Stability of two IMEX methods, CNLF and BDF2-AB2, for uncoupling systems of evolution equations

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Abstract

Stability is proven for two second order, two step methods for uncoupling a system of two evolution equations with exactly skew symmetric coupling: the Crank-Nicolson Leap Frog (CNLF) combination and the BDF2-AB2 combination. The form of the coupling studied arises in spatial discretizations of the Stokes-Darcy problem. For CNLF we prove stability for the coupled system under the time step condition suggested by linear stability theory for the Leap-Frog scheme. This seems to be a first proof of a widely believed result. For BDF2-AB2 we prove stability under a condition that is better than the one suggested by linear stability theory for the individual methods.

Key words: partitioned methods, IMEX methods, CNLF, Stokes-Darcy coupling

1 Introduction

In this note we prove stability of two, second order IMEX methods for uncoupling two evolution equations with exactly skew symmetric coupling:

$$\begin{aligned} \frac{du}{dt} + A_1 u + C\phi = f(t), &\ \text{ for } t > 0 \text{ and } u(0) = u_0 \\ \frac{d\phi}{dt} + A_2\phi - C^T u = g(t), &\ \text{ for } t > 0 \text{ and } \phi(0) = \phi_0. \end{aligned}$$

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This problem occurs, for example, after spatial discretization of the evolutionary Stokes-Darcy problem, e.g., [21,15,24,22]. Here

$$u : [0, \infty) \to \mathbb{R}^N, \phi : [0, \infty) \to \mathbb{R}^M,$$

and f, g, u_0, ϕ_0 and the matrices $A_{1/2}, C$ have compatible dimensions (and in particular C is $N \times M$). Note especially the exactly skew symmetric coupling linking the two equations. We assume that **the A_i are SPD**. Our analysis extends to the case of A_i non-symmetric with positive definite symmetric part or even nonlinear with $\langle A(v), v\rangle \geq Const.|v|^2$. The case where $A_{1/2}$ are exactly skew symmetric, relevant to wave propagation problems with both fast and slow waves, is treated in a remark below. With superscript denoting the time step number, the first method is CNLF, the combination of Crank-Nicolson and Leap Frog given by: for $n \geq 2$

$$\begin{aligned} \frac{u^{n+1} - u^{n-1}}{2\triangle t} + A_1 \frac{u^{n+1} + u^{n-1}}{2} + C\phi^n = f^n, \\ \frac{\phi^{n+1} - \phi^{n-1}}{2\triangle t} + A_2 \frac{\phi^{n+1} + \phi^{n-1}}{2} - C^T u^n = g^n. \end{aligned} \tag{CNLF}$$

Since the stability region of LF is the interval $-1 < \mathrm{Im}(z) < +1$, from the scalar case we expect a stability restriction of the form $\triangle t\sqrt{\lambda_{\max}(C^T C)} \leq 1$. Interestingly, it seems that sufficiency in the non-commutative case is not yet proven Verwer [28], Remark 3.1, page 6. We prove in Section 2 that CNLF is indeed stable under (1), exactly the condition suggested by the linear stability theory.

For vectors of the same length, denote the usual euclidean inner product and norm by $\langle u, v\rangle := u^T v$, $|\phi|^2 := \langle \phi, \phi\rangle$. We denote the weighted norms by

$$|u|^2_{A_1} := u^T A_1 u, \text{ and } |\phi|^2_{A_2} := \phi^T A_2 \phi.$$

Theorem 1 (Stability of CNLF) *Consider CNLF. Suppose the time step restriction holds:*

$$\triangle t\sqrt{\lambda_{\max}(C^T C)} \leq \alpha < 1, \textit{ for some } \alpha < 1. \tag{1}$$

Then for any $n \geq 2$

$$\begin{aligned} &\frac{1-\alpha}{2}\left[|u^{n+1}|^2 + |\phi^{n+1}|^2 + |u^n|^2 + |\phi^n|^2\right] \\ &+ \triangle t \sum_{\ell=1}^{n} \frac{1}{4}\left(|u^{\ell+1} + u^{\ell-1}|^2_{A_1} + |\phi^{\ell+1} + \phi^{\ell-1}|^2_{A_2}\right) \\ &\leq \frac{1}{2}\left[|u^1|^2 + |\phi^1|^2 + |u^0|^2 + |\phi^0|^2\right] + \triangle t\left[\langle C\phi^0, u^1\rangle - \langle C\phi^1, u^0\rangle\right] \\ &\quad + \triangle t \sum_{\ell=1}^{n}\left(\lambda^{-1}_{\min}(A_1)|f^\ell|^2 + \lambda^{-1}_{\min}(A_2)|g^\ell|^2\right). \end{aligned}$$

Next we establish the stability of BDF2 with explicit AB2 coupling

$$\begin{aligned}
&\frac{3u^{n+1} - 4u^n + u^{n-1}}{2\triangle t} + A_1 u^{n+1} + C(2\phi^n - \phi^{n-1}) = f^{n+1}, \qquad \text{(BDF2-AB2)}\\
&\frac{3\phi^{n+1} - 4\phi^n + \phi^{n-1}}{2\triangle t} + A_2 \phi^{n+1} - C^T(2u^n - u^{n-1}) = g^{n+1}.
\end{aligned}$$

The stability region of AB2 suggests that this combination is strictly worse than CNLF. However, we prove that the combination inherits enough stability from BDF2 to be stable under a time step condition that in many cases is better than the one for CNLF.

Theorem 2 (Stability of BDF2-AB2) *Consider BDF2-AB2. Suppose that the time step restriction holds*

$$\Delta t \max\{\lambda_{\max}(A_1^{-1}CC^T), \lambda_{\max}(A_2^{-1}C^TC)\} \leq \alpha < 1, \quad \textit{for some } \alpha > 0, \tag{2}$$

then BDF2-AB2 is stable:

$$|u^n|^2 + |\phi^n|^2 \leq C(\textit{initial data}, \textit{forcing terms}), \textit{ for any } n \geq 2.$$

More precisely, for all $n \geq 1$, we have that

$$\begin{aligned}
\frac{1}{2}\Big(|u^{n+1}|^2 + |\phi^{n+1}|^2\Big) + \frac{1}{2}\Big(|2u^{n+1} - u^n|^2 + |2\phi^{n+1} - \phi^n|^2\Big) + \Delta t \sum_{\ell=1}^{n} \frac{1}{2}\left(\mathcal{R}^{\ell+1} + \mathfrak{R}^{\ell+1}\right)\\
\leq \frac{1}{2}\Big(|u^1|^2 + |\phi^1|^2\Big) + \frac{1}{2}\Big(|2u^1 - u^0|^2 + |2\phi^1 - \phi^0|^2\Big)\\
+ \Delta t \sum_{\ell=1}^{n} \frac{1}{2(1-\alpha)}\Big(\frac{|f^{\ell+1}|^2}{\lambda_{\min}(A_1)} + \frac{|g^{\ell+1}|^2}{\lambda_{\min}(A_2)}\Big),
\end{aligned}$$

where we have denoted

$$\begin{aligned}
\mathcal{R}^{\ell+1} =& \Big|\sqrt{\Delta t}C^T u^{\ell+1} - \frac{1}{2\sqrt{\Delta t}}(\phi^{\ell+1} - 2\phi^\ell + \phi^{\ell-1})\Big|^2\\
&+ \Big|\sqrt{\Delta t}C\phi^{\ell+1} + \frac{1}{2\sqrt{\Delta t}}(u^{\ell+1} - 2u^\ell + u^{\ell-1})\Big|^2,\\
\mathfrak{R}^{\ell+1} =& \Big|\lambda_{\min}^{1/2}(A_1 - \Delta t CC^T)u^{\ell+1} - \frac{1}{2\lambda_{\min}^{1/2}(A_1 - \Delta t CC^T)} f^{\ell+1}\Big|^2\\
&+ \Big|\lambda_{\min}^{1/2}(A_2 - \Delta t C^TC)\phi^{\ell+1} - \frac{1}{2\lambda_{\min}^{1/2}(A_2 - \Delta t C^TC)} g^{\ell+1}\Big|^2.
\end{aligned}$$

Note that (2) assumes that $A_1 - \Delta t C^TC, A_2 - \Delta t CC^T$ are SPD.

Both methods use 3 levels; approximations are needed at the first two time steps to begin. We suppose these are computed to appropriate accuracy, Verwer [28].

Because the problem and methods are linear, stability immediately implies that the error is bounded by its consistency error.

Remark 1 (The case when A_i are skew symmetric.) *Suppose $A_{1/2}$ are skew symmetric and $f = g = 0$. The same proof shows that CNLF remains stable under exactly the same time step condition:*

$$(1-\alpha)\frac{1}{2}\left[|u^{\boldsymbol{n}}|^2 + |\boldsymbol{\phi}^{\boldsymbol{n}}|^2 + |u^{n-1}|^2 + |\phi^{n-1}|^2\right] \leq$$
$$\frac{1}{2}\left[|u^1|^2 + |\boldsymbol{\phi}^1|^2 + |u^0|^2 + |\phi^0|^2\right] + \triangle t[\left\langle C\phi^0, u^1\right\rangle - \left\langle C\phi^1, u^0\right\rangle].$$

The stability proof of BDF2-AB2 is less clear in this case. It includes dissipation terms, which, while possibly large, do not seem to control the coupling terms. The result is a worst case stability bound with exponential growth of $O(exp(\beta t_n))$ where $\beta = 4\triangle t\lambda_{\max}(C^TC)$. Our tests do show cases where the numerical dissipation dominates and drives the BDF2-AB2 approximate solution to zero.

1.1 Previous work

When A_i are SPD, IMEX methods, like CNLF and BDF2-AB2 require the solution of two, smaller SPD systems per time step (which can be done by legacy codes for the independent sub-problems) as compared to one larger, nonsymmetric system for monolithically coupled methods. Given this potentially large simplification, it is not surprising that IMEX methods (and associated partitioned schemes) have been used extensively in the computational practice of multi-domain, multiphysics applications. The theory of IMEX methods is also developing; see [14,27,5,2,1] and [16] for early papers and [3,4,13] and particularly [28] and the book [19] for recent work. CNLF is itself a classic (e.g. [20]) combination of methods in computational fluid dynamics with wide practical use, including in the dynamic core of the NCAR climate model, [25].

Partitioned methods are often motivated by available codes for subproblems [26] and tend to be application specific. Examples of partitioned methods include ones designed for fluid-structure interaction [6,7,10], Maxwell's equations [29] and atmosphere-ocean coupling [11,13,12]. The block system we study arises in evolutionary groundwater-surface water coupling, e.g., [9,8,15,21]. Mu and Zhu [22] gave the first (in 2010) numerical analysis of a partitioned method based on the backward Euler-forward Euler IMEX scheme; this has been extended to, so-called, asynchronous time stepping (different time steps for different system components) in [24]. Our work herein is motivated by the search for partitioned methods for the Stokes-Darcy problem with higher accuracy and better stability.

2 Proof of stability of CNLF

This section gives a complete proof of Theorem 1.

Lemma 3 *We estimate*

$$\langle C\phi, u\rangle = \frac{1}{2}|C\phi|^2 + \frac{1}{2}|u|^2 - \frac{1}{2}|u - C\phi|^2$$

and, if A_i are SPD

$$\begin{aligned}
&|u| \leq \lambda_{\min}^{-1/2}(A_1)|u|_{A_1}, |\phi| \leq \lambda_{\min}^{-1/2}(A_2)|\phi|_{A_2},\\
&|C\phi| \leq \sqrt{\lambda_{\max}(C^T C)}|\phi|.
\end{aligned}$$

Thus

$$|\langle C\phi, u\rangle| \leq \frac{1}{2}\sqrt{\lambda_{\max}(C^T C)}|\phi|^2 + \frac{1}{2}\sqrt{\lambda_{\max}(C^T C)}|u|^2.$$

Proof. The first claim is the polarization identity. The second inequality is elementary while the fourth follows by inserting the third into the first. For the third, we have

$$|C\phi| = \langle C\phi, C\phi\rangle^{1/2} = \langle C^T C\phi, \phi\rangle^{1/2} \leq \lambda_{\max}^{1/2}(C^T C)|\phi|.$$

■

The **first** of three main steps in the proof of Theorem 1 is to take the inner product of CNLF with $u^{n+1} + u^{n-1}$ and $\phi^{n+1} + \phi^{n-1}$ and add:

$$\begin{aligned}
&\frac{1}{2\triangle t}\left[|u^{n+1}|^2 + |\phi^{n+1}|^2\right] - \frac{1}{2\triangle t}\left[|u^{n-1}|^2 + |\phi^{n-1}|^2\right]\\
&\quad +\frac{1}{2}\left[|u^{n+1} + u^{n-1}|_{A_1}^2 + |\phi^{n+1} + \phi^{n-1}|_{A_2}^2\right]\\
&\quad +\langle C\phi^n, u^{n+1} + u^{n-1}\rangle - \langle C^T u^n, \phi^{n+1} + \phi^{n-1}\rangle\\
&\quad = \langle f^n, u^{n+1} + u^{n-1}\rangle + \langle g^n, u^{n+1} + u^{n-1}\rangle.
\end{aligned} \tag{3}$$

The **second** step is to rearrange the coupling terms as an exact difference between two time levels: *Coupling* $= \langle C\phi^n, u^{n+1} - u^{n-1}\rangle - \langle C^T u^n, \phi^{n+1} - \phi^{n-1}\rangle = C^{n+1/2} - C^{n-1/2}$, where

$$C^{n+1/2} := \langle C\phi^n, u^{n+1}\rangle - \langle C\phi^{n+1}, u^n\rangle,$$
$$C^{n-1/2} := \langle C\phi^{n-1}, u^n\rangle - \langle C\phi^n, u^{n-1}\rangle.$$

The **third** step is to add and subtract $|\boldsymbol{u^n}|^2 + |\boldsymbol{\phi^n}|^2$ to the control the energy at level t_n:

$$\frac{1}{2\triangle t}\left[|u^{n+1}|^2 + |\phi^{n+1}|^2 + |\boldsymbol{u^n}|^2 + |\boldsymbol{\phi^n}|^2\right] - \frac{1}{2\triangle t}\left[|\boldsymbol{u^n}|^2 + |\boldsymbol{\phi^n}|^2 + |u^{n-1}|^2 + |\phi^{n-1}|^2\right]$$
$$+\frac{1}{2}\left[|u^{n+1}+u^{n-1}|_{A_1}^2 + |\phi^{n+1}+\phi^{n-1}|_{A_2}^2\right] + C^{n+1/2} - C^{n-1/2}$$
$$= \langle f^n, u^{n+1}+u^{n-1}\rangle + \langle g^n, u^{n+1}+u^{n-1}\rangle \equiv \mathbf{RHS}.$$

Using Lemma 3 we treat **RHS** in a standard way:

$$\mathbf{RHS} \le |f^n|\lambda_{\min}^{-1/2}(A_1)|u^{n+1}+u^{n-1}|_{A_1} + |g^n|\lambda_{\min}^{-1/2}(A_2)|\phi^{n+1}+\phi^{n-1}|_{A_2}$$
$$\le \left(\lambda_{\min}^{-1}(A_1)|f^n|^2 + \lambda_{\min}^{-1}(A_2)|g^n|^2\right) + \frac{1}{4}(|u^{n+1}+u^{n-1}|_{A_1}^2 + |\phi^{n+1}+\phi^{n-1}|_{A_2}^2).$$

Thus, define the system energy

$$E^{n+1/2} := \frac{1}{2}\left[|u^{n+1}|^2 + |\phi^{n+1}|^2 + |u^n|^2 + |\phi^n|^2\right] + \triangle t C^{n+1/2}.$$

Collecting terms we obtain

$$E^{n+1/2} - E^{n+1/2} + \triangle t\left(|u^{n+1}+u^{n-1}|_{A_1}^2 + |\phi^{n+1}+\phi^{n-1}|_{A_2}^2\right)$$
$$\le \triangle t(\lambda_{\min}^{-1}(A_1)|f^n|^2 + \lambda_{\min}^{-1}(A_2)|g^n|^2).$$

Obviously, $E^{n+1/2} - E^{n-1/2} + \{\text{positive\_terms}\} \le RHS$ immediately implies stability provided only that $E^{n+1/2} > 0$ for every n. We have (using Lemma 3 to bound the coupling terms)

$$E^{n+1/2} \ge \frac{1}{2}\left[|u^{n+1}|^2 + |\phi^{n+1}|^2 + |u^n|^2 + |\phi^n|^2\right]$$
$$-\frac{\triangle t}{2}\sqrt{\lambda_{\max}(C^TC)}\left[|u^{n+1}|^2 + |u^n|^2 + |\phi^{n+1}|^2 + |\phi^n|^2\right].$$

This is positive (completing the proof) provided

$$\triangle t\sqrt{\lambda_{\max}(C^TC)} < 1.$$

3 Proof of stability of BDF2-AB2

We proceed to prove Theorem 2. Take the inner product of BDF2-AB2 with u^{n+1}, ϕ^{n+1}, respectively, and add. There are two keys to the proof of stability.

The **first key** is the treatment of the *BDF2 term.* Apply the identity

$$\left[\frac{a^2}{4}+\frac{(2a-b)^2}{4}\right]-\left[\frac{b^2}{4}+\frac{(2b-c)^2}{4}\right]+\frac{(a-2b+c)^2}{4}=\frac{1}{2}(3a-4b+c)a$$

with $a=u^{n+1}, b=u^n, c=u^{n-1}$, and once with $a=\phi^{n+1}, b=\phi^n, c=\phi^{n-1}$. This gives

$$\begin{aligned}
&\frac{1}{4\Delta t}\Big(|u^{n+1}|^2+|2u^{n+1}-u^n|^2\Big)-\frac{1}{4\Delta t}\Big(|u^n|^2+|2u^n-u^{n-1}|^2\Big)\\
&\qquad+\frac{1}{4\Delta t}|u^{n+1}-2u^n+u^{n-1}|^2\\
&+\frac{1}{4\Delta t}\Big(|\phi^{n+1}|^2+|2\phi^{n+1}-\phi^n|^2\Big)-\frac{1}{4\Delta t}\Big(|\phi^n|^2+|2\phi^n-\phi^{n-1}|^2\Big)\\
&\qquad+\frac{1}{4\Delta t}|\phi^{n+1}-2\phi^n+\phi^{n-1}|^2\\
&+|u^{n+1}|^2_{A_1}+|\phi^{n+1}|^2_{A_2}+\langle C(2\phi^n-\phi^{n-1}),u^{n+1}\rangle-\langle C^T(2u^n-u^{n-1}),\phi^{n+1}\rangle\\
&=\langle f^{n+1},u^{n+1}\rangle+\langle g^{n+1},\phi^{n+1}\rangle.
\end{aligned}\tag{4}$$

The **second key** is to *rearrange the coupling terms.* We use the skew-symmetry of the coupling term and the polarization identity (Lemma 3) to write it as follows:

$$\begin{aligned}
&\mathit{Coupling}=\langle C(2\phi^n-\phi^{n-1}),u^{n+1}\rangle-\langle C^T(2u^n-u^{n-1}),\phi^{n+1}\rangle\\
&=-\langle C(\phi^{n+1}-2\phi^n+\phi^{n-1}),u^{n+1}\rangle+\langle C^T(u^{n+1}-2u^n+u^{n-1}),\phi^{n+1}\rangle\\
&=-\frac{1}{4\Delta t}|\phi^{n+1}-2\phi^n+\phi^{n-1}|^2-\Delta t|u^{n+1}|^2_{CC^T}\\
&\qquad-\frac{1}{4\Delta t}|u^{n+1}-2u^n+u^{n-1}|^2-\Delta t|\phi^{n+1}|^2_{C^TC}+\mathcal{R}^{n+1}.
\end{aligned}\tag{5}$$

Then (4) and (5) give

$$\begin{aligned}
&\frac{1}{4\Delta t}\Big(|u^{n+1}|^2+|2u^{n+1}-u^n|^2\Big)-\frac{1}{4\Delta t}\Big(|u^n|^2+|2u^n-u^{n-1}|^2\Big)\\
&\quad+\frac{1}{4\Delta t}\Big(|\phi^{n+1}|^2+|2\phi^{n+1}-\phi^n|^2\Big)-\frac{1}{4\Delta t}\Big(|\phi^n|^2+|2\phi^n-\phi^{n-1}|^2\Big)\\
&\quad+|u^{n+1}|^2_{A_1}+|\phi^{n+1}|^2_{A_2}-\Delta t|u^{n+1}|^2_{CC^T}-\Delta t|\phi^{n+1}|^2_{C^TC}+\mathcal{R}^{n+1}\\
&=\langle f^{n+1},u^{n+1}\rangle+\langle g^{n+1},\phi^{n+1}\rangle.
\end{aligned}\tag{6}$$

Using again the polarization identity yields

$$
\begin{aligned}
&\frac{1}{4\Delta t}\Big(|u^{n+1}|^2 + |2u^{n+1} - u^n|^2\Big) - \frac{1}{4\Delta t}\Big(|u^n|^2 + |2u^n - u^{n-1}|^2\Big) \\
&\quad + \frac{1}{4\Delta t}\Big(|\phi^{n+1}|^2 + |2\phi^{n+1} - \phi^n|^2\Big) - \frac{1}{4\Delta t}\Big(|\phi^n|^2 + |2\phi^n - \phi^{n-1}|^2\Big) \\
&\quad + |u^{n+1}|^2_{A_1} + |\phi^{n+1}|^2_{A_2} - \Delta t|u^{n+1}|^2_{CC^T} - \Delta t|\phi^{n+1}|^2_{C^TC} + \mathcal{R}^{n+1} \\
&\quad = \lambda_{\min}(A_1 - \Delta t CC^T)|u^{n+1}|^2 + \frac{1}{4\lambda_{\min}(A_1 - \Delta t CC^T)}|f^{n+1}|^2 \\
&\qquad + \lambda_{\min}(A_2 - \Delta t C^TC)|\phi^{n+1}|^2 + \frac{1}{4\lambda_{\min}(A_2 - \Delta t C^TC)}|g^{n+1}|^2 - \mathfrak{R}^{n+1},
\end{aligned}
$$

which by summation implies the stability result

$$
\begin{aligned}
&\frac{|u^{n+1}|^2}{4\Delta t} + \frac{1}{4\Delta t}|2u^{n+1} - u^n|^2 + \frac{|\phi^{n+1}|^2}{4\Delta t} + \frac{1}{4\Delta t}|2\phi^{n+1} - \phi^n|^2 + \sum_{\ell=1}^{n}(\mathcal{R}^{\ell+1} + \mathfrak{R}^{\ell+1}) \\
&\leq \frac{|u^1|^2}{4\Delta t} + \frac{1}{4\Delta t}|2u^1 - u^0|^2 + \frac{|\phi^1|^2}{4\Delta t} + \frac{1}{4\Delta t}|2\phi^1 - \phi^0|^2 \\
&\quad + \sum_{\ell=1}^{n}\Big(\frac{1}{4(1-\alpha)\lambda_{\min}(A_1)}|f^{n+1}|^2 + \frac{1}{4(1-\alpha)\lambda_{\min}(A_2)}|g^{n+1}|^2\Big).
\end{aligned}
$$

4 Numerical verification of the Theorems

We give two numerical tests that confirm the theory (showing in particular that the restriction (1) is sharp). The examples also illustrate that there are cases where each method's time step restriction is better than the other method.

In all test cases, the initial conditions are

$$u^0 = (1,1)^T \quad \text{and} \quad \phi^0 = (1,1)^T$$

and u^1, ϕ^1 are computed using the implicit backward Euler. We take $f = g = 0$, so that *any growth* in the energy is an instability.

Test 1. In the first case the matrices are

$$A_1 = \begin{pmatrix} 10 & 0 \\ 0 & 20 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 30 & 0 \\ 0 & 50 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$$

yielding the following time step restrictions

$$\Delta t_{\text{CNLF}} = 0.1361, \qquad \Delta t_{\text{BDFAB}} = 0.2990.$$

With the time step

$$\Delta t = 0.99 * \Delta t_{\text{CNLF}}$$

both methods are observed to be stable, Figure 1). With the time step $\Delta t =$

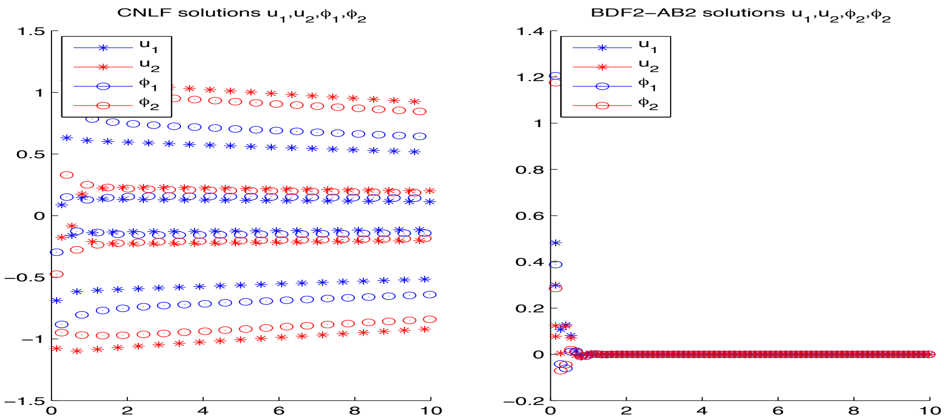

Fig. 1. Both methods stable, as predicted.

$1.01 * \Delta t_{\text{CNLF}}$ the CNLF approximations exhibit growth and thus are unstable Since $1.01 * \Delta t_{\text{CNLF}} < \Delta t_{BDFAB}$ the theory predicts BDF2-AB2 to be stable and this is indeed seen in Figure 2.

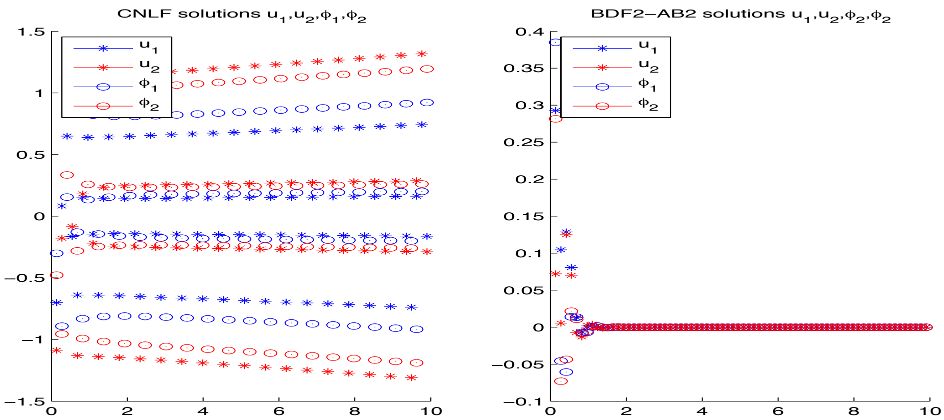

Fig. 2. CNLF unstable, BDF2-AB2 stable, as predicted.

Test 2. With matrices

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$$

the time step restrictions are

$$\Delta t_{\text{CNLF}} = 0.1361, \qquad \Delta t_{\text{BDFAB}} = 0.0299.$$

With time step $\Delta t = .99 * \Delta t_{\mathrm{CNLF}}$ the CNLF converges, while with BDF2-AB2 the solution is unstable, Figure 3.

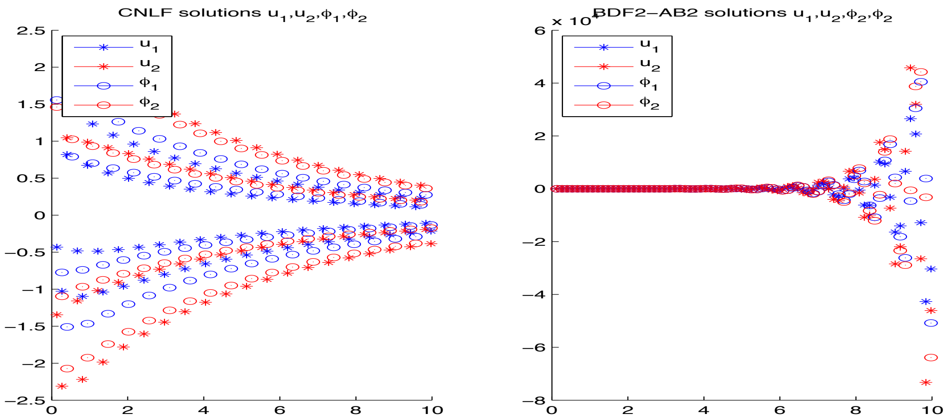

Fig. 3. CNLF stable, BDF2-AB2 unstable, as predicted.

In Figure 4 we plot the energy at the final time (vertical axis) against the time step (horizontal axis) for CNLF and BDF2-AB2 versus time-step for Test 1 and Test 2, respectively. The time interval is $[0, 10]$. The x-axis length is $\max\{\Delta t_{BDFAB}, \Delta t_{CNLF}\} * 1.1$.

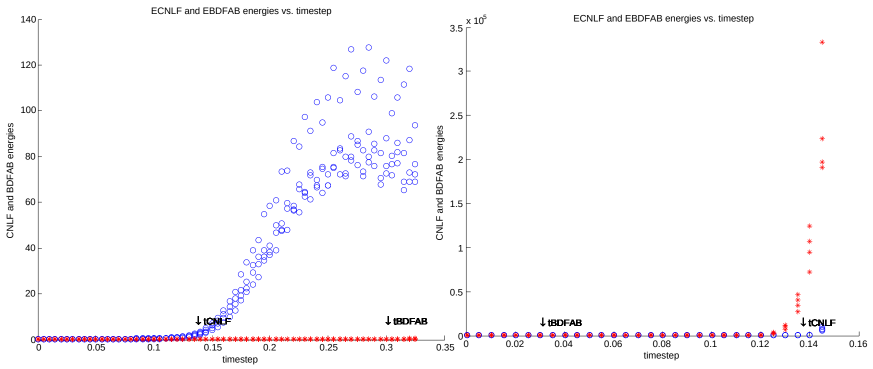

Fig. 4. BDF2-AB2 and CNLF energies: Test 1 and Test 2.

4.1 *Tests when A_i are skew symmetric*

Following the remark in the introduction we test the skew symmetric case. The tests confirm the predicted stability of CNLF. The stability result for BDF2-AB2 is less clear however. Test 3 shows a case where BDF2-AB2 solutions grow and test 4 shows a case when the large numerical dissipation term $\mathcal{R}^{n+1}$ drives the BDF2-AB2 solution to zero.

Test 3. Figure 5 plots the solutions and energy versus time step for

$$A_{1/2} = \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}, \triangle t = t_{\mathrm{CNLF}} \cdot .99,\ \triangle t = t_{\mathrm{CNLF}} \cdot 1.1.$$

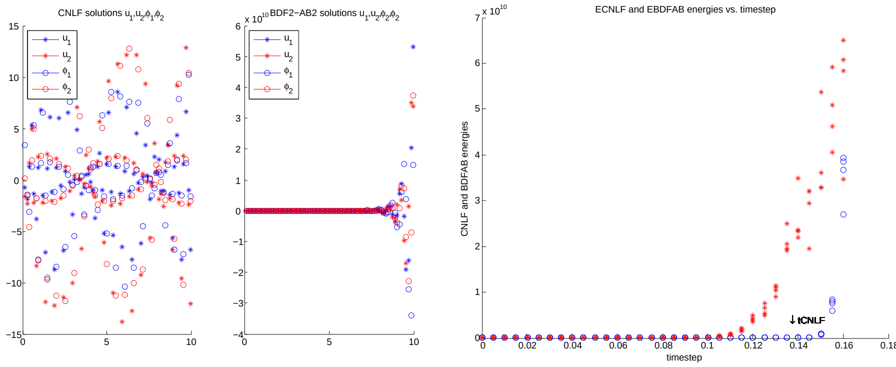

Fig. 5. Test 3: CNLF, BDF2-AB2 solutions, and BDF2-AB2/CNLF energies.

Test 4. Figure 6 plots solutions and energy versus time step for

$$A_{1/2} = \begin{bmatrix} 0 & -50 \\ 50 & 0 \end{bmatrix}, \triangle t = t_{\mathrm{CNLF}} \cdot .99,\ \triangle t = t_{\mathrm{CNLF}} \cdot 1.1.$$

The first figure in Figure 6 suggests incorrectly that CNLF experiences growth. Plotting the CNLF solution over a longer time corrects this impression; see the plot over $[0, 60]$ in Figure 7.

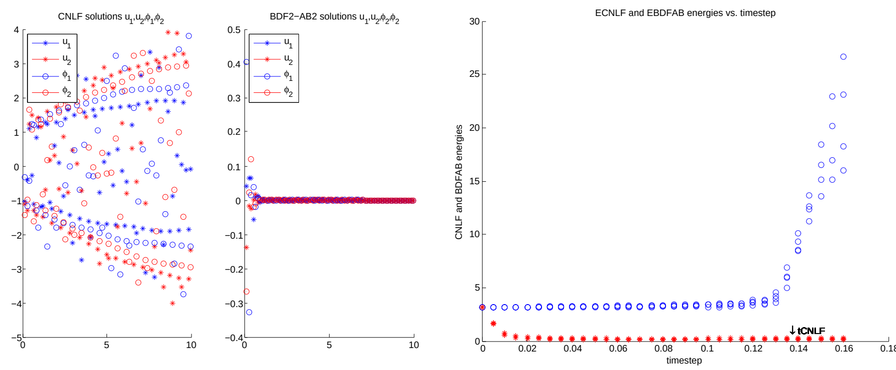

Fig. 6. Test 4: CNLF, BDF2-AB2 solutions, and BDF2-AB2/CNLF energies.

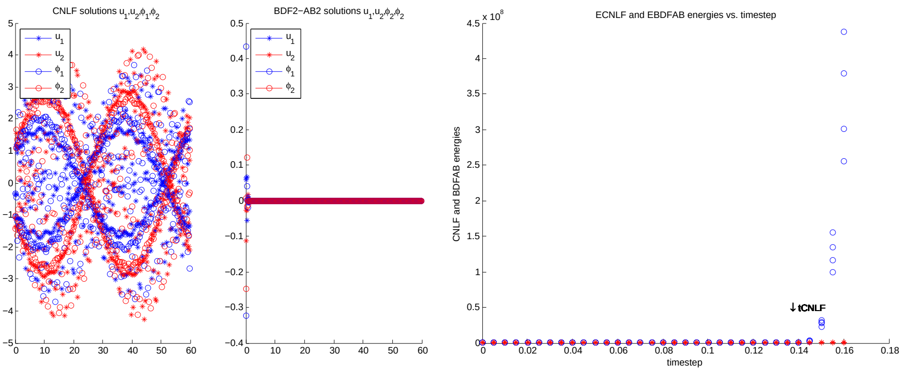

Fig. 7. Test 4: CNLF and BDF2-AB2 solutions, and BDFAB/CNLF energies.

References

[1] G. Akrivis, M. Crouzeix, C. Makridakis, Implicit-explicit multistep finite element methods for nonlinear parabolic problems, Math. Comp. 67 (222) (1998) 457–477.

[2] G. Akrivis, M. Crouzeix, C. Makridakis, Implicit-explicit multistep methods for quasilinear parabolic equations, Numer. Math. 82 (4) (1999) 521–541.

[3] G. Akrivis, Y.-S. Smyrlis, Implicit-explicit BDF methods for the Kuramoto-Sivashinsky equation, Appl. Numer. Math. 51 (2-3) (2004) 151–169.

[4] M. Anitescu, F. Pahlevani, W. J. Layton, Implicit for local effects and explicit for nonlocal effects is unconditionally stable, Electron. Trans. Numer. Anal. 18 (2004) 174–187 (electronic).

[5] U. M. Ascher, S. J. Ruuth, B. T. R. Wetton, Implicit-explicit methods for time-dependent partial differential equations, SIAM J. Numer. Anal. 32 (3) (1995) 797–823.

[6] E. Burman, M. A. Fernández, Stabilized explicit coupling for fluid-structure interaction using Nitsche's method, C. R. Math. Acad. Sci. Paris 345 (8) (2007) 467–472.

[7] E. Burman, M. A. Fernández, Stabilization of explicit coupling in fluid-structure interaction involving fluid incompressibility, Comput. Methods Appl. Mech. Engrg. 198 (5-8) (2009) 766–784.

[8] Y. Cao, M. Gunzburger, X. Hu, F. Hua, X. Wang, W. Zhao, Finite element approximations for Stokes-Darcy flow with Beavers-Joseph interface conditions, SIAM J. Numer. Anal. 47 (6) (2010) 4239–4256.

[9] Y. Cao, M. Gunzburger, F. Hua, X. Wang, Coupled Stokes-Darcy model with Beavers-Joseph interface boundary condition, Commun. Math. Sci. 8 (1) (2010) 1–25.

[10] P. Causin, J. F. Gerbeau, F. Nobile, Added-mass effect in the design of partitioned algorithms for fluid-structure problems, Comput. Methods Appl. Mech. Engrg. 194 (42-44) (2005) 4506–4527.

[11] J. M. Connors, J. Howell, A fluid-fluid interaction method using decoupled subproblems and differing timesteps, Tech. rep., Univ. of Pittsburgh (2010).

[12] J. M. Connors, J. S. Howell, W. J. Layton, Decoupled time stepping methods for fluid-fluid interaction, submitted SIAM J. Numer. Anal.

[13] J. M. Connors, A. Miloua, Partitioned time discretization for parallel solution of coupled ode systems, BIT Numerical Mathematics (2010) 1–2110.1007/s10543-010-0295-z.

[14] M. Crouzeix, Une méthode multipas implicite-explicite pour l'approximation des équations d'évolution paraboliques, Numer. Math. 35 (3) (1980) 257–276.

[15] M. Discacciati, E. Miglio, A. Quarteroni, Mathematical and numerical models for coupling surface and groundwater flows, Appl. Numer. Math. 43 (1-2) (2002) 57–74, 19th Dundee Biennial Conference on Numerical Analysis (2001).

[16] J. Frank, W. Hundsdorfer, J. Verwer, Stability of Implicit-Explicit linear multistep methods, Tech. rep., Centrum Wiskunde and Informatica (CWI) (1996).

[17] W. Hundsdorfer, S. J. Ruuth, On monotonicity and boundedness properties of linear multistep methods, Math. Comp. 75 (254) (2006) 655–672 (electronic).

[18] W. Hundsdorfer, S. J. Ruuth, IMEX extensions of linear multistep methods with general monotonicity and boundedness properties, J. Comput. Phys. 225 (2) (2007) 2016–2042.

[19] W. Hundsdorfer, J. Verwer, Numerical solution of time-dependent advection-diffusion-reaction equations, vol. 33 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 2003.

[20] O. Johansson, H.-O. Kreiss, Über das Verfahren der zentralen Differenzen zur Lösung des Cauchy problems für partielle Differentialgleichungen, Nordisk Tidskr. Informations-Behandling 3 (1963) 97–107.

[21] W. J. Layton, F. Schieweck, I. Yotov, Coupling fluid flow with porous media flow, SIAM J. Numer. Anal. 40 (6) (2002) 2195–2218 (2003).

[22] M. Mu, X. Zhu, Decoupled schemes for a non-stationary mixed Stokes-Darcy model, Math. Comp. 79 (270) (2010) 707–731.

[23] S. J. Ruuth, W. Hundsdorfer, High-order linear multistep methods with general monotonicity and boundedness properties, J. Comput. Phys. 209 (1) (2005) 226–248.

[24] L. Shan, H. Zheng, W. Layton, Decoupled schemes with different time sizes for the time-dependent Stokes-Darcy model, Tech. rep., Univ. Pittsburgh (http://www.math.pitt.edu/technical-reports, 2011).

[25] S. J. Thomas, R. D. Loft, The NCAR spectral element climate dynamical core: semi-implicit Eulerian formulation, J. Sci. Comput. 25 (1-2) (2005) 307–322.

[26] A. Valli, G. Carey, A. Coutinho, On decoupled time step/subcycling and iteration strategies for multiphysics problems, Commun. Numer. Methods Eng. 24 (12) (2008) 1941–1952.

[27] J. Varah, Stability restrictions on second order, three level finite difference schemes for parabolic equations, SIAM J. Numer. Anal. 17 (2) (1980) 300–309.

[28] J. Verwer, Convergence and component splitting for the Crank-Nicolson Leap-Frog integration method, Tech. rep., Centrum Wiskunde and Informatica (CWI) (2009).

[29] J. Verwer, Component splitting for semi-discrete Maxwell equations, BIT Numerical Mathematics (2010) 1–1910.1007/s10543-010-0296-y.