(n)} + \mathscr{B}(\Phi^{(n)}, \Phi^{(n)}) = f^{(n)} + \Psi^{(n)} \tag{4.1}$$

obtained from (2.7) by a backward Euler discretization in time and the periodic condition

$$\Phi^{(0)} = \Phi^{(N)}. \tag{4.2}$$

Optimization is achieved by means of the minimization of the discretized-in-time functional

$$J^N(\mathbf{\Phi}, \mathbf{\Psi}) = \frac{1}{2}\Delta t \sum_{n=1}^{N} \|\Phi^{(n)} - \Phi^{\bullet(n)}\|_J^2 + \frac{\ell}{2}\Delta t \sum_{n=1}^{N} |\Psi^{(n)}|^2. \tag{4.3}$$

This functional results from applying the right-point discretization rule in time to the continuous functional J. The discrete-in-time approximate optimal control problem is then given by

$$\begin{array}{l} \textit{given } \Delta t = T/N, \;\; \phi^{\bullet} \in L^2(0,T;V) \textit{ find } (\mathbf{\Phi}, \mathbf{\Psi}) \textit{ in } \mathbf{D}(\mathscr{A}) \times \mathbf{H} \textit{ such that} \\ (\mathbf{\Phi}, \mathbf{\Psi}) \textit{ is the solution of } (4.1) \textit{ and the functional } (4.3) \textit{ is minimized.} \end{array} \tag{P^N}$$

As in the continuous case we have the following result on the existence of a optimal pair:

Theorem 4.1. *Given $T > 0, \Delta t = T/N$ there exists at least one optimal solution $(\mathbf{\Phi}^*, \mathbf{\Psi}^*) \in \mathbf{D}(\mathscr{A}) \times \mathbf{H}$ of the semidiscrete optimal control problem.*

Now we can prove the convergence of the semidiscrete optimal control problem.

Theorem 4.2. *For $\Delta t \to 0$ the solution $\{(\Phi^{*(n)}, \Psi^{*(n)})\}_{n=1}^N$ of the semidiscrete-in-time optimal control problem tends to the solution (Φ^*, Ψ^*) of the corresponding continuous optimal control problem.*

Proof. Using the similar computations as in the previous section (see [3,4,8]) we obtain easily that $\{(\Phi^{*N}, \Psi^{*N})\}_{N=1}^{\infty}$ is uniformly bounded in $L^2(0,T;D(\mathscr{A})) \cap L^{\infty}(0,T;V) \times L^2(0,T;H)$ and $\{\frac{d}{dt}\Phi^{*N}_{\mathrm{pl}}\}$ is uniformly bounded in $L^2(0,T;V)$. Moreover, we have that

$$\|\Phi^{*N} - \Phi^{*N}_{\mathrm{pl}}\|^2_{L^2(0,T;V)} = \frac{\Delta t}{3}\sum_{n=1}^{N} \|\Phi^{*(n)} - \Phi^{*(n-1)}\|^2 \to 0 \quad \text{when } \Delta t \to 0.$$

Hence on subsequences we have that

$$\Psi^{*N} \to \Psi^{*} \text{ weakly in } L^2(0,T;H),$$
$$\Phi^{*N} \to \Phi^{*} \text{ strongly in } L^2(0,T;V), \text{ weak-* in } L^2(0,T;D(\mathscr{A})). \tag{4.4}$$

Eq. (4.1) can be interpreted as

$$\frac{\mathrm{d}\Phi^{*N}_{\mathrm{pl}}}{\mathrm{d}t} + \mathscr{A}\Phi^{*N} + \mathscr{B}(\Phi^{*N},\Phi^{*N}) = f^N + \Psi^{*N}$$

and as we pass $N\to\infty$ we find that the solution of the semidiscrete problem (P^N) converges to the corresponding solution of the continuous optimal control problem (P). □

Due to the lack of differentiability in the application $\mathbf{\Psi} \to \mathbf{\Phi}(\mathbf{\Psi})$ (see e.g. [7]) we will replace problem (P^N) by a sequence of approximating problems (P^N_ε), for which we can compute necessary conditions of optimality.

For each ε consider the following optimization problem: minimize

$$J^N_\varepsilon(\mathbf{\Phi},\mathbf{\Psi},\boldsymbol{\xi}) = \frac{\Delta t}{2}\sum_{n=1}^{N} \|\Phi^{(n)} - \Phi^{\bullet(n)}\|_J^2 + \frac{\ell}{2}\Delta t \sum_{n=1}^{N} |\Psi^{(n)}|^2 + \frac{\Delta t}{2\varepsilon}\sum_{n=1}^{N} |\xi^{(n)}|^2 \quad (P^N_\varepsilon)$$

over $(\mathbf{\Phi},\mathbf{\Psi},\boldsymbol{\xi}) \in \mathbf{D}(\mathscr{A}) \times \mathbf{H} \times \mathbf{H}$ satisfying

$$\frac{1}{\Delta t}(\Phi^{(n)} - \Phi^{(n-1)}) + \mathscr{A}\Phi^{(n)} + \mathscr{B}(\Phi^{(n)},\Phi^{(n)}) = f^{(n)} + \Psi^{(n)} + \xi^{(n)},$$
$$\Phi^{(0)} = \Phi^{(N)}. \tag{4.5}$$

By Theorem 4.1, for each $\varepsilon > 0$ problem (P^N_ε) has at least one solution $(\mathbf{\Phi}_\varepsilon,\mathbf{\Psi}_\varepsilon,\boldsymbol{\xi}_\varepsilon)$.

Lemma 4.1. *For* $\varepsilon \to 0$ *we have*

$$\Phi^{(n)}_\varepsilon \to \Phi^{*(n)} \text{ weakly in } D(\mathscr{A}), \text{ strongly in } V,$$
$$\Psi^{(n)}_\varepsilon \to \Psi^{*(n)} \text{ weakly in } V, \text{ strongly in } H,$$
$$\varepsilon^{-1/2}\xi^{(n)}_\varepsilon \to 0 \text{ weakly in } H \tag{4.6}$$

for all $n = 1,\dots,N$ *and*

$$\lim_{\varepsilon\to 0}\left\{\inf_{(\mathbf{\Phi},\Psi,\xi)} (P^N_\varepsilon)\right\} = \inf_{(\mathbf{\Phi},\Psi)} (P^N).$$

For $\lambda \in R$, $\Phi^{(n)} \in D(\mathscr{A})$, $\forall n = 0,\dots,N$ with $\Phi^{(0)} = \Phi^{(N)}$ we denote

$$\zeta^{(n)}_\lambda = \zeta^{(n)}_\varepsilon + \lambda\left(\frac{\Phi^{(n)} - \Phi^{(n-1)}}{\Delta t} + \mathscr{A}\Phi^{(n)} + \mathscr{B}(\Phi^{(n)}_\varepsilon,\Phi^{(n)}) + \mathscr{B}(\Phi^{(n)},\Phi^{(n)}_\varepsilon) \right.$$
$$\left. + \lambda\mathscr{B}(\Phi^{(n)},\Phi^{(n)}) - \Psi^{(n)}\right)$$

and by the optimality of $(\Phi_\varepsilon^{(n)}, \Psi_\varepsilon^{(n)}, \zeta_\varepsilon^{(n)})$ we get

$$\begin{aligned}&\Delta t \sum((\Phi_\varepsilon^{(n)} - \Phi^{\bullet(n)}, \Phi^{(n)}))_J + \ell\Delta t \sum(\Psi_\varepsilon^{(n)}, \Psi^{(n)}) \\ &\quad + \frac{\Delta t}{\varepsilon}\sum_{n=1}^{N}\left(\zeta_\varepsilon^{(n)}, \frac{\Phi^{(n)} - \Phi^{(n-1)}}{\Delta t} + \mathscr{A}\Phi^{(n)} + \mathscr{B}(\Phi_\varepsilon^{(n)}, \Phi^{(n)}) + \mathscr{B}(\Phi^{(n)}, \Phi_\varepsilon^{(n)})\right. \\ &\quad \left. + \lambda\mathscr{B}(\Phi^{(n)}, \Phi^{(n)}) - \Psi^{(n)}\right) \geqslant 0 \qquad (4.7)\end{aligned}$$

for all $\{\Phi^{(n)}\}_{n=0}^{N} \subset D(\mathscr{A})^{N+1}$ with $\Phi^{(0)} = \Phi^{(N)}$, and for all $\Psi^{(n)} \in H, n = 1, \ldots, N$. We set $q_\varepsilon^{(n)} = \frac{1}{\varepsilon}\zeta_\varepsilon^{(n)}$. Using an argument similar to the continuous case, it can be easily proved the following maximum principle for the semidiscrete-in-time optimal control problem (P^N).

Theorem 4.3. *If the pair* $(\mathbf{\Phi}^*, \mathbf{\Psi}^*)$ *is optimal in problem* (P^N) *then there is* $\mathbf{q} \in \mathbf{D}(\mathscr{A})$ *such that*

$$\begin{aligned}&\frac{1}{\Delta t}(q^{(n)} - q^{(n-1)}) - \mathscr{A}q^{(n)} - \mathscr{B}_0(\Phi_\varepsilon^{(n)}, q^{(n)}, \Psi^{(n)}) - \mathscr{B}_0(\Psi^{(n)}, q^{(n)}, \Phi_\varepsilon^{(n)}) \\ &\quad = \mathscr{A}_J(\Phi^{*(n)} - \Phi^{\bullet(n)}),\end{aligned}$$

$$\Psi^{*(n)} = \frac{1}{\ell}q^{(n)}$$

for all $n = 1, \ldots, N$, *with* $q^{(0)} = q^{(N)}$.

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