# INTERNAL OPTIMAL CONTROL OF THE PERIODIC EULER-BERNOULLI EQUATION

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**ABSTRACT:** This paper is concerned with the existence and the maximum principle for optimal control problems governed by the periodic Euler-Bernoulli equation on $(0,\pi)\times(0,T)$. The case of internal controllers supported on $\omega\subset(0,\pi)$ is examined.

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## 1. INTRODUCTION

We study here the optimal control problem

$$\text{minimize}\int_Q (g(y(x,t)+y_0(x,t))+h(u(x,t)))\,dxdt \tag{1.1}$$

subject to $u\in L^2(Q),\ Q=(0,\pi)\times(0,T)$ and

$$\begin{aligned}
&y_{tt}(x,t)+y_{xxxx}(x,t)=m(x)u(x,t)+f(x,t), && x\in(0,\pi),\ t\in\mathbb{R},\\
&y(x,t)=y(x,t+T), && x\in(0,\pi),\ t\in\mathbb{R},\\
&y(0,t)=y_x(0,t)=0,\ y_{xx}(\pi,t)=y_{xxx}(\pi,t)=0, && t\in\mathbb{R},
\end{aligned}\tag{1.2}$$

where $m\in L^\infty(0,\pi),\ m(x)\ge\rho>0$ on $\omega\subset(0,\pi),\ m=0$ in rest. Here $g:\mathbb{R}\to\overline{\mathbb{R}}$, $h:\mathbb{R}\to\overline{\mathbb{R}}=]-\infty,\ +\infty]$ are lower semicontinuous convex functions, $\omega$ is an open subset of $(0,\pi)$.

By solution to state system (1.2) we mean a weak solution, i.e., $y\in L^2(Q)$ and

$$\int_Q y(x,t)(\varphi_{tt}(x,t)+\varphi_{xxxx}(x,t))\,dxdt=\int_Q(m(x)u(x,t)+f(x,t))\varphi(x,t)\,dxdt, \tag{1.3}$$

for all

$$\begin{aligned}
\varphi\in X=\{\ &\varphi\in C^4([0,\pi]\times[0,T]);\ \varphi(x,0)=\varphi(x,T)=0,\ \varphi_t(x,T)=\varphi_t(x,0),\\
&\varphi(0,t)=\varphi_x(0,t)=0,\ \varphi_{xx}(\pi,t)=\varphi_{xxx}(\pi,t)=0,\ \ \forall(x,t)\in(0,\pi)\times(0,T)\}\,.
\end{aligned}$$

Let $\mathcal{A} : L^2(Q) \to L^2(Q)$ be the Euler-Bernoulli operator (1.2), i.e.,

$$\mathcal{A}y = f, \quad \mathrm{for}[y, f] \in D(\mathcal{A}) \times R(\mathcal{A}) \mathrm{if, and only if,} \tag{1.4}$$

$$\int_Q (\varphi_{tt} + \varphi_{xxxx}) y \, dxdt = \int_Q f\varphi \, dxdt, \qquad \forall \varphi \in X. \tag{1.5}$$

In terms of $\mathcal{A}$ a weak solution to (1.2) is a solution to operator equation $\mathcal{A}y = mu + f$. Equation (1.2) models the motion of the Euler-Bernoulli beam fixed at $x = 0$ and without torque and lateral force at free-end point $x = \pi$ (see Russel [8] and Krabs [5]).
We shall study some properties of $\mathcal{A}$ under the following assumption

$$\inf \left\{ \left| \left(k + \frac{1}{2}\right)^4 - \left(\frac{2\pi j}{T}\right)^2 \right| ; k, j \in \mathbb{Z} \right\} > 0. \tag{H}$$

We note that for $T = 2\pi$, (H) is fulfilled with the infimum=1/16, while for $T = 8\pi$ the hypothesis does not hold.

**Proposition 1.1.** *Assume that hypothesis* (H) *hold. Then the operator $\mathcal{A}$ is self-adjoint, $R(\mathcal{A})$(the range of $\mathcal{A}$) is closed, $N(\mathcal{A})$(the null space of $\mathcal{A}$) is finite-dimensional and $\mathcal{A}^{-1} \in L(R(\mathcal{A}), R(\mathcal{A}))$. Moreover, $\mathcal{A}^{-1}$ is compact on $R(\mathcal{A})$,*

$$\|\mathcal{A}^{-1} f\|_{H^{2,1}(Q)} \le C \|f\|_{L^2(Q)}, \ \forall f \in R(\mathcal{A}), \tag{1.6}$$

*and, in addition,*

$$\|y\|_{L^2(Q)} \le C \|my\|_{L^2(Q)}, \ \forall y \in N(\mathcal{A}). \tag{1.7}$$

**Proof.** We seek for solutions to (1.2) of the form

$$y(x, t) = \sum_{j,k \in \mathbb{Z}} y_{jk} \psi_j(t) \varphi_k(x), \tag{1.8}$$

where $\psi_j(t) = \sqrt{\frac{2}{T}} \sin \omega_j t$, $\omega_j = 2\pi j / T$, and $\varphi_k$, $\lambda_k^4$ are the eigenfunctions, respectively the eigenvalues of $\frac{\partial^4 y}{\partial x^4}$ with the boundary conditions (1.2). We have $\varphi_k(x) = c_{1k} e^{\lambda_k x} + c_{2k} e^{-\lambda_k x} + c_{3k} \cos \lambda_k x + c_{4k} \sin \lambda_k x$.

Imposing that the eigenfunctions to be nonconstants, it follows that the eigenvalues $\{\lambda_k;\ k \in \mathbf{Z}\}$ are pairs $\{(\lambda_k^1,\ \lambda_k^2);\ k \in \mathbf{Z}\}$ from intervals $]2k + 2^{-1}, 2k + 3/2[$. Moreover, we have $(\lambda_k^1 - (2k + 2^{-1})) \searrow 0$ and $(\lambda_k^2 - (2k + 3/2)) \nearrow 0$ when $|k| \to \infty$.
If

$$\mathcal{A}y = f, \qquad \mathrm{for}[y, f] \in R(\mathcal{A}) \times R(\mathcal{A}),$$

then

$$y(x,t)=\sum_{j,k\in I}\frac{f_{jk}}{\lambda_k^4-\omega_j^2}\psi_j(t)\varphi_k(x), \tag{1.10}$$

where $I=\{j,k\in\mathbf{Z};\ \omega_j^2\neq\lambda_k^4\}$ and $f_{jk}$ are the Fourier coefficients of $f$.
By (H) it follows that $\mathcal{A}^{-1}\in L\left(R\left(\mathcal{A}\right),R\left(\mathcal{A}\right)\right)$, i.e.,

$$\|y\|_{L^2(Q)}\le C\|f\|_{L^2(Q)},\qquad \forall[y,f]\in R(\mathcal{A})\times R(\mathcal{A}), \tag{1.11}$$

where $C=\|\mathcal{A}^{-1}\|=\left(\inf_{k,j\in\mathbb{Z}}\left\{\left|\lambda_k^4-\omega_j^2\right|\right\}\right)^{-1}=\left(\inf_{k,j\in\mathbb{Z}}\left\{\left|\left(k+\frac{1}{2}\right)^4-\omega_j^2\right|\right\}\right)^{-1}$.
As regards (1.6), we have

$$\|y_{xx}\|_{L^2(Q)}^2+\|y_t\|_{L^2(Q)}^2=\int_Q\left(\sum_{j,k\in I}\frac{f_{jk}}{\lambda_k^4-\omega_j^2}\psi_j(\varphi_k)_{xx}\right)^2+\left(\sum_{j,k\in I}\frac{f_{jk}}{\lambda_k^4-\omega_j^2}(\psi_j)_t\varphi_k\right)^2dxdt$$

$$=\sum_{\lambda_k^2>|\omega_j|}\frac{\lambda_k^4+\omega_j^2}{\left(\lambda_k^4-\omega_j^2\right)^2}f_{jk}^2+\sum_{\lambda_k^2<|\omega_j|}\frac{\lambda_k^4+\omega_j^2}{\left(\lambda_k^4-\omega_j^2\right)^2}f_{jk}^2,\quad \forall[y,f]\in R(\mathcal{A})\times R(\mathcal{A}).$$

If $\lambda_k^2>|\omega_j|$, $\lambda_k^4+\omega_j^2<2\lambda_k^4$, while $\left(\lambda_k^4-\omega_j^2\right)^2=\left(\lambda_k^2-\omega_j\right)^2\left(\lambda_k^2+\omega_j\right)^2\ge\lambda_k^4$. Thus

$$\sum_{\lambda_k^2>|\omega_j|}\frac{\lambda_k^4+\omega_j^2}{\left(\lambda_k^4-\omega_j^2\right)^2}f_{jk}^2\le 2\sum_{\lambda_k^2>|\omega_j|}|f_{jk}|^2$$

Similarly, $\lambda_k^2>|\omega_j|$ implies $\lambda_k^4-\omega_j^2\ge\omega_j^2$. Consequently we get $\|y\|_{H^{2,1}(Q)}^2\le C\|f\|_{L^2(Q)}^2$, i.e., R$(\mathcal{A})$ is closed.

Now, if $y\in N(\mathcal{A})$, then

$$y(x,t)=\sum_{j,k\in J}y_{jk}\psi_j(t)\varphi_k(x), \tag{1.12}$$

where $J=\{j,k\in\mathbb{Z};\ \omega_j^2=\lambda_k^4\}$. Suppose that card$J=\aleph_0$. Therefore there are subsequences denoted again $k,j\in\mathbb{Z}$ such that $\omega_j^2=\lambda_k^4$. Since

$$\left|\left(2k+\frac{1}{2}\right)^4-\omega_j^2\right|\le\left|\left(\lambda_k^1\right)^4-\left(2k+\frac{1}{2}\right)^4\right|+\left|\left(\lambda_k^1\right)^4-\omega_j^2\right|\to 0$$

contradicts (H), we deduce that $N(\mathcal{A})$ is finite-dimensional.

Suppose that (1.7) does not hold. Hence there is $\{y^n\}\subset N(\mathcal{A})$ such that

$$\|y^n\|_{L^2(Q)}=1,\quad \forall n\in\mathbb{N}, \tag{1.13}$$

$$\|my^n\|_{L^2(Q)} \longrightarrow 0, \quad \text{when} n \to \infty. \tag{1.14}$$

Therefore

$$\|y^n\|_{L^2(Q)} = \sum_{j\in J} (y^n_{jk})^2 = 1 \tag{1.15}$$

and

$$\rho^2 \sum_{j\in J} (y^n_{jk})^2 \int_\omega \varphi_k^2(x)\,dx \le \sum_{j\in J} (y^n_{jk})^2 \int_\omega m^2(x)\varphi_k^2(x)\,dx \to 0. \tag{1.16}$$

Since $N(\mathcal{A})$ is finite-dimensional, $\int_\omega \varphi_k^2(x)\,dx$ is uniformly bounded from below by a positive constant, and so by (1.15)-(1.16) we are led to a contradiction.

**Remark 1.1.** If in (1.2) we take as boundary conditions

$$y(0,t) = y_x(0,t) = 0,\ y(\pi,t) = y_x(\pi,t) = 0, \qquad t \in \mathbb{R},$$

we see that the eigenvalues $\{\lambda_k;\ k \in \mathbb{Z}\}$ satisfies $((\lambda_k^1 - (2k - \frac{1}{2})) \searrow 0, (\lambda_k^2 - (2k + \frac{1}{2})) \nearrow 0$ and Proposition 1.1 still hold.

The closed range theorem implies that $L^2(Q) = R(\mathcal{A}) \oplus N(\mathcal{A})$ and the solutions to (1.4) are in the form $y^1 + N(\mathcal{A})$, with $y^1 \in R(\mathcal{A})$ unique. Note that the spectrum of $\mathcal{A}$ is discrete and we can boundedly invert it on the orthogonal complement of its null space (as in the case of wave equation, see Rabinowitz [8]).

In Section 2, the existence of optimal controllers and the maximum principle for (1.1) are obtained. Section 3 is devoted to the existence for the Hamiltonian system associated to (1.1). An important case is that where $m$ is the characteristic function of an open set $\omega$. This corresponds to the situation of the Euler-Bernoulli equation with internal controller supported on $\omega \subset (0,\pi)$. The controllability and stabilizability problem (with initial data) of Euler-Bernoulli beam were approached in Littman and Markus [6], [7]. The boundary control problem, with initial data and the control acting as a lateral force was treated in Russel [9].

## 2. EXISTENCE AND THE MAXIMUM PRINCIPLE

Here, we assume that
(i) $g : \mathbb{R} \to \overline{\mathbb{R}},\ h : \mathbb{R} \to \overline{\mathbb{R}}$ are lower semicontinuous, convex function and

$$g(y) \ge \alpha|y|^2 + \beta, \quad \forall y \in \mathbb{R}, \tag{2.1}$$

$$h(u) \ge \gamma|u|^2 + \delta, \quad \forall u \in \mathbb{R}, \tag{2.2}$$

where $\alpha,\ \gamma > 0$ and $\beta,\ \delta \in \mathbb{R}$.

**Theorem 2.1.** *Assume that hypothesis* (i) *hold and there is at least one admissible pair* $(y,u)$. *Then, problem* (1.1) *has at least one solution* $(y^*,u^*) \in L^2(Q)\times L^2(Q)$.

The argument is standard (see Barbu [1] and Barbu and Precupanu [4]), but we outline it for the reader's convenience. Let $(y_n,u_n)\in L^2(Q)\times L^2(Q)$ be such that $\mathcal{A}y_n=mu_n+f$ and

$$\inf(1.1)=d\le\int_Q(g(y_n+y_0)+h(u_n))\,dxdt\le d+1/n. \tag{2.3}$$

By (2.1), (2.2) we see that

$$\|y_n\|_{L^2(Q)}+\|u_n\|_{L^2(Q)}\le C. \tag{2.4}$$

Then, on a subsequence again denoted $n$, we have

$$y_n\to y^*,\quad u_n\to u^*\quad \text{weaklyin}L^2(Q). \tag{2.5}$$

Recalling that by Proposition 1.1 $R(\mathcal{A})$ is closed, we infer that $\mathcal{A}y^*=mu^*+f$. Since the convex integrand is weakly lower semicontinuous, we get

$$d=\int_Q(g(y^*+y_0)+h(u^*))\,dxdt, \tag{2.6}$$

i.e., $(y^*,u^*)$ is optimal in problem (1.1).

In order to get the maximum principle for problem (1.1) we shall use the following assumptions
(j) the function $g:\mathbb{R}\to\mathbb{R}$ is convex and continuous.
(jj) $h:\mathbb{R}\to\overline{\mathbb{R}}$ is convex, lower semicontinuous and

$$h(u)\le a|u|^2+b,\quad \forall u\in\mathbb{R}, \tag{2.7}$$

where $a>0,\ b\in\mathbb{R}$.

**Theorem 2.2.** *In addition to hypothesis* (H) *assume that* (j)-(jj) *hold. Then, the pair* $(y^*,u^*)\in L^2(Q)\times L^2(Q)$ *is optimal in problem (1.1) if, and only if, there are* $p\in L^2(Q)$ *and* $w\in L^2(Q)$ *such that*

$$p_{tt}+p_{xxxx}=-w\quad 0,\pi)\times(0,T)$$
$$p(0,t)=p_x(0,t)=0,\ p_{xx}(\pi,t)=p_{xxx}(\pi,t)=0,\quad \forall t\in\mathbb{R} \tag{2.8}$$
$$p(x,0)=p(x,T),\ p_t(x,0)=p_t(x,T),\quad \forall x\in(0,\pi),$$

$$w(x,t)\in\partial g(y^*(x,t)+y_0(x,t)),\quad \text{a.e. } (x,t)\in Q, \tag{2.9}$$

$$u^*(x,t)\in\partial h^*(m(x)p(x,t)),\quad \text{a.e. } (x,t)\in Q. \tag{2.10}$$

Here, $\partial g : \mathbb{R} \to 2^{\mathbb{R}}$, $\partial h^* : \mathbb{R} \to 2^{\mathbb{R}}$ are the subdifferentials of $g$ and $h^*$, respectively, $h^*$ is the conjugate function of $h$.
The solution $p \in L^2(Q)$ of (2.8) should be considered in the weak sense, i.e.,

$$\mathcal{A}p = -w.$$

**Proof.** It is readily seen that (2.8)-(2.10) are sufficient for optimality. To prove necessity, we fix an optimal pair $(y^*, u^*)$ and consider the approximating control problem

$$\min \left\{ \int_Q (g_\varepsilon(y + y_0) + h_\varepsilon(u))\, dx\, dt + 2^{-1} \int_Q (|y^* - y|^2 + |u^* - u|^2)\, dx dt; \right. \\ \left. (y, u) \in L^2(Q) \times L^2(Q),\ \mathcal{A}y = mu + f \right\}, \tag{2.11}$$

where $g_\varepsilon \in C^1(\mathbb{R})$ is the convex regularization on $g$, i.e.,

$$g_\varepsilon(r) = \inf \left\{ \frac{|r - s|^2}{2\varepsilon} + g(s);\ s \in \mathbb{R} \right\},\ \forall r \in \mathbb{R}.$$

The function $h_\varepsilon$ is similarly defined.
Since by Proposition 1.1 the affine manifold $\{(y, u) \in L^2(Q) \times L^2(Q);\ \mathcal{A}y = mu + f\}$ is closed and the cost functional in (2.11) is strictly convex and coercive, problem (2.11) has a unique solution $(y_\varepsilon, u_\varepsilon) \in L^2(Q) \times L^2(Q)$. We have

$$\int_Q (g_\varepsilon(y_\varepsilon + y_0) + h_\varepsilon(u_\varepsilon) + 2^{-1}|y^* - y_\varepsilon|^2 + 2^{-1}|u^* - u_\varepsilon|^2)\, dx\, dt \le \int_Q (g(y^* + y_0) + h(u^*))\, dx\, dt.$$

By a standard device (see [1] and [2]) we have

$$\lim_{\varepsilon \to 0} \int_Q (|y_\varepsilon - y^*|^2 + |u_\varepsilon - u^*|^2)\, dx\, dt = 0 \tag{2.12}$$

and

$$\int_Q (g_\varepsilon'(y_\varepsilon + y_0)z + (y_\varepsilon - y^*)z + h_\varepsilon'(u_\varepsilon)u + (u_\varepsilon - u^*)u)\, dx\, dt = 0 \tag{2.13}$$

for all $(z, u) \in L^2(Q) \times L^2(Q)$, such that $\mathcal{A}z = mu$. In particular, for $u = 0$, the latter yields

$$g_\varepsilon'(y_\varepsilon + y_0) + y_\varepsilon - y^* \in N(\mathcal{A})^\perp = R(\mathcal{A}). \tag{2.14}$$

Hence, there is $p_\varepsilon \in L^2(Q)$ such that

$$\mathcal{A}p_\varepsilon = -g_\varepsilon'(y_\varepsilon + y_0) - y_\varepsilon + y^*. \tag{2.15}$$

Substituting the latter into (2.13) we get

$$\int_Q ((-mp_\varepsilon + u_\varepsilon - u^*)u + h'_\varepsilon(u_\varepsilon)u)\, dx\, dt = 0,\ \forall u \in Y,$$

where

$$Y = \{v \in L^2(Q);\ mv \in R(\mathcal{A})\}.$$

Note that the orthogonal compliment $Y^\perp$ of $Y$ in $L^2(Q)$ is precisely the space $\{mv;\ v \in N(\mathcal{A})\}$. Therefore

$$mp_\varepsilon + u^* - u_\varepsilon = h'_\varepsilon(u_\varepsilon) + m\eta_\varepsilon, \text{a.e.in} Q,$$

where $\eta_\varepsilon \in N(\mathcal{A})$. If we denote again by $p_\varepsilon$ the function $p_\varepsilon - \eta_\varepsilon$, we get

$$mp_\varepsilon + u^* - u_\varepsilon = h'_\varepsilon(u_\varepsilon), \text{a.e.in} Q. \tag{2.16}$$

Since $L^2(Q) = R(\mathcal{A}) \oplus N(\mathcal{A})$, we write $p_\varepsilon = p^1_\varepsilon + p^2_\varepsilon$, where $p^1_\varepsilon \in R(\mathcal{A}),\ p^2_\varepsilon \in N(\mathcal{A})$. Then, by the local boundedness of $\partial g$ on $\mathbb{R} \times \mathbb{R}$ and Proposition 1.1, we have

$$\|p^1_\varepsilon\|_{L^2(Q)} \le C, \quad \forall \varepsilon > 0. \tag{2.17}$$

On the other hand, (2.16) implies

$$\int_Q (h^*_\varepsilon(mp_\varepsilon + u^* - u_\varepsilon) + h_\varepsilon(u_\varepsilon))\, dxdt = \int_Q (mp_\varepsilon + u^* - u_\varepsilon)u_\varepsilon\, dxdt. \tag{2.18}$$

By assumption (jj) and (2.12), the latter yields

$$\|mp_\varepsilon\|_{L^2(Q)} \le C, \forall \varepsilon > 0. \tag{2.19}$$

Hence $\{mp^2_\varepsilon\}$ is bounded in $L^2(Q)$, and by Proposition 1.1 we get

$$\|p^2_\varepsilon\|_{L^2(Q)} \le C, \ \ \forall \varepsilon > 0. \tag{2.20}$$

We may assume that, on a subsequence, again denoted $\varepsilon$, we have

$$\begin{array}{ll} y_\varepsilon \to y^*,\ u_\varepsilon \to u^* & \text{in} L^2(Q), \\ p_\varepsilon \to p & \text{weaklyin} L^2(Q), \\ g'_\varepsilon(y_\varepsilon + y_0) \to w & \text{weaklyin} L^2(Q). \end{array}$$

Since $y \to \partial g(y + y_0),\ z \to \partial h^*(z)$ are maximal monotone (and therefore strongly-weakly closed in $L^2(Q) \times L^2(Q)$), letting $\varepsilon$ tend to zero in (2.15), (2.18), we get (2.8)-(2.10) as desired. This completes the proof of Theorem 2.2.

**Remark 2.1.** Theorems 2.1 and 2.2 remain true in the nonhomogenous case, i.e.,

$$
\begin{aligned}
&y_{tt} + y_{xxxx} = m(x)u + f, && x \in (0,\pi),\ t \in \mathbb{R},\\
&y(x,t) = y(x,t+T), && x \in (0,\pi),\ t \in \mathbb{R},\\
&y(0,t) = y_x(0,t) = 0, && \\
&y_{xx}(\pi,t) = g_0(t),\ y_{xxx}(\pi,t) = -g_1(t), && t \in \mathbb{R},
\end{aligned}
\tag{1.2'}
$$

where $g_0,\ g_1$ are $T$-periodic functions (the torque, respectively the lateral force at $x = \pi$).

## 3. PERIODIC HAMILTONIAN SYSTEM ASOCIATED TO $\mathcal{A}$

In this section, we are concerned with the existence for the Hamiltonian system

$$
\begin{aligned}
&y_{tt}(x,t) + y_{xxxx}(x,t) \in \partial_p h(y(x,t),\ p(x,t)) + f(x,t) && \text{in} Q,\\
&p_{tt}(x,t) + p_{xxxx}(x,t) \in \partial_y h(y(x,t),\ p(x,t)) + g(x,t) && \text{in} Q,\\
&y(x,0) = y(x,T),\ y_t(x,0) = y_t(x,T), && x \in (0,\pi)\\
&p(x,0) = p(x,T),\ p_t(x,0) = p_t(x,T), && x \in (0,\pi)\\
&y(0,t) = y_x(0,t) = 0,\ y_{xx}(\pi,t) = y_{xxx}(\pi,t) = 0, && t \in \mathbb{R},\\
&p(0,t) = p_x(0,t) = 0,\ p_{xx}(\pi,t) = p_{xxx}(\pi,t) = 0, && t \in \mathbb{R},
\end{aligned}
\tag{3.1}
$$

where $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a convex continuous function, $\partial h = (\partial_y h, \partial_p h)$ is its subdifferential, and $f,\ g \in L^2(Q)$. The solution $(y,p) \in L^2(Q) \times L^2(Q)$ to (3.1) should be considered in the weak sense, i.e.,

$$
\mathcal{A}y \in \partial_p h(y,p) + g,\ \mathcal{A}p \in \partial_y h(y,p) + f. \tag{3.1'}
$$

Let $f = f^1 + f^2,\ g = g^1 + g^2$, where $f^1,\ g^1 \in R(\mathcal{A}),\ f^2,\ g^2 \in N(\mathcal{A})$.
Here we assume that
(k) $\quad h : \mathbb{R}\times \to \mathbb{R}$ is convex, continuous and satisfy the growth condition

$$
\gamma_1|y| + \gamma_2|p| + C_1 \le h(y,p) \le \gamma_3(|y|^2 + |p|^2) + C_2, \text{a.e.in} Q, \tag{3.2}
$$

where $\gamma_1, \gamma_2, \gamma_3 > 0$.
(kk) $\quad \|f^2\|_{L^\infty(Q)} < \gamma_1,\ \ \|g^2\|_{L^\infty(Q)} < \gamma_2,$
where $\nu(Q)$ is the Lebesgue measure of $Q$.

**Theorem 3.1.** *Assume that (k), (kk) hold and*

$$
\inf\left\{\left|\left(k+\frac{1}{2}\right)^4 - \left(\frac{2\pi j}{T}\right)^2\right|;\ k,j \in \mathbb{Z}\right\} > 2\gamma_3. \tag{3.3}
$$

*Then system (3.1) has at least one optimal solution $(y,p) \in L^2(Q) \times L^2(Q)$.*

**Proof.** The proof is as in Barbu [3], but it will sketched for the reader's convenience. Let $h^*$ be the conjugate function of $h$, i.e.,

$$h^*(v,u) = \sup\{vp + uq - h(p,q);\ (p,q) \in \mathbb{R} \times \mathbb{R}\}. \tag{3.4}$$

Then $(3.1)'$ can be written as

$$(y,p) \in \partial h^*(\mathcal{A}p - f, \mathcal{A}y - g),\ \text{a.e.in} Q, \tag{3.5}$$

or equivalently

$$(\mathcal{A}^{-1}u + \xi, \mathcal{A}^{-1}v + \eta) \in \partial h^*(v - f, u - g),\ \text{a.e.in} Q, \tag{3.6}$$

where

$$u, v \in R(\mathcal{A}),\quad \xi, \eta \in N(\mathcal{A}). \tag{3.7}$$

Throughout what follows we denote by $\|\cdot\|$ the usual norm in $L^2(Q)$.
We consider the minimization problem

$$\min\left\{\int_Q (h^*(v - f, u - g) - v\mathcal{A}^{-1}u)\,dxdt;\ u, v \in R(\mathcal{A})\right\} \tag{3.8}$$

and we shall prove that the solution $(y^*, u^*)$ to (3.8) satisfies (3.6), (3.7).
Indeed, by (3.2), (3.3) we have

$$\int_Q (h^*(v-f, u-g) - v\mathcal{A}^{-1}u)\,dxdt \geq (4\gamma_3)^{-1}(\|v\|^2_{L^2(Q)} + \|u\|^2_{L^2(Q)}) - C_3 - \int_Q v\mathcal{A}^{-1}u\,dxdt$$

$$\geq \delta(\|u\|^2_{L^2(Q)} + \|v\|^2_{L^2(Q)}) - C_3,\quad \forall u, v \in R(\mathcal{A}),$$

where $\delta > 0$. Since $h^*$ is convex, lower semicontinuous and $\mathcal{A}^{-1}$ is compact on $R(\mathcal{A})$, the latter implies that (3.8) has at least one solution $(u^*, v^*)$.
Consider now the approximating problem

$$\begin{aligned}\min\Big\{\textstyle\int_Q (h^*_\lambda(v - f, u - g) - v\mathcal{A}^{-1}u)\,dxdt + 2^{-1}\|u - u^*\|^2_{L^2(Q)}\\ +2^{-1}\|v - v^*\|^2_{L^2(Q)};\ u, v \in R(\mathcal{A})\Big\}\end{aligned} \tag{3.9}$$

where $h^*_\lambda \in C^1(\mathbb{R} \times \mathbb{R})$ is the convex regularization of $h^*$, i.e.,

$$h^*_\lambda(v,u) = \inf\left\{\frac{1}{2\lambda}(|v - \bar{v}|^2 + |u - \bar{u}|^2) + h^*(\bar{v}, \bar{u});\ (\bar{v}, \bar{u}) \in \mathbb{R} \times \mathbb{R}\right\},\ \lambda > 0. \tag{3.10}$$

Recalling that

$$\begin{aligned}h^*_\lambda(v - f, u - g) = h^*((I + \lambda\partial h^*)^{-1}(v - f, u - g))\\ +\tfrac{1}{2\lambda}|(I + \lambda\partial h^*)^{-1}(v - f, u - g) - (v - f, u - g)|^2,\end{aligned}$$

since $h^*$ is bounded from below by an affine function, by (3.2), (3.3) we infer that (3.9) has a solution $(u_\lambda, v_\lambda) \in R(\mathcal{A})\times R(\mathcal{A})$. By a standard device (see Barbu [1]) it follows that

$$u_\lambda \to u^*, \quad v_\lambda \to v^* \text{strongly in} L^2(Q). \tag{3.11}$$

Moreover, $(u_\lambda, v_\lambda)$ satisfies the first-order optimality system

$$\nabla h_\lambda^*(v_\lambda - f, u_\lambda - g) - (\mathcal{A}^{-1}u_\lambda, \mathcal{A}^{-1}v_\lambda) + (v_\lambda - v^*, u_\lambda - u^*) = (\eta_\lambda, \xi_\lambda), \text{a.e.in} Q \tag{3.12}$$

$$\eta_\lambda,\ \xi_\lambda \in N(\mathcal{A}) = R(\mathcal{A})^\perp. \tag{3.13}$$

By (3.11) we have

$$\begin{matrix} \mathcal{A}^{-1}u_\lambda \to y = \mathcal{A}^{-1}u^* \\ \mathcal{A}^{-1}v_\lambda \to p = \mathcal{A}^{-1}v^* \end{matrix} \quad \text{strongly in} L^2(Q) \text{as} \lambda \to 0. \tag{3.14}$$

Then, by (3.12), we have

$$\int_Q (\eta_\lambda + \mathcal{A}^{-1}u_\lambda - v_\lambda + v^*)(v_\lambda - w_1 - f^1) + (\xi_\lambda + \mathcal{A}^{-1}v_\lambda - u_\lambda + u^*)(u_\lambda - w_2 - g^2)\, dxdt$$

$$\geq \int_Q (h_\lambda^*(v_\lambda - f, u_\lambda - g) - h_\lambda^*(w_1 - f^2, w_2 - g^2))\, dxdt$$

for all $w_1,\ w_2 \in N(\mathcal{A}),\ \|w_1\|_{L^\infty(Q)} = \|w_2\|_{L^\infty(Q)} = \varepsilon$. We get

$$\varepsilon \left( \|\eta_\lambda\|_{L^2(Q)} + \|\xi_\lambda\|_{L^2(Q)} \right) \leq \int_Q h_\lambda^*(w_1 - f^2, w_2 - g^2)\, dxdt + C_4, \quad \forall \lambda > 0. \tag{3.15}$$

On the other hand, by (3.1), (3.2) and (3.4) we infer that

$$\int_Q h_\lambda^*(w_1 - f^2, w_2 - g^2)\, dxdt \leq \int_Q h^*(w_1 - f^2, w_2 - g^2)\, dxdt$$

$$\leq \sup \left\{ \int_Q ((w_1 - f^2)p + (w_2 - g^2)q - h(p,q))\, dxdt;\ (p,q) \in L^2(Q)\times L^2(Q) \right\}$$

$$\leq \|p\|_{L^1(Q)} \left(\varepsilon + \|f^2\|_{L^\infty(Q)} - \gamma_1\right) + \|q\|_{L^1(Q)} \left(\varepsilon + \|g^2\|_{L^\infty(Q)} - \gamma_2\right) - C_1.$$

Then, for $\varepsilon$ sufficiently small we have by (k) and (kk) that

$$\int_Q h_\lambda^*(w_1 - f^2, w_2 - g^2)\, dxdt \leq C_6. \tag{3.16}$$

Next, by (3.15) and (3.16) it follows that

$$\|\eta_\lambda\|_{L^2(Q)} + \|\xi_\lambda\|_{L^2(Q)} \leq C_6, \forall\lambda > 0.$$

Since $N(\mathcal{A})$ is finite dimensional, we have that, on a subsequence,

$$\eta_\lambda \to \eta,\ \xi_\lambda \to \xi \text{ stronglyin} L^2(Q). \tag{3.17}$$

Then, letting $\varepsilon$ tend to zero in (3.12), by (3.11), (3.14) and (3.17) we see that $(u^*, v^*)$ satisfies (3.6), (3.7) and the proof is complete.

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