# Optimal control of an elliptic equation under periodic conditions

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**Abstract**

In this paper we find explicitly the optimal control for an elliptic equation with respect to each of the cost functional (1.1) or (1.7).

## 1 Introduction

In this paper we are concerned with the following two control problems

$$(\mathrm{P}_1) \qquad \text{Minimize } L_1(y,u) = \int_0^T (y(0,t) - g_1(t))^2 \, dt \tag{1.1}$$

subject to

$$y_{tt}(x,t) + (u(x)y_x(x,t))_x = f(x,t) \text{ in } Q = (0,1) \times (0,T) \tag{1.2}$$

$$y_x(0,t) = 0, \ y(1,t) = 0 \text{ in } (0,T) \tag{1.3}$$

$$y(x,0) = y(x,T), \ y_t(x,0) = y_t(x,T) \text{ in } (0,1) \tag{1.4}$$

where $T$ is an arbitrary positive number and the set $U$ of admissible controls is given by

$$U = \left\{ \begin{array}{l} u \in Lip(0,1); |u'(x)| \le \rho \ \text{ a.e. in } (0,1), \\ 0 < a \le u(x) \le b, \ \forall x \in [0,1], \ u(0) = \alpha, \ u(1) = \beta \end{array} \right\} \tag{1.5}$$

with $a < \alpha < \beta < b, \ \rho \ge \alpha + \beta - 2a > 0$ and $g_1 \in C^2([0,T])$.
Here $f \in L^2(Q)$ guarantees the existence and uniqueness of an $L^2(0,T;H)$ solution of (1.1)-(1.4), $H = L^2(0,1)$.

The second problem is

$$(\mathrm{P}_2) \qquad \text{Minimize } L_2(y,u) = \int_0^1 (y(x,0) - g_2(t))^2 \, dx \tag{1.6}$$

subject to (1.2)-(1.4), where $g_2 \in L^2(0,1)$. The hypotheses on $f$ are

$$f \in C^1(Q), \ f(x,t) > 0, \ f_x(x,t) < 0 \text{ in } Q. \tag{$H1$}$$

The problem (1.2)-(1.4) can be viewed as a stationary heat conductor model for a strip conductor $(0,1) \times \mathbb{R}$ with periodic conditions in the variable $t$, with zero temperature at $x = 1$ (see [2]). Our goal is to determine the optimal control $u^*$ corresponding to the cost functionals of the form (1.1), (1.7). Other cost functionals can be treated in a similar way, but the determination of optimal control will be likely more complicated (or simply not possible). For example the following cost functional $L(y,u)$ was considered in place of $L_i(y,u), i = 1,2$ (see [2])

$$L(y,u) = \int_0^1 y(x,0)dx.$$

Note that problems $(\mathrm{P}_1)$ and $(\mathrm{P}_2)$ can be viewed as identification problems, i.e.,

How should we choose the coefficient $u$ in the state equation (1.2) (under boundary condition (1.3)-(1.4)) to achieve some prescribed goal (e.g. to achieve the minimum of $L_i$)?

The main result of this paper is that the function $u^*$ given by (2.30) is the unique (solution) optimal control for both $(P_1)$ and $(P_2)$. This is done in Sections 2 and 3 (respectively).

## 2 Problem ($P_1$)

First let explain why the problem (1.2)-(1.4) is well-posed. For each $u \in U$ set

$$D(A(u)) = \{H^2(0,1);\ y_x(0) = 0,\ y(1) = 0\} \tag{2.1}$$

$$(A(u)y)(x) = (u(x)y_x(x))_x,\ y \in D(A(u)), \tag{2.2}$$

where $H^i(0,1)$ denotes the usual Sobolev spaces ($i = 1, 2$).

Observe that $y(1) = 0$ yields

$$|y(x)| \leq \int_0^1 |y_x(s)|ds \leq |y_x|_{L^2(0,1)},\ y \in D(A(u)) \tag{2.3}$$

so

$$|y|_{L^2(0,1)} \leq |y_x|_{L^2(0,1)},\ y \in D(A(u)). \tag{2.4}$$

Integrating by parts is easy to check that $A(u) : L^2(0,1) \to L^2(0,1)$ is strictly negative definite (strictly dissipative), namely

$$\langle A(u)y, y\rangle_{L^2(0,1)} = -\int_0^1 u(x)y_x^2(x)dx \leq -a|y_x|^2 \leq -a|y|^2,\ \forall y \in D(A(u)) \tag{2.5}$$

with $a > 0$ as in (1.6) and $|y| = |y|_{L^2(0,1)}$. Moreover, $\langle A(u)y, z\rangle_{L^2(0,1)} = \langle y, A(u)z\rangle_{L^2(0,1)}$ for all $y, z \in D(A(u))$ and therefore $A(u)$ is selfadjoint.

Define the realization $\widetilde{A}$ of $A(u)$ in $L^2(0,T;H)$ by

$$D(\widetilde{A}) = \left\{\widetilde{y} \in L^2(0,T;H);\ \widetilde{y}(t) \in D(A(u)) \text{ a.e. in } (0,T);\ t \to A(u)\widetilde{y}(t) \in L^2(0,T;H)\right\}$$

and

$$(\widetilde{A}\widetilde{y})(t) = A(u)\widetilde{y}(t),\ \text{a.e. in } (0,T) \text{ for } \widetilde{y} \in D(\widetilde{A}) \tag{2.6}$$

with $\widetilde{A} = \widetilde{A}(u)$.

We note that $\widetilde{A}$ is also strictly dissipative and selfadjoint in $L^2(0,T;H)$. Now let us define

$$D(B) = \left\{z \in H^2(0,T;H),\ z(0) = z(T),\ z_t(0) = z_t(T)\right\} \tag{2.7}$$

and

$$Bz = z_{tt},\ z \in D(B). \tag{2.8}$$

We have that $B$ is dissipative

$$\langle Bz, z\rangle_{L^2(0,T;H)} = -\int_0^T |z_t(s)|_H^2 ds \leq 0,\ \forall z \in D(B), \tag{2.9}$$

and is easy to check that $B$ is maximal dissipative (see [1]).

On the other hand, the sum $\widetilde{A}+B$ is a selfadjoint strictly negative definite operator in $L^2(0,T;H)$ so it is maximal dissipative onto and one-to-one. Therefore for every $f \in L^2(0,T;H)$ there is a unique $\widetilde{y} \in D(\widetilde{A}\,) \cap D(B)$ such that

$$B\widetilde{y} + \widetilde{A}(u)\widetilde{y} = f. \tag{2.10}$$

which means that $\widetilde{y} = \widetilde{y}(x,t)$ is the unique solution to (1.2)-(1.4) in $L^2(0,T;H)$. Summarizing, we have proved the following theorem.

**Theorem 2.1** *For every $u \in U$ and $f \in L^2(Q)$, the problem* (1.2)-(1.4) *has a unique solution $y^u = y(x,t) \in D(\widetilde{A}) \cap D(B)$.*

In order to define precisely the optimal control $u^*$ for Problem ($P_1$), introduce the set $M$ of admissible pairs $(y,u)$.

$$M = \{(y,u), y \in D(\widetilde{A}(u)) \cap D(B), u \in U, (y,u) \text{ related as in } (2.10)\}. \tag{2.11}$$

In other words, $(y,u) \in M$ means: $y$ is the $L^2(Q)$ solution of (1.2)-(1.4) corresponding to $u \in U$.

By definition, an optimal pair $(y^*,u^*) \in M$ satisfies

$$L_1(y^*,u^*) = \inf\{L_1(y,u),\ (y,u) \in M\} \tag{2.12}$$

with $L_1$ as in (1.1). Here $u^*$ is an optimal control and $y^* = y_{u^*}$ is the corresponding optimal arc.

Now let us rewrite (1.2)-(1.4) as

$$y_{tt} + A(u)y = f \text{ in } L^2(0,T;H) = L^2(Q) \tag{2.13}$$

$$y(0) = y(T),\ y_t(0) = y_t(T). \tag{2.14}$$

and introduce the tangential (contingent) cone to $M$ at $(y,u) \in M$

$$\begin{aligned} \text{Tan } M(y,u) = \big\{&(v,w),\ v \in L^2(Q),\ w \in \text{Tan } U(u); \\ &v_{tt} + A(u)v + A(w)y = 0 \text{ in } L^2(Q) \\ &v(0) = v(T),\ v_t(0) = v_t(T)\big\}. \end{aligned} \tag{2.15}$$

The normal cone $NU(u)$ to $U$ at $u \in U$ is defined by

$$NU(u) = \{w^* \in C^*([0,1]);\ w^*(v) \le 0,\ \forall v \in \text{Tan } U(u)\}. \tag{2.16}$$

In view of the convexity of $U$ it follows that

$$NU(u) = \{w^* \in C^*([0,1]);\ w^*(z-u) \le 0,\ \forall z \in U\}. \tag{2.17}$$

The existence of an optimal pair $(y^*,u^*)$ follows by standard arguments and it is not our goal here. It is our purpose here to determine explicitly the optimal control $u^*$.

Recall that if

$$G(F) = \inf_{y \in M} G(y)$$

with $G : M \to \mathbb{R}$ Fréchet differentiable, then necessarily

$$\partial G(F)(Y) \ge 0 \text{ for all } Y \in \text{Tan } M(F), \tag{2.18}$$

where $\partial G(F)$ stands for the Fréchet derivative of $G$ at $F \in M$.

Applying the optimum principle given by (2.18), it follows that (in our case $G(Y) = L_1(y,u)$)

$$\int_0^T (y^*(0,t) - g(t))\, v(0,t)dt \ge 0 \text{ for all } (v,w) \in \text{ Tan } M(y^*,u^*). \tag{2.19}$$

In order to take advantage of (2.19) (which is a necessary condition for the optimality of $(y^*,u^*)$), let consider the adjoint (dual) problem

$$p_{tt} + \left((u^*(x))p_x(x,t)\right)_x = 0 \text{ in } Q \tag{2.20}$$

$$p_x(0,t) = y^*(0,t) - g_1(t),\ p(1,t) = 0 \text{ in } [0,T] \tag{2.21}$$

$$p(x,0) = p(x,T),\ p_t(x,0) = p_t(x,T) \text{ in } (0,1). \tag{2.22}$$

The existence and uniqueness of solution to (2.20)$-$(2.22) is discussed by the end of this section. Multiplying (2.20) by $v$ and integrating over $[0,T] \times [0,1]$ we derive successively

$$-\int_0^T u^*(0)p_x(0,t)v(0,t)dt + \int_0^T \langle p, v_{tt} + A(u^*)v\rangle_{L^2(0,1)}\, dt = 0$$

for all $w \in \text{Tan } U(u^*)$, i.e.,

$$\int_0^1 w(x) \int_0^T y_x^*(x,t) p_x(x,t)\, dt\, dx = \int_0^T u^*(0)\, (y^*(0,t) - g(t))\, v(0,t)\, dt \tag{2.23}$$

for all $w \in \text{Tan } U(u^*)$. A simple combination of (2.16), (2.19) and (2.23) shows that the function

$$\varphi(x) = -\int_0^T y_x^*(x,t) p_x(x,t) dt, \quad x \in (0,1) \tag{2.24}$$

is an element of $NU(u^*)$.

Summarizing, we have proved the following result.

**Lemma 2.1** *Let $f \in C^1(Q)$ and let $(y^*, u^*)$ be an optimal pair for* (P$_1$)*. Then $\varphi \in NU(u^*)$.*

Next we prove that $\varphi$ holds a constant sign. Precisely, we have the following lemma.

**Lemma 2.2** *Let f satisfy the sign condition in* (H$_1$) *and $g_1 \in C^2([0,T])$, with $g_1(0) = g_1(T)$ and*

$$g_1(t) < \inf_{u \in U} y^u(0,t), \quad t \in [0,T], \tag{2.25}$$

*where $y^u$ is the solution of* $(1.2) - (1.4)$ *corresponding to $u \in U$. Then $\varphi$ given by* (2.24) *satisfies $\varphi(x) < 0$ for $x \in (0,1)$.*

*Proof.* First we prove that the solution $y$ to (2.13)-(2.14) satisfies the sign condition

$$y(x,t) < 0, \quad y_x(x,t) > 0 \ \text{ in } Q. \tag{2.26}$$

Since $f > 0$ in Q, by the maximum principle for elliptic equations we get

$$\max_{\overline{Q}} y(x,t) = \max_{\partial Q} y(x,t)$$

while

$$\max_{\partial Q} y(x,t) = \max_{0 \le t \le T} y(1,t) = 0$$

so $y(x,t) < 0$ in $Q$. Indeed, if we assume that $\max\limits_{\partial Q} y(x,t) = y(0,t_0)$, with $0 < t_0 < T$, then $\frac{\partial y}{\partial v_x}(0,t_0) > 0$ which contradicts (1.3). In the same manner, if we assume by contradiction that

$$\max_{\partial Q} y(x,t) = y(x_0,0) = y(x_0,T), \quad 0 < x_0 < 1$$

we get in conflict with (1.4). In order to prove the second part of (2.26), set

$$z(x,t) = u(x) y_x(x,t) \ \text{ in } Q$$

Since $y(x,t) < 0$ in $Q$, by (1.2)-(1.4) and (H$_1$) we get

$$z_{tt} + u(x) z_{xx} = u(x) f_x(x,t) < 0 \ \text{ in } Q$$

$$z(0,t) = 0; \quad z(1,t) \ge 0 \ \text{ in } (0,T)$$

$$z(x,0) = z(x,T); \quad z_t(x,0) = z_t(x,T) \ \text{ in } (0,1)$$

so again by the minimum principle

$$\min_{\overline{Q}} z(x,t) = \min_{\partial Q} z(x,t).$$

Using the same argument as above, $z$ cannot assume its minimum on the subboundary $\{(x,0),(x,T), 0 < x < 1\}$ of $\partial Q$, which implies $z(x,t) = u(x) y_x(x,t) > 0$ in $Q$.

Next we seek $p_x(x,t) > 0$ in $Q$, where $p$ is the solution of (2.20)−(2.22). Clearly this can be accomplished if $p_x(0,t) > 0$ in [0,T], i.e.,

$$g_1(t) < y^*(0,t) \quad \text{in } [0,T] \tag{2.27}$$

which is obviously implied by (2.25).
Indeed, by the maximum principle

$$\max_{\overline{Q}} p(x,t) = \max_{\partial Q} p(x,t).$$

In view of (2.27), $\max_{\overline{Q}} p$ cannot be assumed on $\{(0,t),\ 0 < t < T\}$ as $\max_{\overline{Q}} p = p(0,t)$ would imply $p_x(0,t) \leq 0$ for $0 < t < T$ (which is in conflict with $p_x(0,t) > 0$ required by (2.27)). Moreover, $\max_{\overline{Q}} p(x,t)$ cannot occur on $\{t = T\}$ as $\max_{\overline{Q}} p(x,t) = p(x,0) = p(x,T)$ would contradict (2.22). Therefore $\max_{\overline{Q}} p(x,t) = p(1,t) = 0$ for some $t \in [0,T]$ so

$$p(x,t) < 0 \quad \text{in } Q. \tag{2.28}$$

Set $q(x,t) = u^*(x)p_x(x,t)$. It follows from (2.20) – (2.22) and (2.28) that

$$q_{tt} + u^*(x)q_{xx} = 0 \quad \text{in } Q$$

$$q(0,t) > 0,\ q(1,t) \geq 0 \quad \text{in } (0,T)$$

$$q(x,0) = q(x,T),\ q_t(x,0) = q_t(x,T) \quad \text{in } (0,1).$$

(Note that (2.28) and $p(1,t) = 0$ imply $p_x(1,t) \geq 0$, i.e., $q(1,t) \geq 0$.)

According to the minimum principle we can derive that $\min_{\overline{Q}} p(x,t) = q(0,t)$ or $\min_{\overline{Q}} p(x,t) = q(1,t)$ for some $t \in [0,T)$. Therefore $\min_{\overline{Q}} q(x,t) \geq 0$, i.e., $q(x,t) > 0$ which implies

$$p_x(x,t) > 0 \quad \text{in } Q$$

as desired. This completes the proof of Lemma 2.2. □

The next result gives the structure of those $L^1(0,1)$-functions which belong to $NU(u)$, with $U$ regarded as a subset of $L^\infty(0,1)$. So we now view $NU(u)$ as

$$NU(u) = \{w \in (L^\infty(0,1))^*,\ w(z,u) \leq 0,\ \forall z \in U\}.$$

**Lemma 2.3** *A function $w \in L^1(0,1)$ belongs to the normal cone $N_U(u)$ to $U$ at $u \in U$ if and only if*

$$w = -\theta' + \eta \text{ in } \mathcal{D}'(0,1) \tag{2.29}$$

*with $\theta \in L^1(0,1), \eta \in L^1(0,1)$ satisfying*

$$\begin{aligned} &\theta(x) = 0 \text{ a.e. in } \{x \in (0,1); |u'(x)| < \rho\} \\ &\theta(x) = \lambda(x)u'(x) \text{ a.e. in } \{x \in (0,1); |u'(x)| = \rho\} \end{aligned}$$

*where $\lambda \in L^1(0,1)$, $\lambda(x) \geq 0$ a.e. in $(0,1)$*

$$\begin{aligned} &\eta(x) = 0, \text{ a.e. in } \{x \in (0,1); a < u(x) < b\} \\ &\eta(x) \leq 0, \text{ a.e. in } \{x \in (0,1); u(x) = a\} \\ &\eta(x) \geq 0, \text{ a.e. in } \{x \in (0,1); u(x) = b\}. \end{aligned}$$

The proof of Lemma 2.3 can be found in [5].

We are now in position to state the main result of this paper, i.e.,

**Theorem 2.2** *Let $f \in C^1(Q)$ satisfy the sign condition in* $(\mathrm{H}_1)$ *and $g_1 \in C^2([0,T])$, with $g_1(0) = g_1(T)$ satisfy* (2.25)*. Then the optimal control $u^*$ of* $(\mathrm{P}_1)$ *is given by*

$$u^*(x) = \begin{cases} \alpha - \rho x, & \text{for } x \in [0, (\alpha - a)\rho^{-1}] \\ a, & \text{for } x \in [(\alpha - a)\rho^{-1}, (a + \rho - \beta)\rho^{-1}] \\ \beta + \rho(x-1), & \text{for } x \in [(a+\rho-\beta)\rho^{-1}, 1] \end{cases} \tag{2.30}$$

*with $a, \alpha, \beta$ as in* (1.6).

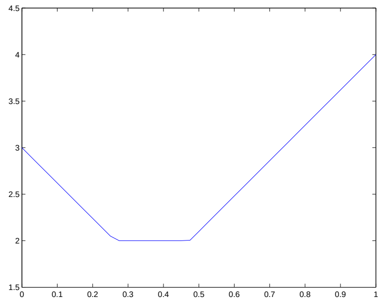

Figure 1. The optimal control $u^*$

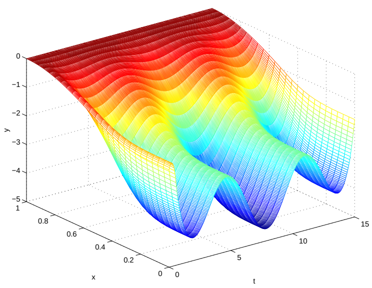

Figure 2. The optimal state $y^*$

(We have set $a = 2,\ \alpha = 3,\ \beta = 4,\ b = 5,\ \rho = 3.8,\ T = 15,\ f = (2-x) * (2 + \sin t)$.)

*Proof.* By hypothesis, $u^* \in U$ so $u^*(0) = \alpha > a$. Therefore the open set $M_a = \{x \in [0,1];\ u^*(x) > a\}$ is nonempty.

By Lemma 2.2 we have that $\varphi(x) < 0$ in $[0,1]$. Or, it was proved in [5] that if $NU(u^*)$ contains a negative function, then $u^*$ is convex. Indeed, in view of Lemmas 2.1, 2.2 and 2.3, the function $\varphi$ given by (2.24) can be written as $\eta - \theta'$, so $-\theta' + \eta = \varphi < 0$ in $M_a$ with $\eta \geq 0$ in $M_a$. Thus $-\theta' < 0$ in $M_a$, which implies $|u^{*\prime}(x)| = \rho$ in $M_a$. Suppose now that $u^*$ is not convex on $[0,1]$. This means that there is an interval $[0, x_0] \subset M_a$ such that $u^*(x) = \alpha + \rho x_0 - \rho(x - x_0)$, $x \in [x_0, x_0 + \varepsilon]$, with $\varepsilon > 0$, $\varepsilon$ sufficiently small.

Choose now $v$ such that

$$v(x) = \begin{cases} \alpha + \frac{u^*(x_0+\varepsilon) - \alpha}{x_0 + \varepsilon} x, & \text{for } 0 < x \leq x_0 + \varepsilon \\ u^*(x), & \text{for } x_0 + \varepsilon \leq x \leq 1. \end{cases}$$

Obviously $v(x) < u^*(x)$ for $0 < x < x_0 + \varepsilon$ and so $u^* - v \in H_0^1(0,1)$, with $u^* - v > 0$. Since $-\theta' < 0$ in $M_a$, we must have

$$\int_0^{x_0+\varepsilon} -\theta'(u^* - v)dx < 0. \tag{2.31}$$

But $\theta(x) = \lambda(x) u^{*\prime}(x)$ in $M_a$, so

$$\begin{aligned} \int_0^{x_0+\varepsilon} -\theta'(u^*(x) - v(x))dx &= \int_0^{x_0+\varepsilon} \lambda(x) u^{*\prime}(x)(u^{*\prime}(x) - v'(x))dx \\ &= \int_0^{x_0+\varepsilon} \lambda(x)(\rho^2 - u^{*\prime}(x) v'(x))dx \geq 0 \end{aligned} \tag{2.32}$$

as $v'(x) = \frac{\rho(x_0 - \varepsilon)}{x_0 + \varepsilon} < \rho$. The contradiction between (2.31) and (2.32) shows that $u^*$ must be convex in $M_a$. Similarly one proves that $u^*$ is convex on the whole interval $[0,1]$. It follows that $u^*$ must have the form as indicated in formula (2.30). This completes the proof. $\square$

Now let us motivate the existence of the solution $p = p(x,t)$ of (2.20) – (2.22) under the hypotheses

$$g_1 \in C^2([0,1]), \quad g_1(0) = g_1(T), \quad g_1'(0) = g_1'(T).$$

Set

$$w(x,t) = p(x,t) + (g_1(t) - y^*(0,t))(x-1) \ \text{ in } Q.$$

Therefore $p$ is the solution to (2.20) – (2.22) if and only if $w$ is the solution to the problem

$$w_{tt} + (u^* w_x)_x = h(x,t) \ \text{ in } Q$$

$$w_x(0,t) = 0, \ w(1,t) = 0$$

$$w(x,0) = w(x,T), \ w_t(x,0) = w_t(x,T)$$

where

$$h(x,t) = (g_1''(t) - y_{tt}^*(0,t))(x-1) + (g_1(t) - y^*(0,t))\, u_x^*(x)$$

**Remark 2.1** *The condition* (2.25) *makes sense, since the function* $(t,u) \to y^u(0,t)$ *is continuous on* $[0,T] \times U$ *and* $U$ *is obviously compact in* $C([0,1])$.

# 3 Problem (P$_2$)

With the same notations as above, in this case the analogous to (2.20) – (2.22) (i.e., the dual problem) is

$$p_{tt} + (u^* p_x)_x = 0 \ \text{ in } Q \tag{3.1}$$

$$p_x(0,t) = 0, \ p(1,t) = 0 \ \text{ in } [0,T] \tag{3.2}$$

$$p(x,0) = p(x,T), \ p_t(x,T) - p_t(x,0) = y^*(x,0) - g_2(x) \ \text{ in } (0,1). \tag{3.3}$$

In this case, the necessary condition for the optimality of $(y^*, u^*)$ (i.e., the analogous of (2.19)) is given by

$$\int_0^1 (y^*(x,0) - g_2(x))\, v(x,0) dx \geq 0 \tag{3.4}$$

for all $(v,w) \in \text{Tan } M(y^*, u^*)$ as in (2.15).
Multiplying (3.1) by $v$ in the sense of $L^2(0,1)$ and integrating over $[0,T]$, we derive successively

$$\int_0^1 v(x,0)\,(p_t(x,T) - p_t(x,0))\, dx + \int_0^T \langle p, v_{tt} + A(u^*)v \rangle_{L^2(0,1)}\, dt = 0.$$

Keeping in mind that

$$w(0) = w(1) = 0$$

and integrating by parts over $[0,1]$ we find

$$\int_0^1 w(x,0) \int_0^T p_x(x,t) y_x^*(x,t)\, dt dx = \int_0^1 v(x,0)\,(p_t(x,T) - p_t(x,0))\, dx \tag{3.5}$$

for all $w \in \text{Tan } U(u^*)$ and $(v,w) \in \text{Tan } M(y^*, u^*)$. It follows from (3.3) – (3.5) and the definition of $NU(u^*)$ (see (2.16)) that the function

$$\psi(x) = -\int_0^T p_x(x,t) y_x^*(x,t) dt, \ x \in [0,1] \tag{3.6}$$

is an element of $NU(u^*)$.

As in the case of $\varphi$ given by (2.24), we need $\psi < 0$ in $[0,1]$ (in order to use the proof of Theorem 2.2). We already know from (2.26) that $y_x^*(x,t) > 0$ in $Q$. Therefore we seek $p_x(x,t) > 0$ in $Q$.

Clearly, this can be accomplished if $p_t(x,T) < p_t(x,0)$ in $[0,T]$, i.e., if

$$y^*(x,0) < g_2(x), \ \text{ in } [0,1]. \tag{3.7}$$

Indeed, by the maximum principle

$$\max_{\overline{Q}} p(x,t) = \max\{p(x,t);\ \{x=0\} \cup \{x=1\} \cup \{t=T\}\}.$$

But $\max_{\overline{Q}} p(x,t)$ cannot be assumed on $\{x = 0\}$ as $\max_{\overline{Q}} p = p(0,t)$ would imply $\frac{\partial p}{\partial \nu_x}(0,t) = -p_x(0,t) > 0$ for $0 \le t \le T$ which is not the case (as $p_x(0,t) = 0$ by (3.2)). Moreover, $\max_{\overline{Q}} p(x,t)$ cannot occur on $\{t = T\}$ as $\max_{\overline{Q}} p(x,t) = p(x_0,T) = p(x_0,0)$ would imply

$$\frac{\partial p}{\partial \nu_t}(x_0,0) = -p_t(x_0,0) > 0, \quad \frac{\partial p}{\partial \nu_t}(x_0,T) = p_t(x_0,T) > 0 \tag{3.8}$$

respectively. Or, (3.8) is in conflict with (3.3) and (3.7). Therefore the only possibility is $\max_{\overline{Q}} p(x,t) = p(1,t)$ so

$$p(x,t) < 0 \text{ in } Q. \tag{3.9}$$

Set $q(x,t) = u^*(x)p_x(x,t)$. It follows from (3.1) – (3.3) and (3.9) that

$$q_{tt} + u^*(x)q_{xx} = 0 \text{ in } Q$$

$$q(0,t) = 0,\ q(1,t) \ge 0 \text{ in } (0,T)$$

$$q(x,0) = q(x,T),\ q_t(x,T) - q_t(x,0) = u^*(x)(y_x^*(x,0) - g_2'(x)) \text{ in } (0,1).$$

According to the minimum principle we can derive as above that $\min_{\overline{Q}} q(x,t) = q(0,t)$ or $\min_{\overline{Q}} q(x,t) = q(1,t)$, for some $t \in [0,T)$, provided that

$$y_x^*(x,0) > g_2'(x) \text{ in } [0,1]. \tag{3.10}$$

Therefore, $\min_Q q(x,t) > 0$, which implies $p_x(x,t) > 0$ in $Q$ as desired.

Now we are in position to state the main result of this section, i.e.

**Theorem 3.1** *Let $f$ satisfy $(H1)$ and $g_2 \in C^1([0,1])$, with*

$$\sup_{u \in U} y^u(x,0) < g_2(x),\ \inf_{u \in U} y_x^u(x,0) > g_2'(x), \quad x \in [0,1] \tag{3.11}$$

*where $y^u$ is the solution of (1.2) – (1.4) corresponding to $u \in U$. Then the optimal control $u^*$ of ($P_2$) is given by (2.30).*

*Proof.* Obviously (3.11) imply (3.7) and (3.10), therefore $p_x > 0$ in $Q$. Thus the function $\psi$ given by (3.6) is negative and it belongs to $NU(u^*)$. According to the proof of Theorem 2.2 it follows that $u^*$ is the one given by (2.30), which completes the proof. □

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