Optimal control of an elliptic equation
under periodic conditions

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Abstract

In this paper we find explicitly the optimal control for an elliptic equation with respect to each of the cost functional (1.1) or (1.7).

1 Introduction

In this paper we are concerned with the following two control problems

(P1) \[ \text{Minimize } L_1(y, u) = \int_0^T (y(0, t) - g_1(t))^2 \, dt \]

subject to

\[ y_{tt}(x, t) + (u(x)y_x(x, t))_x = f(x, t) \text{ in } Q = (0, 1) \times (0, T) \]
\[ y_x(0, t) = 0, \quad y(1, t) = 0 \text{ in } (0, T) \]
\[ y(x, 0) = y(x), \quad y_t(x, 0) = y_t(x, T) \text{ in } (0, 1) \]

where \( T \) is an arbitrary positive number and the set \( U \) of admissible controls is given by

\[ U = \left\{ u \in \text{Lip}(0, 1); |u'(x)| \leq \rho \text{ a.e. in } (0, 1),
0 < a \leq u(x) \leq b, \forall x \in [0, 1], \ u(0) = \alpha, \ u(1) = \beta \right\} \]

with \( a < \alpha < \beta < b, \rho \geq \alpha + \beta - 2a > 0 \) and \( g_1 \in C^2([0, T]) \).

Here \( f \in L^2(Q) \) guarantees the existence and uniqueness of an \( L^2(0, T; H) \) solution of (1.1)-(1.4), \( H = L^2(0, 1) \).

The problem (1.2)-(1.4) can be viewed as a stationary heat conductor model for a strip conductor \((0, 1) \times \mathbb{R}\) with periodic conditions in the variable \( t \), with zero temperature at \( x = 1 \) (see [2]). Our goal is to determine the optimal control \( u^* \) corresponding to the cost functionals of the form (1.1), (1.7). Other cost functionals can be treated in a similar way, but the determination of optimal control will be likely more complicated (or simply not possible). For example the following cost functional \( L(y, u) \) was considered in place of \( L_i(y, u), i = 1, 2 \) (see [2])

\[ L(y, u) = \int_0^1 y(x, 0) \, dx. \]

Note that problems (P1) and (P2) can be viewed as identification problems, i.e.,

How should we choose the coefficient \( u \) in the state equation (1.2) (under boundary condition (1.3)-(1.4)) to achieve some prescribed goal (e.g. to achieve the minimum of \( L_i \))?

The main result of this paper is that the function \( u^* \) given by (2.30) is the unique (solution) optimal control for both (P1) and (P2). This is done in Sections 2 and 3 (respectively).
2 Problem (P_1)

First let explain why the problem (1.2)-(1.4) is well-posed. For each \( u \in U \) set

\[
D(A(u)) = \{ H^2(0, 1); \ y_x(0) = 0, \ y(1) = 0 \}
\]

(2.1)

\[
(A(u)y)(x) = (u(x)y_x(x))_x, \ y \in D(A(u)),
\]

(2.2)

where \( H^4(0, 1) \) denotes the usual Sobolev spaces \((i = 1, 2)\).

Observe that \( y(1) = 0 \) yields

\[
|y(x)| \leq \int_0^1 |y_x(s)|ds \leq |y_x|_{L^2(0, 1)}, \ y \in D(A(u))
\]

(2.3)

so

\[
|y|_{L^2(0, 1)} \leq |y_x|_{L^2(0, 1)}, \ y \in D(A(u)).
\]

(2.4)

Integrating by parts is easy to check that \( A(u) : L^2(0, 1) \to L^2(0, 1) \) is strictly negative definite (strictly dissipative), namely

\[
\langle A(u)y, y \rangle_{L^2(0, 1)} = - \int_0^1 u(x)y_x^2(x)dx \leq -a|y_x|^2, \ \forall y \in D(A(u))
\]

(2.5)

with \( a > 0 \) as in (1.6) and \(|y| = |y|_{L^2(0, 1)}\). Moreover, \( \langle A(u)y, z \rangle_{L^2(0, 1)} = \langle y, A(u)z \rangle_{L^2(0, 1)} \) for all \( y, z \in D(A(u)) \) and therefore \( A(u) \) is selfadjoint.

Define the realization \( \tilde{A} \) of \( A(u) \) in \( L^2(0, T; H) \) by

\[
D(\tilde{A}) = \{ \tilde{y} \in L^2(0, T; H); \tilde{y}(t) \in D(A(u)) \text{ a.e. in } (0, T); \ t \to A(u)\tilde{y}(t) \in L^2(0, T; H) \}
\]

and

\[
(\tilde{A}\tilde{y})(t) = A(u)\tilde{y}(t), \ \text{a.e. in } (0, T) \text{ for } \tilde{y} \in D(\tilde{A})
\]

(2.6)

with \( \tilde{A} = \tilde{A}(u) \).

We note that \( \tilde{A} \) is also strictly dissipative and selfadjoint in \( L^2(0, T; H) \). Now let us define

\[
D(B) = \{ z \in H^2(0, T; H), \ z(0) = z(T), \ z_t(0) = z_t(T) \}
\]

(2.7)

and

\[
Bz = z_{tt}, \ z \in D(B).
\]

(2.8)

We have that \( B \) is dissipative

\[
\langle Bz, z \rangle_{L^2(0, T; H)} = - \int_0^T |z_t(s)|^2_H ds \leq 0, \ \forall z \in D(B),
\]

(2.9)

and is easy to check that \( B \) is maximal dissipative (see [1]).

On the other hand, the sum \( \tilde{A} + B \) is a selfadjoint strictly negative definite operator in \( L^2(0, T; H) \) so it is maximal dissipative onto and one-to-one. Therefore for every \( f \in L^2(0, T; H) \) there is a unique \( \tilde{y} \in D(\tilde{A}) \cap D(B) \) such that

\[
B\tilde{y} + \tilde{A}(u)\tilde{y} = f
\]

(2.10)

which means that \( \tilde{y} = \tilde{y}(x, t) \) is the unique solution to (1.2)-(1.4) in \( L^2(0, T; H) \). Summarizing, we have proved the following theorem.

**Theorem 2.1** For every \( u \in U \) and \( f \in L^2(Q) \), the problem (1.2)-(1.4) has a unique solution \( y^u = y(x, t) \in D(\tilde{A}) \cap D(B) \).
In order to define precisely the optimal control \( u^* \) for Problem (P1), introduce the set \( M \) of admissible pairs \((y, u)\).

\[
M = \{(y, u), y \in D(A(u)) \cap D(B), u \in U, (y, u) \ \text{related as in (2.10)}\}. \tag{2.11}
\]

In other words, \((y, u) \in M\) means: \(y\) is the \(L^2(Q)\) solution of (1.2)-(1.4) corresponding to \(u \in U\).

By definition, an optimal pair \((y^*, u^*) \in M\) satisfies

\[
L_1(y^*, u^*) = \inf \{L_1(y, u), (y, u) \in M\} \tag{2.12}
\]

with \(L_1\) as in (1.1). Here \(u^*\) is an optimal control and \(y^* = y_{u^*}\) is the corresponding optimal arc.

Now let us rewrite (1.2)-(1.4) as

\[
y_{tt} + A(u)y = f \ \text{in} \ L^2(0, T; H) = L^2(Q) \tag{2.13}
\]

\[
y(0) = y(T), \ y_t(0) = y_t(T). \tag{2.14}
\]

and introduce the tangential (contingent) cone to \(M\) at \((y, u) \in M\)

\[
\text{Tan} \ M(y, u) = \{(v, w), \ v \in L^2(Q), \ w \in \text{Tan} \ U(u); \ v_{tt} + A(u)v + A(u)y = 0 \ \text{in} \ L^2(Q)
\]

\[
v(0) = v(T), \ u_t(0) = u_t(T)\}. \tag{2.15}
\]

The normal cone \(NU(u)\) to \(U\) at \(u \in U\) is defined by

\[
NU(u) = \{w^* \in C^*[0, 1]; \ w^*(v) \leq 0, \ \forall v \in \text{Tan} \ U(u)\}. \tag{2.16}
\]

In view of the convexity of \(U\) it follows that

\[
NU(u) = \{w^* \in C^*[0, 1]; \ w^*(z - u) \leq 0, \ \forall z \in U\}. \tag{2.17}
\]

The existence of an optimal pair \((y^*, u^*)\) follows by standard arguments and it is not our goal here. It is our purpose here to determine explicitly the optimal control \(u^*\).

Recall that if

\[
G(F) = \inf_{y \in M} G(y)
\]

with \(G : M \to \mathbb{R}\) Fréchet differentiable, then necessarily

\[
\partial G(F)(Y) \geq 0 \ \forall Y \in \text{Tan} \ M(F), \tag{2.18}
\]

where \(\partial G(F)\) stands for the Fréchet derivative of \(G\) at \(F \in M\).

Applying the optimum principle given by (2.18), it follows that (in our case \(G(Y) = L_1(y, u)\))

\[
\int_0^T (y^*(0, t) - g(t))v(0, t)dt \geq 0 \ \forall (v, w) \in \text{Tan} \ M(y^*, u^*). \tag{2.19}
\]

In order to take advantage of (2.19) (which is a necessary condition for the optimality of \((y^*, u^*)\)), let consider the adjoint (dual) problem

\[
p_{tt} + ((u^*(x))p_x(x, t))_x = 0 \ \text{in} \ Q \tag{2.20}
\]

\[
p_x(0, t) = y^*(0, t) - g_1(t), \ p(1, t) = 0 \ \text{in} \ [0, T] \tag{2.21}
\]

\[
p(x, 0) = p(x, T), \ p_t(x, 0) = p_t(x, T) \ \text{in} \ (0, 1). \tag{2.22}
\]

The existence and uniqueness of solution to (2.20)–(2.22) is discussed by the end of this section. Multiplying (2.20) by \(v\) and integrating over \([0, T] \times [0, 1]\) we derive successively

\[
-\int_0^T u^*(0)p_x(0, t)v(0, t)dt + \int_0^T \langle p, v_{tt} + A(u^*)v \rangle_{L^2(0, 1)}dt = 0
\]
for all $w \in \text{Tan } U(u^*)$, i.e.,
\[
\int_0^1 w(x) \int_0^T y^*_w(x,t) p_w(x,t) \, dt \, dx = \int_0^T u^*(0) \left( y^*(0, t) - g(t) \right) v(0,t) \, dt
\]  
(2.23)
for all $w \in \text{Tan } U(u^*)$. A simple combination of (2.16), (2.19) and (2.23) shows that the function
\[
\varphi(x) = - \int_0^T y^*_w(x,t) p_w(x,t) \, dt, \quad x \in (0,1)
\]  
(2.24)
is an element of $NU(u^*)$.

Summarizing, we have proved the following result.

**Lemma 2.1** Let $f \in C^1(Q)$ and let $(y^*, u^*)$ be an optimal pair for $(P_1)$. Then $\varphi \in NU(u^*)$.

Next we prove that $\varphi$ holds a constant sign. Precisely, we have the following lemma.

**Lemma 2.2** Let $f$ satisfy the sign condition in (H1) and $g_1 \in C^2([0,T])$, with $g_1(0) = g_1(T)$ and
\[
g_1(t) < \inf_{u \in U} y^u(0, t), \quad t \in [0,T],
\]  
(2.25)
where $y^u$ is the solution of (1.2) – (1.4) corresponding to $u \in U$. Then $\varphi$ given by (2.24) satisfies $\varphi(x) < 0$ for $x \in (0,1)$.

**Proof.** First we prove that the solution $y$ to (2.13)-(2.14) satisfies the sign condition
\[
y(x,t) < 0, \quad y_x(x,t) > 0 \quad \text{in } Q.
\]  
(2.26)
Since $f > 0$ in $Q$, by the maximum principle for elliptic equations we get
\[
\max_Q y(x,t) = \max_{\partial Q} y(x,t)
\]
while
\[
\max_{\partial Q} y(x,t) = \max_{0 \leq t \leq T} y(1,t) = 0
\]
so $y(x,t) < 0$ in $Q$. Indeed, if we assume that $\max_{\partial Q} y(x,t) = y(0,t_0)$, with $0 < t_0 < T$, then
\[
\frac{\partial y}{\partial n}(0,t_0) > 0 \quad \text{which contradicts (1.3).}
\]
In the same manner, if we assume by contradiction that
\[
\max_{\partial Q} y(x,t) = y(x_0,0) = y(x_0,T), \quad 0 < x_0 < 1
\]
we get in conflict with (1.4). In order to prove the second part of (2.26), set
\[
z(x,t) = u(x)y_x(x,t) \quad \text{in } Q
\]
Since $y(x,t) < 0$ in $Q$, by (1.2)-(1.4) and (H1) we get
\[
z_{tt} + u(x)z_{xx} = u(x)f_x(x,t) < 0 \quad \text{in } Q
\]
\[
z(0,t) = 0; \quad z(1,t) \geq 0 \quad \text{in } (0,T)
\]
\[
z(x,0) = z(x,T); \quad z_t(x,0) = z_t(x,T) \quad \text{in } (0,1)
\]
so again by the minimum principle
\[
\min_Q z(x,t) = \min_{\partial Q} z(x,t).
\]
Using the same argument as above, $z$ cannot assume its minimum on the subboundary $\{(x,0), (x,T), 0 < x < 1\}$ of $\partial Q$, which implies $z(x,t) = u(x)y_x(x,t) > 0$ in $Q$. 

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Next we seek \( p_x(x, t) > 0 \) in \( Q \), where \( p \) is the solution of (2.20)–(2.22). Clearly this can be accomplished if \( p_x(0, t) > 0 \) in \([0,T]\), i.e.,

\[
g_1(t) < y^*(0, t) \quad \text{in} \quad [0,T]
\]  

(2.27)

which is obviously implied by (2.25).

Indeed, by the maximum principle

\[
\max_{\partial Q} p(x, t) = \max_{\partial Q} p(x, t).
\]

In view of (2.27), \( \max \) over \( \partial Q \) cannot be assumed on \( \{0, t \}, 0 < t < T \) as \( \max \) over \( \partial Q \) would imply \( p_x(0, t) \leq 0 \) for \( 0 < t < T \) (which is in conflict with \( p_x(0, t) > 0 \) required by (2.27)). Moreover, \( \max \) over \( \partial Q \) cannot occur on \( \{t = T\} \) as \( \max \) over \( \partial Q \) would contradict (2.22). Therefore \( \max \) over \( \partial Q \) is not a possibility. This completes the proof of Lemma 2.2.

\[\square\]

The next result gives the structure of those \( L^1(0, 1) \)-functions which belong to \( NU(u) \), with \( U \) regarded as a subset of \( L^\infty(0, 1) \). So we now view \( NU(u) \) as

\[
NU(u) = \{ w \in (L^\infty(0, 1))^* \mid w(z, u) \leq 0, \forall z \in U \}.
\]

**Lemma 2.3** A function \( w \in L^1(0, 1) \) belongs to the normal cone \( NU(u) \) to \( U \) at \( u \in U \) if and only if

\[
w = -\theta' + \eta \quad \text{in} \quad \mathcal{D}'(0, 1)
\]  

(2.29)

with \( \theta \in L^1(0, 1), \eta \in L^1(0, 1) \) satisfying

\[
\begin{align*}
\theta(x) &= 0 \quad \text{a.e. in} \quad \{ x \in (0, 1); |u'(x)| < \rho \} \\
\theta(x) &= \lambda(x)u'(x) \quad \text{a.e. in} \quad \{ x \in (0, 1); |u'(x)| = \rho \}
\end{align*}
\]

where \( \lambda \in L^1(0, 1), \lambda(x) \geq 0 \quad \text{a.e. in} \quad (0, 1) \)

\[
\begin{align*}
\eta(x) &= 0 \quad \text{a.e. in} \quad \{ x \in (0, 1); a < u(x) < b \} \\
\eta(x) &\leq 0 \quad \text{a.e. in} \quad \{ x \in (0, 1); u(x) = a \} \\
\eta(x) &\geq 0 \quad \text{a.e. in} \quad \{ x \in (0, 1); u(x) = b \}.
\end{align*}
\]

The proof of Lemma 2.3 can be found in [5].

We are now in position to state the main result of this paper, i.e.,
Theorem 2.2. Let \( f \in C^1(\Omega) \) satisfy the sign condition in \((H_1)\) and \( g_1 \in C^2([0, T]) \), with \( g_1(0) = g_1(T) \) satisfy (2.25). Then the optimal control \( u^* \) of \((P_1)\) is given by

\[
u^*(x) = \begin{cases} 
\alpha - px, & \text{for } x \in [0, (\alpha - a)p^{-1}] \\
a, & \text{for } x \in [(\alpha - a)p^{-1}, (a + \rho - \beta)(a - \rho^{-1})] \\
\beta + \rho(x - 1), & \text{for } x \in [(a + \rho - \beta)(a - \rho^{-1}), 1]
\end{cases}
\]  

(2.30)

with \( a, \alpha, \beta \) as in (1.6).

Figure 1. The optimal control \( u^* \)

Figure 2. The optimal state \( y^* \)

(We have set \( a = 2, \alpha = 3, \beta = 4, b = 5, \rho = 3.8, T = 15, \) \( f = (2 - x)(2 + \sin t) \).)

Proof. By hypothesis, \( u^* \in U \) so \( u^*(0) = \alpha > a \). Therefore the open set \( M_a = \{x \in [0, 1]; u^*(x) > a \} \) is nonempty.

By Lemma 2.2 we have that \( \varphi(x) < 0 \) in \([0, 1] \). Or, it was proved in [5] that if \( NU(u^*) \) contains a negative function, then \( u^* \) is convex. Indeed, in view of Lemmas 2.1, 2.2 and 2.3, the function \( \varphi \) given by (2.24) can be written as \( \eta - \theta^* \), so \( \theta^* + \eta = 0 \) in \( M_a \) with \( \eta \geq 0 \) in \( M_a \). Thus \( \theta^* < 0 \) in \( M_a \), which implies \( |u^*(x)| = \rho \) in \( M_a \). Suppose now that \( u^* \) is not convex on \([0, 1] \). This means that there is an interval \([0, x_0] \subset M_a \) such that \( u^*(x) = \alpha + \rho x_0 - \rho(x - x_0), x \in [x_0, x_0 + \varepsilon], \) with \( \varepsilon > 0, \varepsilon \) sufficiently small.

Choose now \( v \) such that

\[
v(x) = \begin{cases} 
\alpha + u^*(x_0 + \varepsilon) - \alpha, & \text{for } 0 < x \leq x_0 + \varepsilon \\
u^*(x), & \text{for } x_0 + \varepsilon < x \leq 1
\end{cases}
\]

Obviously \( v(x) < u^*(x) \) for \( 0 < x < x_0 + \varepsilon \) and so \( u^* - v \in H_0^1[0, 1] \), with \( u^* - v > 0 \). Since \( \theta^* < 0 \) in \( M_a \), we must have

\[
\int_{x_0 + \varepsilon}^{x_0 + \varepsilon} -\theta'(u^* - v)dx < 0.
\]

(2.31)

But \( \theta(x) = \lambda(x)u^*(x) \) in \( M_a \), so

\[
\int_{x_0 + \varepsilon}^{x_0 + \varepsilon} -\theta'(u^*(x) - v(x))dx = \int_{x_0 + \varepsilon}^{x_0 + \varepsilon} \lambda(x)u^*(x)(u^*(x) - v'(x))dx
\]

\[
= \int_{x_0 + \varepsilon}^{x_0 + \varepsilon} \lambda(x)(\rho^2 - u^*(x)v'(x))dx \geq 0
\]

(2.32)

as \( v'(x) = \frac{\rho(x_0 - \varepsilon)}{\rho(x_0 - \varepsilon)} < \rho \). The contradiction between (2.31) and (2.32) shows that \( u^* \) must be convex in \( M_a \). Similarly one proves that \( u^* \) is convex on the whole interval \([0, 1] \). It follows that \( u^* \) must have the form as indicated in formula (2.30). This completes the proof.

□

Now let us motivate the existence of the solution \( p = p(x, t) \) of (2.20) - (2.22) under the hypotheses

\[
g_1 \in C^2([0, 1]), \ g_1(0) = g_1(T), \ g_1'(0) = g_1'(T).
\]
Set
\[ w(x, t) = p(x, t) + (g_1(t) - y^*(0, t))(x - 1) \in Q. \]
Therefore \( p \) is the solution to (2.20) – (2.22) if and only if \( w \) is the solution to the problem
\[
\begin{align*}
& w_{tt} + (u^*w_x)_x = h(x, t) \text{ in } Q \\
& w_x(0, t) = 0, \ w(1, t) = 0 \\
& w(x, 0) = w(x, T), \ w_t(x, 0) = w_t(x, T)
\end{align*}
\]
where
\[ h(x, t) = (g_1^*(t) - y^*_t(0, t))(x - 1) + (g_1(t) - y^*(0, t))u^*_x(x) \]

**Remark 2.1** The condition (2.25) makes sense, since the function \((t, u) \to y^*(0, t)\) is continuous on \([0, T] \times U\) and \(U\) is obviously compact in \(C([0, 1])\).

### 3 Problem \((P_2)\)

With the same notations as above, in this case the analogous to (2.20) – (2.22) (i.e., the dual problem) is
\[
\begin{align*}
& p_{tt} + (u^*_p p_x)_x = 0 \text{ in } Q \tag{3.1} \\
& p_x(0, t) = 0, \ p(1, t) = 0 \text{ in } [0, T] \tag{3.2} \\
& p(x, 0) = p(x, T), \ p_t(x, 0) = y^*(x, 0) - g_2(x) \text{ in } (0, 1). \tag{3.3}
\end{align*}
\]
In this case, the necessary condition for the optimality of \((y^*, u^*)\) (i.e., the analogous of (2.19)) is given by
\[
\int_0^1 (y^*(x, 0) - g_2(x))v(x, 0)dx \geq 0 \tag{3.4}
\]
for all \((v, w) \in \text{Tan } M(y^*, u^*)\) as in (2.15).

Multiplying (3.1) by \(v\) in the sense of \(L^2(0, 1)\) and integrating over \([0, T]\), we derive successively
\[
\int_0^1 v(x, 0) (p_t(x, T) - p_t(x, 0)) dx + \int_0^T \langle p, v_{tt} + A(u^*)v \rangle_{L^2(0, 1)} dt = 0.
\]
Keeping in mind that
\[ w(0) = w(1) = 0 \]
and integrating by parts over \([0, 1]\) we find
\[
\int_0^1 w(x, 0) \int_0^T p_x(x, t) y^*_x(x, t) dt dx = \int_0^1 v(x, 0) (p_t(x, T) - p_t(x, 0)) dx \tag{3.5}
\]
for all \(w \in \text{Tan } U(u^*)\) and \((v, w) \in \text{Tan } M(y^*, u^*)\). It follows from (3.3) – (3.5) and the definition of \(NU(u^*)\) (see (2.16)) that the function
\[
\psi(x) = -\int_0^T p_x(x, t) y^*_x(x, t) dt, \ x \in [0, 1] \tag{3.6}
\]
is an element of \(NU(u^*)\).

As in the case of \(\varphi\) given by (2.24), we need \(\psi < 0\) in \([0, 1]\) (in order to use the proof of Theorem 2.2). We already know from (2.26) that \(y^*_x(x, t) > 0\) in \(Q\). Therefore we seek \(p_x(x, t) > 0\) in \(Q\).

Clearly, this can be accomplished if \(p_t(x, T) < p_t(x, 0)\) in \([0, T]\), i.e., if
\[
y^*(x, 0) < g_2(x), \text{ in } [0, 1]. \tag{3.7}
\]
Indeed, by the maximum principle
\[
\max_{Q} p(x,t) = \max \{ p(x,t): \{ x = 0 \} \cup \{ x = 1 \} \cup \{ t = T \} \}.
\]
But \( \max_{Q} p(x,t) \) cannot be assumed on \( \{ x = 0 \} \) as \( \max_{Q} p = p(0,t) \) would imply \( \frac{\partial p}{\partial x}(0,t) = -p_x(0,t) > 0 \) for \( 0 \leq t \leq T \) which is not the case (as \( p_x(0,t) = 0 \) by (3.2)). Moreover, \( \max_{Q} p(x,t) \) cannot occur on \( \{ t = T \} \) as \( \max_{Q} p(x,t) = p(x_0,T) = p(x,0,0) \) would imply
\[
\frac{\partial p}{\partial x}(x_0,0) = -p_x(x_0,0) > 0, \quad \frac{\partial p}{\partial t}(x_0,T) = p_t(x_0,0) > 0
\]
respectively. Or, (3.8) is in conflict with (3.3) and (3.7). Therefore the only possibility is \( \max_{Q} p(x,t) = p(1,t) \) so
\[
p(x,t) < 0 \text{ in } Q.
\]
Set \( q(x,t) = u^*(x)p_x(x,t) \). It follows from (3.1) – (3.3) and (3.9) that
\[
q_t + u^*(x)q_{xx} = 0 \text{ in } Q
\]
\[
q(0,t) = 0, \quad q(1,t) \geq 0 \text{ in } (0,T)
\]
\[
q(x,0) = q(x,T), \quad q_t(x,T) - q_t(x,0) = u^*(x)(g_x^*(x,0) - g_x^2(x)) \text{ in } (0,1).
\]
According to the minimum principle we can derive as above that \( \min_{Q} q(x,t) = q(0,t) \) or \( \min_{Q} q(x,t) = q(1,t) \), for some \( t \in [0,T) \), provided that
\[
y_x^*(x,0) > g_x^2(x) \text{ in } [0,1].
\]
Therefore, \( \min_{Q} q(x,t) > 0 \), which implies \( p_x(x,t) > 0 \) in \( Q \) as desired.

Now we are in position to state the main result of this section, i.e.

**Theorem 3.1** Let \( f \) satisfy (H1) and \( g_2 \in C^1([0,1]) \), with
\[
\sup_{u \in U} y^u(x,0) < g_2(x), \quad \inf_{u \in U} y^u(x,0) > g_2^*(x), \quad x \in [0,1]
\]
where \( y^u \) is the solution of (1.2) – (1.4) corresponding to \( u \in U \). Then the optimal control \( u^* \) of (P_2) is given by (2.30).

**Proof.** Obviously (3.11) imply (3.7) and (3.10), therefore \( p_x > 0 \) in \( Q \). Thus the function \( \psi \) given by (3.6) is negative and it belongs to \( NU(u^*) \). According to the proof of Theorem 2.2 it follows that \( u^* \) is the one given by (2.30), which completes the proof. \( \square \)

**References**


