LARGE EDDY SIMULATION FOR TURBULENT MHD FLOWS

A. LABOVSKY† AND C. TRENCHEA†

Abstract. We investigate the mathematical properties of a model for the simulation of large eddies in turbulent, electrically conducting, viscous, incompressible flows. We prove existence and uniqueness of solutions for the simplest (zeroth) closed MHD model (1.7), we show that its solutions converge to the solution of the MHD equations as the averaging radii converge to zero, and derive a bound on the modeling error. Furthermore, we show that the model preserves the properties of the 3D MHD equations: the kinetic energy and the magnetic helicity are conserved, while the cross helicity is approximately conserved and converges to the cross helicity of the MHD equations, and the model is proven to preserve the Alfvén waves, with the velocity converging to that of the MHD, as δ_1, δ_2 tend to zero. We perform computational tests that verify the accuracy of the method and compare the conserved quantities of the model to those of the averaged MHD.

Key words. Large eddy simulation, magnetohydrodynamics, deconvolution

AMS subject classifications. 65M12, 76F65, 76W05

1. Introduction. Magnetically conducting fluids arise in important applications including plasma physics, geophysics and astronomy. In many of these, turbulent MHD (magnetohydrodynamics [2]) flows are typical. The difficulties of accurately modeling and simulating turbulent flows are magnified many times over in the MHD case. They are evinced by the more complex dynamics of the flow due to the coupling of Navier-Stokes and Maxwell equations via the Lorenz force and Ohm's law.

The flow of an electrically conducting fluid is affected by Lorentz forces, induced by the interaction of electric currents and magnetic fields in the fluid. The Lorentz forces can be used to control the flow and to attain specific engineering design goals such as flow stabilization, suppression or delay of flow separation, reduction of near-wall turbulence and skin friction, drag reduction and thrust generation. There is a large body of literature dedicated to both experimental and theoretical investigations on the influence of electromagnetic force on flows (see e.g., [23, 35, 36, 22, 52, 16, 53, 24, 46, 8]). The MHD equations are related to engineering problems such as plasma confinement, controlled thermonuclear fusion, liquid-metal cooling of nuclear reactors, electromagnetic casting of metals, MHD sea water propulsion. The MHD effects arising from the macroscopic interaction of liquid metals with applied currents and magnetic fields are exploited in metallurgical processes to control the flow of metallic melts: the electromagnetic stirring of molten metals [37], electromagnetic turbulence control in induction furnaces [54], electromagnetic damping of buoyancy-driven flow during solidification [41], and the electromagnetic shaping of ingots in continuous casting [43].

The turbulent flow of an electrically and magnetic conducting fluid and is more complex than the turbulent flow of a nonconducting fluid and has more parameter regimes. The invariants of 3D MHD are the total energy (velocity and magnetic field), the magnetic and cross helicity (see [14, 28]). Although the kinetic helicity is a rugged invariant for 3D Euler flows, it is not one for MHD systems, but still an important quantity (see [39]). The magnetic helicity is not conserved when a mean magnetic field is present, see e.g. [34, 47, 48, 7, 38]. Note that a strong alignment of the vorticity with the Lorentz force or the velocity and the curl of the Lorentz force is likely to produce a sizabile change in $u\cdot(\nabla\times u)$. Also, a flow that is instantaneously nonhelical

†Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260

and/or irrotational will not remain so if $\nabla \times (j \times B)$ has a non-zero projection on the velocity.

The mathematical description of the problem proceeds as follows. Assuming the fluid to be viscous and incompressible, the governing equations are the Navier-Stokes and pre-Maxwell equations, coupled via the Lorentz force and Ohm's law (see e.g. [45]). Let $\Omega = (0, L)^3$ be the flow domain, and $u(t,x), p(t,x), B(t,x)$ be the velocity, pressure, and the magnetic field of the flow, drived by the velocity body force f and magnetic field force $\operatorname{curl} g$. Then u, p, B satisfy the MHD equations:

$$\begin{aligned}
&u_t + \nabla\cdot(uu^T) - \frac{1}{\mathrm{Re}}\Delta u + \frac{S}{2}\nabla(B^2) - S\nabla\cdot(BB^T) + \nabla p = f,\\
&B_t + \frac{1}{\mathrm{Re_m}}\operatorname{curl}(\operatorname{curl} B) + \operatorname{curl}(B \times u) = \operatorname{curl} g,\\
&\nabla\cdot u = 0, \nabla\cdot B = 0,
\end{aligned} \tag{1.1}$$

in $Q = (0,T)\times\Omega$, with the initial data:

$$u(0,x) = u_0(x), \quad B(0,x) = B_0(x) \quad \text{in } \Omega, \tag{1.2}$$

and with periodic boundary conditions (with zero mean):

$$\Phi(t, x + L\mathrm{e}_i) = \Phi(t,x), i = 1,2,3, \quad \int_\Omega \Phi(t,x)dx = 0, \tag{1.3}$$

for $\Phi = u, u_0, p, B, B_0, f, g$.

Here Re, $\mathrm{Re_m}$, and S are nondimensional constants that characterize the flow: the Reynolds number, the magnetic Reynolds number and the coupling number, respectively. For derivation of (1.1), physical interpretation and mathematical analysis, see [12, 10, 26, 44, 21] and the references therein.

If $\overline{}^{\delta_1}$, $\overline{}^{\delta_2}$ denote two local, spacing averaging operators that commute with the differentiation, then averaging (1.1) gives the following non-closed equations for $\overline{u}^{\delta_1}$, $\overline{B}^{\delta_2}$, $\overline{p}^{\delta_1}$ in $(0,T)\times\Omega$:

$$\begin{aligned}
&\overline{u}_t^{\delta_1} + \nabla\cdot(\overline{uu^T}^{\delta_1}) - \frac{1}{\mathrm{Re}}\Delta\overline{u}^{\delta_1} - S\nabla\cdot(\overline{BB^T}^{\delta_1}) + \nabla\Big(\frac{S}{2}\overline{B^2}^{\delta_1} + \overline{p}^{\delta_1}\Big) = \overline{f}^{\delta_1},\\
&\overline{B}_t^{\delta_2} + \frac{1}{\mathrm{Re_m}}\operatorname{curl}(\operatorname{curl}\overline{B}^{\delta_2}) + \nabla\cdot(\overline{Bu^T}^{\delta_2}) - \nabla\cdot(\overline{uB^T}^{\delta_2}) = \operatorname{curl}\overline{g}^{\delta_2},\\
&\nabla\cdot\overline{u}^{\delta_2} = 0, \quad \nabla\cdot\overline{B}^{\delta_2} = 0.
\end{aligned} \tag{1.4}$$

The usual closure problem which we study here arises because $\overline{uu^T}^{\delta_1} \neq \overline{u}^{\delta_1}\,\overline{u}^{\delta_1}$, $\overline{BB^T}^{\delta_1} \neq \overline{B}^{\delta_1}\,\overline{B}^{\delta_1}$, $\overline{uB^T}^{\delta_2} \neq \overline{u}^{\delta_1}\,\overline{B^T}^{\delta_2}$. To isolate the turbulence closure problem from the difficult problem of wall laws for near wall turbulence, we study (1.1) hence (1.4) subject to (1.3). The closure problem is to replace the tensors $\overline{uu^T}^{\delta_1}$, $\overline{BB^T}^{\delta_1}$, $\overline{uB^T}^{\delta_2}$ with tensors $\mathscr{T}(\overline{u}^{\delta_1}, \overline{u}^{\delta_1})$, $\mathscr{T}(\overline{B}^{\delta_2}, \overline{B}^{\delta_2})$, $\mathscr{T}(\overline{u}^{\delta_1}, \overline{B}^{\delta_2})$, respectively, depending only on $\overline{u}^{\delta_1}, \overline{B}^{\delta_2}$ and not u, B. There are many closure models proposed in large eddy simulation reflecting the centrality of closure in turbulence simulation. Calling w, q, W the resulting approximations to $\overline{u}^{\delta_1}, \overline{p}^{\delta_1}, \overline{B}^{\delta_2}$, we are led to considering the following model

$$
\begin{aligned}
&w_t + \nabla\cdot\mathscr{T}(w,w) - \frac{1}{\mathrm{Re}}\Delta w - S\mathscr{T}(W,W) + \nabla q = \overline{f}^{\delta_1}\\
&W_t + \frac{1}{\mathrm{Re_m}}\operatorname{curl}(\operatorname{curl} W) + \nabla\cdot\mathscr{T}(w,W) - \nabla\cdot\mathscr{T}(W,w) = \operatorname{curl}\overline{g}^{\delta_2},\\
&\nabla\cdot w = 0, \quad \nabla\cdot W = 0.
\end{aligned}
\tag{1.5}
$$

With any reasonable averaging operator, the true averages $\overline{u}^{\delta_1}, \overline{B}^{\delta_2}, \overline{p}^{\delta_1}$ are smoother than u, B, p. We consider the simplest, accurate closure model that is exact on constant flows (i.e., $\overline{u}^{\delta_1} = u, \overline{B}^{\delta_2} = B$) is

$$
\begin{aligned}
\overline{uu^T}^{\delta_1} &\approx \overline{\overline{u}^{\delta_1}\,\overline{u^T}^{\delta_1}}^{\delta_1} =: \mathscr{T}(\overline{u}^{\delta_1}, \overline{u}^{\delta_1}),\\
\overline{BB^T}^{\delta_1} &\approx \overline{\overline{B}^{\delta_2}\,\overline{B^T}^{\delta_2}}^{\delta_1} =: \mathscr{T}(\overline{B}^{\delta_2}, \overline{B}^{\delta_2}),\\
\overline{uB^T}^{\delta_2} &\approx \overline{\overline{u}^{\delta_1}\,\overline{B^T}^{\delta_2}}^{\delta_2} =: \mathscr{T}(\overline{u}^{\delta_1}, \overline{B}^{\delta_2}),
\end{aligned}
\tag{1.6}
$$

leading to

$$
w_t + \nabla\cdot(\overline{ww^T}^{\delta_1}) - \frac{1}{\mathrm{Re}}\Delta w - S\nabla\cdot(\overline{W\,W^T}^{\delta_1}) + \nabla q = \overline{f}^{\delta_1}, \tag{1.7a}
$$

$$
W_T + \frac{1}{\mathrm{Re_m}}\operatorname{curl}(\operatorname{curl} W) + \nabla\cdot(\overline{Ww^T}^{\delta_2}) - \nabla\cdot(\overline{wW^T}^{\delta_2}) = \operatorname{curl}\overline{g}^{\delta_2}, \tag{1.7b}
$$

$$
\nabla\cdot w = 0, \quad \nabla\cdot W = 0, \tag{1.7c}
$$

subject to $w(x,0) = \overline{u}_0^{\delta_1}(x), W(x,0) = \overline{B}_0^{\delta_2}(x)$ and periodic boundary conditions (with zero means).

The first to introduce a regularization of the 3D Navier-Stokes equations was Leray [29], who proved that its solution converge to the weak solution of the 3D NSE. Recently such analysis was done for numerous regularizations in [27]. For the MHD turbulence, Linshiz and Titi [32] studied the NS-α regularization of the momentum equation, with no averaging of the other MHD system's couple equations. The Lagrangian-averaged magnetohydrodynamics-α model proposed in [19] is also conserving the Alfvén waves.

In this report we show that the LES MHD model (1.7) has the mathematical properties (conservation of kinetic energy, magnetic helicity, approximate conservation of the cross helicity, preservation of Alfvén waves) expected of a model derived from the MHD equations by an averaging operation.

The model considered can be developed for quite general averaging operators, see e.g. [1, 42, 25, 9, 30, 31]. The choice of averaging operator in (1.7) is a differential filter due to Germano [17]. Let the $\delta > 0$ denote the averaging radius, related to the finest computationally feasible mesh. (We use different lengthscales for the Navier-Stokes and Maxwell equations, see e.g. [40] for the treatment of large eddy simulation of stratified flows). Given $\phi \in L_0^2(\Omega)$, $\overline{\phi}^\delta \in H^2(\Omega)\cap L_0^2(\Omega)$ is the unique solution of

$$
A_\delta\overline{\phi}^\delta := -\delta^2\Delta\overline{\phi}^\delta + \overline{\phi}^\delta = \phi \quad \text{in } \Omega, \tag{1.8}
$$

subject to periodic boundary conditions. Under periodic boundary conditions, this averaging operator commutes with differentiation, and with this averaging operator, the model (1.6) has consistency $O(\delta^2)$, i.e.,

$$\begin{aligned}
\overline{uu^T}^{\delta_1} &= \overline{\overline{u}^{\delta_1}\,\overline{u^T}^{\delta_1}} + O({\delta_1}^2),\\
\overline{BB^T}^{\delta_1} &= \overline{\overline{B}^{\delta_2}\,\overline{B^T}^{\delta_2}}^{\delta_1} + O({\delta_2}^2),\\
\overline{uB^T}^{\delta_2} &= \overline{\overline{u}^{\delta_1}\,\overline{B^T}^{\delta_2}}^{\delta_2} + O({\delta_1}^2+{\delta_2}^2),
\end{aligned}$$

for smooth u, B. We prove that the model (1.7) has a unique, weak solution w, W that converges in the appropriate sense $w \to u$, $W \to B$, as $\delta_1, \delta_2 \to 0$.

In Section 2 we address the question of global existence and uniqueness of the solution for the closed MHD model. Section 2.3 treats the limit consistency of the model and verifiability. The conservation of the kinetic energy and helicity for the approximate deconvolution model is presented in Section 3. Section 4 shows that the model preserves the Alfén waves, with the velocity tending to the velocity of Alfvén waves in the MHD, as the radii δ_1, δ_2 tend to zero. Finally, Section 5 presents the computational results: we apply the LES-MHD model to the two-dimensional Chorin's problem and verify the predicted accuracy of the model. We also compare the conserved quantities: plot the energy of the model vs. the energy of the averaged MHD.

2. Well-posedness of the LES-MHD model.

2.1. Notations and preliminaries. We shall use the standard notations for function spaces in the space periodic case (see [51]). Let $H_p^m(\Omega)$ denote the space of functions (and their vector valued counterparts also) that are locally in $H^m(\mathbb{R}^3)$, are periodic of period L and have zero mean, i.e. satisfy (1.3). We recall the solenoidal space $\mathscr{D}(\Omega) = \{\phi \in C^\infty(\Omega) : \phi \text{ periodic with zero mean}, \nabla\cdot\phi = 0\}$, and the closures of $\mathscr{D}(\Omega)$ in the usual $L^2(\Omega)$ and $H^1(\Omega)$ norms:

$$H = \{\phi \in H_2^0(\Omega), \nabla\cdot\phi = 0 \text{ in } \mathscr{D}(\Omega)'\}^2, \quad V = \{\phi \in H_2^1(\Omega), \nabla\cdot\phi = 0 \text{ in } \mathscr{D}(\Omega)'\}^2.$$

Definition 2.1. *Let* $(\overline{u_0}^{\delta_1}, \overline{B_0}^{\delta_2}) \in H$, $\overline{f}^{\delta_1}, curl\overline{g}^{\delta_2} \in L^2(0,T;V')$. *The measurable functions* $w, W : [0,T]\times\Omega \to \mathbb{R}^3$ *are the weak solutions of (1.7) if* $w, W \in L^2(0,T;V)\cap L^\infty(0,T;H)$, *and* w, W *satisfy*

$$\begin{aligned}
&\int_\Omega w(t)\phi dx + \int_0^t\int_\Omega \frac{1}{\mathrm{Re}}\nabla w(\tau)\nabla\phi + \overline{w(\tau)\cdot\nabla w(\tau)}^{\delta_1}\phi - S\overline{W(\tau)\cdot\nabla W(\tau)}^{\delta_1}\phi\, dxd\tau\\
&\qquad = \int_\Omega \overline{u_0}^{\delta_1}\phi dx + \int_0^t\int_\Omega \overline{f(\tau)}^{\delta_1}\phi\, dxd\tau, \qquad (2.1)\\
&\int_\Omega W(t)\psi dx + \int_0^t\int_\Omega \frac{1}{\mathrm{Re_m}}\nabla W(\tau)\nabla\psi + \overline{w(\tau)\cdot\nabla W(\tau)}^{\delta_2}\psi - \overline{W(\tau)\cdot\nabla w(\tau)}^{\delta_2}\psi\, dxd\tau\\
&\qquad = \int_\Omega \overline{B_0}^{\delta_2}\psi dx + \int_0^t\int_\Omega curl\overline{g(\tau)}^{\delta_2}\psi\, dxd\tau,
\end{aligned}$$

$\forall t \in [0,T), \phi, \psi \in \mathscr{D}(\Omega)$.

Also, it is easy to show that for any $u, v \in H^1(\Omega)$ with $\nabla\cdot u = \nabla\cdot v = 0$, the following identity holds

$$\nabla\times(u\times v) = v\cdot\nabla u - u\cdot\nabla v. \qquad (2.2)$$

2.2. Existence and uniqueness of a solution. The first result states that the weak solution of the MHD LES model (1.7) exists globally in time, for large data and general $\mathrm{Re}, \mathrm{Re}_m > 0$ and that it satisfies an energy equality while initial data and the source terms are smooth enough.

THEOREM 2.2. *Let $\delta_1, \delta_2 > 0$ be fixed. For any $(\overline{u_0}^{\delta_1}, \overline{B_0}^{\delta_2}) \in V$ and $(\overline{f}^{\delta_1}, curl\,\overline{g}^{\delta_2}) \in L^2(0,T;H)$, there exists a unique weak solution w, W to (1.7). The weak solution also belongs to $L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$ and $w_t, W_t \in L^2((0,T)\times\Omega)$. Moreover, the following energy equality holds for $t \in [0,T]$:*

$$\mathscr{E}(t) + \int_0^t \varepsilon(\tau)d\tau = \mathscr{E}(0) + \int_0^t \mathscr{P}(\tau)d\tau, \tag{2.3}$$

where

$$\begin{aligned}
&\mathscr{E}(t) = \frac{{\delta_1}^2}{2}\|\nabla w(t,\cdot)\|_0^2 + \frac{1}{2}\|w(t,\cdot)\|_0^2 + \frac{{\delta_2}^2 S}{2}\|\nabla W(t,\cdot)\|_0^2 + \frac{S}{2}\|W(t,\cdot)\|_0^2,\\
&\varepsilon(t) = \frac{{\delta_1}^2}{\mathrm{Re}}\|\Delta w(t,\cdot)\|_0^2 + \frac{1}{\mathrm{Re}}\|\nabla w(t,\cdot)\|_0^2 + \frac{{\delta_2}^2 S}{\mathrm{Re_m}}\|\Delta W(t,\cdot)\|_0^2 + \frac{S}{\mathrm{Re_m}}\|\nabla W(t,\cdot)\|_0^2,\\
&\mathscr{P}(t) = (f(t), w(t)) + S(curl\, g(t), W(t)).
\end{aligned} \tag{2.4}$$

The proof, using the semigroup approach proposed in [6] for the Navier-Stokes equations, is given in the Appendix, along with a regularity result.

REMARK 2.1. *The pressure is recovered from the weak solution via the classical DeRham theorem (see [29]).*

2.3. Accuracy of the model. We address now the question of consistency, i.e., we show that when δ_1, δ_2 go to zero, the solution of the closed model (1.7) converges to a weak solution of the MHD equations (1.1).

THEOREM 2.3. *For any two sequences $\delta_1^n, \delta_2^n \to 0$ as $n \to \infty$, the corresponding solution of (1.7) satisfies*

$$(w_{\delta_1^n}, W_{\delta_2^n}, q_{\delta_1^n}) \to (u, B, p),$$

where $(u,B,p) \in L^\infty(0,T;H) \cap L^2(0,t;V) \times L^{\frac{4}{3}}(0,T;L^2(\Omega))$ is a weak solution of the MHD equations (1.1). The sequences $\{w_{\delta_1^n}\}_{n\in\mathbb{N}}, \{W_{\delta_2^n}\}_{n\in\mathbb{N}}$ converge strongly to u, B in $L^{\frac{4}{3}}(0,T;L^2(\Omega))$ and weakly in $L^2(0,T;H^1(\Omega))$, respectively, while $\{q_{\delta_1^n}\}_{n\in\mathbb{N}}$ converges weakly to p in $L^{\frac{4}{3}}(0,T;L^2(\Omega))$.

Proof. The proof follows that of Theorem 3.1 in [27], and is an easy consequence of Theorem 2.4 and Proposition 2.6.□

Let $\tau_u, \tau_B, \tau_{Bu}$ denote the model's consistency errors

$$\tau_u = \overline{u}^{\delta_1}\overline{u}^{\delta_1} - uu, \quad \tau_B = \overline{B}^{\delta_2}\overline{B}^{\delta_2} - BB, \quad \tau_{Bu} = \overline{B}^{\delta_2}\overline{u}^{\delta_1} - Bu, \tag{2.5}$$

where u, B is a solution of the MHD equations obtained as a limit of a subsequence of the sequence $w_{\delta_1}, W_{\delta_2}$. We prove that $\|\overline{u}^{\delta_1} - w\|_{L^\infty(0,T;L^2(Q))}$, $\|\overline{B}^{\delta_2} - W\|_{L^\infty(0,T;L^2(Q))}$ are bounded by $\|\tau_u\|_{L^2(Q_T)}$, $\|\tau_B\|_{L^2(Q_T)}$, $\|\tau_{Bu}\|_{L^2(Q_T)}$.

THEOREM 2.4. *Under the assumption* $(u,B)\in L^4(0,T;V)$, *the errors* $e=\overline{u}^{\delta_1}-w$, $E=\overline{B}^{\delta_2}-W$ *satisfy*

$$
\begin{aligned}
&\|e(t)\|_0^2+S\|E(t)\|_0^2+\int_0^t\left(\frac{1}{\mathrm{Re}}\|\nabla e(s)\|_0^2+\frac{S}{\mathrm{Re_m}}\|curl\,E(s)\|_0^2\right)ds\\
&\quad\le C\Phi(t)\int_0^t\left(\mathrm{Re}\|\tau_u(s)+S\tau_B(s)\|_0^2+\mathrm{Re_m}\|\tau_{Bu}(s)-{\tau_{Bu}}^T(s)\|_0^2\right)ds,
\end{aligned}
\tag{2.6}
$$

where $\Phi(t)=\exp\left\{\mathrm{Re}^3\int_0^t\|\nabla u\|_0^4ds,{\mathrm{Re_m}}^3\int_0^t\|\nabla u\|_0^4ds+\mathrm{Re_m}\mathrm{Re}^2\int_0^t\|\nabla B\|_0^4\right\}$.

Proof. The errors e,E satisfy the following momentum equation

$$
\begin{aligned}
&e_t+\nabla\cdot(\overline{\overline{u}^{\delta_1}\overline{u}^{\delta_1}-ww}^{\delta_1})-\frac{1}{\mathrm{Re}}\Delta e+S\nabla\cdot(\overline{\overline{B}^{\delta_2}\overline{B}^{\delta_2}-WW}^{\delta_1})+\nabla(\overline{p}^{\delta_1}-q)\\
&\qquad=\nabla\cdot(\overline{\tau}_u^{\delta_1}+S\overline{\tau}_B^{\delta_1}),\\
&E_t+\frac{1}{\mathrm{Re_m}}\mathrm{curl}\,\mathrm{curl}\,E+\nabla\cdot(\overline{\overline{B}^{\delta_2}\overline{u}^{\delta_1}-Ww}^{\delta_2})-\nabla\cdot(\overline{\overline{u}^{\delta_1}\overline{B}^{\delta_2}-wW}^{\delta_2})\\
&\qquad=\nabla\cdot(\overline{\tau}_{Bu}^{\delta_2}-{\overline{\tau}_{Bu}^{\delta_2}}^T),
\end{aligned}
$$

along with the corresponding conservation of mass equation and homogeneous boundary conditions. Taking the inner product with $A_{\delta_1}e, SA_{\delta_2}E$, respectively, we obtain after some calculation that

$$
\begin{aligned}
&\frac{d}{dt}\left(\|e\|_0^2+S\|E\|_0^2+\delta_1^2\|\nabla e\|_0^2+S\delta_2^2\|\mathrm{curl}\,E\|_0^2\right)\\
&\quad+\frac{1}{\mathrm{Re}}\|\nabla e\|_0^2+\frac{S}{\mathrm{Re_m}}\|\mathrm{curl\ \ E}\|_0^2+\frac{\delta_1^2}{\mathrm{Re}}\|\Delta e\|_0^2+\frac{\delta_2^2S}{\mathrm{Re_m}}\|\mathrm{curl}\,\mathrm{curl}\,E\|_0^2\\
&\le\int_\Omega\Big(-e\cdot\nabla\overline{u}^{\delta_1}e-S\nabla\cdot(E\overline{B}^{\delta_2})e-S\nabla\cdot(E\overline{u}^{\delta_1})E+Se\cdot\nabla\overline{B}^{\delta_2}E\Big)dx\\
&\quad+\mathrm{Re}\|\tau_u+S\tau_B\|_0^2+\mathrm{Re_m}\|\tau_{Bu}-{\tau_{Bu}}^T\|_0^2\\
&\le C\Big(\|\nabla e\|_0^{3/2}\|e\|_0^{1/2}\|\nabla\overline{u}^{\delta_1}\|_0+2S\|E\|_0^{1/2}\|\nabla E\|_0^{1/2}\|\nabla\overline{B}^{\delta_2}\|_0\|\nabla e\|_0\\
&\qquad+S\|E\|_0^{1/2}\|\nabla E\|_0^{3/2}\|\nabla\overline{u}^{\delta_1}\|_0\Big)+\mathrm{Re}\|\tau_u+S\tau_B\|_0^2+\mathrm{Re_m}\|\tau_{Bu}-{\tau_{Bu}}^T\|_0^2.
\end{aligned}
$$

Using Young's and Gronwall's inequality we deduce

$$
\begin{aligned}
&\|e(t)\|_0^2+S\|E(t)\|_0^2+\int_0^t\left(\frac{1}{\mathrm{Re}}\|\nabla e(s)\|_0^2+\frac{S}{\mathrm{Re_m}}\|\mathrm{curl}\,E(s)\|_0^2\right)ds\\
&\le C\Psi(t)\int_0^t\left(\mathrm{Re}\|\tau_u(s)+S\tau_B(s)\|_0^2+\mathrm{Re_m}\|\tau_{Bu}(s)-{\tau_{Bu}}^T(s)\|_0^2\right)ds,
\end{aligned}
$$

where

$$
\Psi(t)=\exp\left\{\mathrm{Re}^3\int_0^t\|\nabla\overline{u}^{\delta_1}\|_0^4ds,{\mathrm{Re_m}}^3\int_0^t\|\nabla\overline{u}^{\delta_1}\|_0^4ds+\mathrm{Re_m}\mathrm{Re}^2\int_0^t\|\nabla\overline{B}^{\delta_2}\|_0^4ds\right\}.
$$

The use of the stability bounds $\|\nabla\overline{u}^{\delta_1}\|_0\le\|\nabla u\|_0$, $\|\nabla\overline{B}^{\delta_2}\|_0\le\|\nabla B\|_0$ concludes the proof. ∎

Finally we give bounds on the consistency errors (2.5) as $\delta_1, \delta_2 \to 0$ in $L^1((0,T)\times\Omega)$ and $L^2((0,T)\times\Omega)$.

PROPOSITION 2.5. *Assuming* $(f, curl\, g) \in L^2(0,T;V')$, *then*

$$\begin{aligned}
\|\tau_u\|_{L^1(0,T;L^1(\Omega))} &\le 2^{3/2}\delta_1 T^{1/2}\mathrm{Re}^{1/2}\mathscr{C}(T),\\
\|\tau_B\|_{L^1(0,T;L^1(\Omega))} &\le 2^{3/2}\delta_2 T^{1/2}\frac{\mathrm{Re_m}^{1/2}}{S}\mathscr{C}(T),\\
\|\tau_{Bu}\|_{L^1(0,T;L^1(\Omega))} &\le 2^{1/2}T^{1/2}\frac{1}{S}(\delta_1\mathrm{Re}^{1/2}+\delta_2\mathrm{Re_m}^{1/2})\mathscr{C}(T),
\end{aligned}\tag{2.7}$$

where

$$\mathscr{C}(T) = \left(\|u_0\|_0^2 + S\|B_0\|_0^2 + \mathrm{Re}\|f\|^2_{L^2(0,T;H^{-1}(\Omega))} + \frac{\mathrm{Re_m}}{S}\|\mathrm{curl}\, g\|^2_{L^2(0,T;H^{-1}(\Omega))}\right).$$

Proof. Using the stability bounds we have

$$\begin{aligned}
\|\tau_u\|_{L^1(0,T;L^1(\Omega))} &\le \|u+\overline{u}^{\delta_1}\|_{L^2(0,T;L^2(\Omega))}\|\overline{u}^{\delta_1}-u\|_{L^2(0,T;L^2(\Omega))}\\
&\le 2\|u\|_{L^2(0,T;L^2(\Omega))}\sqrt{2}\delta_1\|\nabla u\|_{L^2(0,T;L^2(\Omega))}.
\end{aligned}$$

Similarly

$$\begin{aligned}
\|\tau_B\|_{L^1(0,T;L^1(\Omega))} &\le \|B+\overline{B}^{\delta_2}\|_{L^2(0,T;L^2(\Omega))}\|\overline{B}^{\delta_2}-B\|_{L^2(0,T;L^2(\Omega))}\\
&\le 2\|B\|_{L^2(0,T;L^2(\Omega))}\sqrt{2}\delta_2\|\nabla B\|_{L^2(0,T;L^2(\Omega))},\\
\|\tau_{Bu}\|_{L^1(0,T;L^1(\Omega))} &\le \|\overline{B}^{\delta_2}-B\|_{L^2(Q)}\|\overline{u}^{\delta_1}\|_{L^2(Q)} + \|B\|_{L^2(Q)}\|\overline{u}^{\delta_1}-u\|_{L^2(Q)}\\
&\le \sqrt{2}\delta_2\|\nabla B\|_{L^2(Q)}\|u\|_{L^2(Q)} + \sqrt{2}\delta_1\|\nabla u\|_{L^2(Q)}\|B\|_{L^2(Q)}.
\end{aligned}$$

The classical energy estimates for the MHD system (1.1) will yield now (2.7). □

Assuming more regularity on (u,B) leads to the sharper bounds on the consistency errors.

REMARK 2.2. *Let* $(u,B) \in L^2(0,T;H^2(\Omega))$. *Then*

$$\begin{aligned}
\|\tau_u\|_{L^1(0,T;L^1(\Omega))} &\le C\delta_1^2,\\
\|\tau_B\|_{L^1(0,T;L^1(\Omega))} &\le C\delta_2^2,\\
\|\tau_{Bu}\|_{L^1(0,T;L^1(\Omega))} &\le C(\delta_1^2+\delta_2^2),
\end{aligned}$$

where $C = C(T, Re, Re_m, \|(u,B)\|_{L^2(0,T;L^2(\Omega))}, \|(u,B)\|_{L^2(0,T;H^2(\Omega))})$.

Proof. The result is obtained as in the proof of Proposition 2.5, using the bounds

$$\begin{aligned}
\|\overline{u}^{\delta_1}-u\|_{L^2(0,T;L^2(\Omega))} &\le \delta_1^2\|\Delta u\|_{L^2(0,T;L^2(\Omega))},\\
\|\overline{B}^{\delta_2}-B\|_{L^2(0,T;L^2(\Omega))} &\le \delta_2^2\|\Delta B\|_{L^2(0,T;L^2(\Omega))},
\end{aligned}$$

which follow from (1.8). □

Next we estimate the L^2-norms of the consistency errors $\tau_u, \tau_B, \tau_{Bu}$, which were used in Theorem 2.4 to estimate the filtering errors e, E.

PROPOSITION 2.6. *If the solution* u, B *of (1.1) satisfies*

$$(u,B) \in L^4((0,T)\times\Omega) \cap L^2(0,T;H^2(\Omega)),$$

then the model consistency errors satisfy the following bound

$$\|\tau_u\|_{L^2(Q)} \leq C\delta_1, \quad \|\tau_B\|_{L^2(Q)} \leq C\delta_2, \quad \|\tau_{Bu}\|_{L^2(Q)} \leq C(\delta_1 + \delta_2),$$

where $C = C(\|(u,B)\|_{L^4((0,T)\times\Omega)}, \|(u,B)\|_{L^2(0,T;H^2(\Omega))})$.

Proof. As in the proof of Proposition 2.5, using the stability bounds we have

$$\begin{aligned}
\|\tau_u\|_{L^2(Q)} &\leq 2\|u\|_{L^4(Q)}\|\overline{u}^{\delta_1} - u\|_{L^4(Q)} \\
&\leq 2^{3/2}\|u\|_{L^4(Q)}\Big(\int_0^T \|\overline{u}^{\delta_1} - u\|_{L^2(\Omega)}\|\nabla(\overline{u}^{\delta_1} - u)\|^3_{L^2(\Omega)}dt\Big)^{1/4} \\
&\leq 2^{3/2}\|u\|_{L^4(Q)}\Big(\int_0^T 4\delta_1^4\|\nabla u\|_{L^2(\Omega)}\|\Delta u\|^3_{L^2(\Omega)}dt\Big)^{1/4} \\
&\leq 4\delta_1\|u\|_{L^4(Q)}\|u\|_{L^2(0,T;H^1(\Omega))}\|u\|_{L^2(0,T;H^2(\Omega))}.
\end{aligned}$$

Similarly we deduce

$$\|\tau_B\|_{L^2(Q)} \leq 4\delta_2\|B\|_{L^4(Q)}\|B\|_{L^2(0,T;H^1(\Omega))}\|B\|_{L^2(0,T;H^2(\Omega))},$$

and

$$\begin{aligned}
\|\tau_{Bu}\|_{L^2(Q)} &\leq \|u\|_{L^4(Q)}\|\overline{B}^{\delta_2} - B\|_{L^4(Q)} + \|B\|_{L^4(Q)}\|\overline{u}^{\delta_2} - u\|_{L^4(Q)} \\
&\leq 2\delta_2\|u\|_{L^4(Q)}\|B\|_{L^2(0,T;H^1(\Omega))}\|B\|_{L^2(0,T;H^2(\Omega))} \\
&\quad + 2\delta_1\|B\|_{L^4(Q)}\|u\|_{L^2(0,T;H^1(\Omega))}\|u\|_{L^2(0,T;H^2(\Omega))}.
\end{aligned}$$

□

As in Remark 2.2, assuming extra regularity on (u,B) leads to the sharper bounds.

REMARK 2.3. *Let*

$$(u,B) \in L^4((0,T)\times\Omega) \cap L^4(0,T;H^2(\Omega)).$$

Then

$$\|\tau_u\|_{L^2(Q)} \leq C\delta_1^2, \quad \|\tau_B\|_{L^2(Q)} \leq C\delta_2^2, \quad \|\tau_{Bu}\|_{L^2(Q)} \leq C(\delta_1^2 + \delta_2^2),$$

where $C = C(\|(u,B)\|_{L^4((0,T)\times\Omega)}, \|(u,B)\|_{L^4(0,T;H^2(\Omega))})$.

3. Conservation laws. It is well known that kinetic energy and helicity are critical in the organization of the flow. We prove now that the model (1.7) inherits some of the original properties of the 3D MHD equations (1.1), namely it conserves the kinetic energy, magnetic helicity and approximates the cross helicity.

The energy $E = \frac{1}{2}\int_\Omega (u(x)\cdot u(x) + SB(x)\cdot B(x))dx$, the cross helicity $H_C = \frac{1}{2}\int_\Omega (u(x)\cdot B(x))dx$ and the magnetic helicity $H_M = \frac{1}{2}\int_\Omega (\mathbb{A}(x)\cdot B(x))dx$ (where $\mathbb{A}$ is the vector potential, $B = \nabla\times\mathbb{A}$) are the three invariants of the MHD equations (1.1) (see e.g., [14]) in the absence of kinematic viscosity and magnetic diffusivity ($\frac{1}{\mathrm{Re}} = \frac{1}{\mathrm{Re_m}} = 0$). Let introduce the characteristic quantities of the model

$$\begin{aligned}
E_{LES} &= \frac{1}{2}[(A_{\delta_1}w, w) + S(A_{\delta_2}W, W)], \\
H_{C,LES} &= \frac{1}{2}(A_{\delta_1}w, A_{\delta_2}W), \\
H_{M,LES} &= \frac{1}{2}(A_{\delta_2}W, \overline{\mathbb{A}}^{\delta_2}), \text{ where } \overline{\mathbb{A}}^{\delta_2} = A_{\delta_2}^{-1}\mathbb{A}.
\end{aligned}$$

This section is devoted to proving that these quantities are conserved by (1.7) with the periodic boundary conditions and $\frac{1}{\mathrm{Re}} = \frac{1}{\mathrm{Re_m}} = 0$. Also, note that

$$E_{LES} \to E,\ H_{C,LES} \to H_C,\ H_{M,LES} \to H_M, \qquad \text{as } \delta_{1,2} \to 0.$$

THEOREM 3.1. *The following conservation laws hold,* $\forall T > 0$

$$E_{LES}(T) = E_{LES}(0), \tag{3.1a}$$

$$H_{C,LES}(T) = H_{C,LES}(0) + C(T) \max_{i=1,2} \delta_i^2, \tag{3.1b}$$

$$H_{M,LES}(T) = H_{M,LES}(0). \tag{3.1c}$$

Proof. Consider (1.7) with $\frac{1}{\mathrm{Re}} = \frac{1}{\mathrm{Re_m}} = 0$. Multiplying (1.7a), (1.7b) by $A_{\delta_1} w$ and $SA_{\delta_2} W$, respectively, and using the identity

$$((\nabla \times v) \times u, w) = (u \cdot \nabla v, w) - (w \cdot \nabla v, u). \tag{3.2}$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\left[(A_{\delta_1} w, w) + S(A_{\delta_2} W, W)\right] = S(W \cdot \nabla W, w) - S(w \cdot \nabla W, W) + S(W \cdot \nabla w, W),$$

which by (1.7c) yields (3.1a):

$$\frac{1}{2}\frac{d}{dt}[(A_{\delta_1} w, w) + S(A_{\delta_2} W, W)] = 0. \tag{3.3}$$

To prove (3.1b), multiply (1.7a), (1.7b) by $A_{\delta_1} W$ and $A_{\delta_2} w$, respectively, and use the identity $(u \cdot \nabla v, w) = -(u \cdot \nabla w, v)$ to get

$$(\frac{\partial A_{\delta_1} w}{\partial t}, W) + (\frac{\partial A_{\delta_2} W}{\partial t}, w) = 0. \tag{3.4}$$

Recall that from (1.8) we have

$$w = A_{\delta_1} w + \delta_1^2 \Delta w, \qquad W = A_{\delta_2} W + \delta_2^2 \Delta W. \tag{3.5}$$

Then (3.4) gives

$$(\frac{\partial A_{\delta_1} w}{\partial t}, A_{\delta_2} W) + (\frac{\partial A_{\delta_2} W}{\partial t}, A_{\delta_1} w) = (\frac{\partial A_{\delta_1} w}{\partial t}, \delta_2^2 \Delta W) + (\frac{\partial A_{\delta_2} W}{\partial t}, \delta_1^2 \Delta w). \tag{3.6}$$

Hence,

$$\frac{d}{dt}(A_{\delta_1} w, A_{\delta_2} W) = \delta_2^2 (\frac{\partial A_{\delta_1} w}{\partial t}, \Delta W) + \delta_1^2 (\frac{\partial A_{\delta_2} W}{\partial t}, \Delta w), \tag{3.7}$$

which proves (3.1b).

Next, we multiply (1.7b) by $A_{\delta_2} \overline{\mathbb{A}}^{\delta_2}$, and integrating over Ω

$$\frac{1}{2}\frac{d}{dt}(\nabla \times A_{\delta_2} \overline{\mathbb{A}}^{\delta_2}, \overline{\mathbb{A}}^{\delta_2}) + (w \cdot \nabla W, \overline{\mathbb{A}}^{\delta_2}) - (W \cdot \nabla w, \overline{\mathbb{A}}^{\delta_2}) = 0. \tag{3.8}$$

Since the cross-product of two vectors is orthogonal to each of them,

$$((\nabla \times \overline{\mathbb{A}}^{\delta_2}) \times w, \nabla \times \overline{\mathbb{A}}^{\delta_2}) = 0,$$

it follows from (3.9) and (3.2) that

$$(w \cdot \nabla \overline{\mathbb{A}}^{\delta_2}, \nabla \times \overline{\mathbb{A}}^{\delta_2}) = ((\nabla \times \overline{\mathbb{A}}^{\delta_2}) \cdot \nabla \overline{\mathbb{A}}^{\delta_2}, w). \tag{3.9}$$

Since $W = \nabla \times \overline{\mathbb{A}}^{\delta_2}$, we obtain from (3.8) and (3.9) that (3.1c) holds. $\square$

4. Alfvén waves. In this section we prove that our model possesses a very important property of the MHD, namely the ability of the magnetic field to transmit transverse inertial waves - Alfvén waves. We follow the argument typically used to prove the existence of Alfvén waves in MHD, see, e.g., [13].

Using the density ρ and permeability μ, we write the equations of the model (1.7) in the form

$$w_t + \nabla \cdot (\overline{ww^T}^{\delta_1}) + \nabla \overline{p}^{\delta_1} = \frac{1}{\rho\mu}\overline{(\nabla \times W) \times W}^{\delta_1} - \nu \nabla \times (\nabla \times w), \tag{4.1a}$$

$$\frac{\partial W}{\partial t} = \overline{\nabla \times (w \times W)}^{\delta_2} - \eta \nabla \times (\nabla \times W), \tag{4.1b}$$

$$\nabla \cdot w = 0, \quad \nabla \cdot W = 0, \tag{4.1c}$$

where $\nu = \frac{1}{\mathrm{Re}}$, $\eta = \frac{1}{\mathrm{Re_m}}$.

Assume a uniform, steady magnetic field W_0, perturbed by a small velocity field w. We denote the perturbations in current density and magnetic field by j_{model} and W_p, with

$$\nabla \times W_p = \mu j_{model}. \tag{4.2}$$

Also, the vorticity of the model is

$$\omega_{model} = \nabla \times w. \tag{4.3}$$

Since $w \cdot \nabla w$ is quadratic in the small quantity w, it can be neglected in the Navier-Stokes equation (4.1a), and therefore

$$\frac{\partial w}{\partial t} + \nabla \overline{p}^{\delta_1} = \frac{1}{\rho\mu}\overline{(\nabla \times W_p) \times W_0}^{\delta_1} - \nu \nabla \times (\nabla \times w). \tag{4.4}$$

The leading order terms in the induction equation (4.1b) are

$$\frac{\partial W_p}{\partial t} = \overline{\nabla \times (w \times W_0)}^{\delta_2} - \eta \nabla \times (\nabla \times W_p). \tag{4.5}$$

Using (4.2), we rewrite (4.4) as

$$\frac{\partial w}{\partial t} + \nabla \overline{p}^{\delta_1} = \frac{1}{\rho}\overline{j_{model} \times W_0}^{\delta_1} + \nu \Delta w. \tag{4.6}$$

Taking the *curl* of (4.6), using the identity (2.2) and $\nabla W_0 = 0$, we obtain from (4.3) that

$$\frac{\partial \omega_{model}}{\partial t} = \frac{1}{\rho}\overline{W_0 \cdot \nabla j_{model}}^{\delta_1} + \nu \Delta \omega_{model}. \tag{4.7}$$

Similarly, taking *curl* of (4.5) and using (4.2),(4.3) yields

$$\mu \frac{\partial j_{model}}{\partial t} = \overline{W_0 \cdot \nabla \omega_{model}}^{\delta_2} + \eta\mu \Delta j_{model}. \tag{4.8}$$

We now eliminate j_{model} from (4.7) by taking the time derivative of (4.7) and substituting for $\frac{\partial j_{model}}{\partial t}$ using (4.8). This yields

$$\frac{\partial^2 \omega_{model}}{\partial t^2} = \frac{1}{\rho}\overline{W_0 \cdot \nabla \Big(\frac{1}{\mu}\overline{W_0 \cdot \nabla \omega_{model}}^{\delta_2} + \eta \Delta j_{model}\Big)}^{\delta_1} + \nu \Delta \frac{\partial \omega_{model}}{\partial t}. \tag{4.9}$$

The linearity of $A_{\delta_1}^{-1}$ implies

$$\frac{\partial^2 \omega_{model}}{\partial t^2} = \frac{1}{\rho\mu}\overline{W_0 \cdot \nabla(\overline{W_0 \cdot \nabla\omega_{model}}^{\delta_2})}^{\delta_1} + \frac{\eta}{\rho}\overline{W_0 \cdot \nabla(\Delta j_{model})}^{\delta_1} + \nu\Delta\frac{\partial \omega_{model}}{\partial t}. \quad (4.10)$$

To eliminate the term containing Δj_{model} from (4.10), we take the Laplacian of (4.7):

$$\Delta\frac{\partial \omega_{model}}{\partial t} = \frac{1}{\rho}\overline{W_0 \cdot \nabla(\Delta j_{model})}^{\delta_1} + \nu\Delta^2\omega_{model}. \quad (4.11)$$

Then from (4.10)-(4.11) we obatin

$$\frac{\partial^2 \omega_{model}}{\partial t^2} = \frac{1}{\rho\mu}\overline{W_0 \cdot \nabla(\overline{W_0 \cdot \nabla\omega_{model}}^{\delta_2})}^{\delta_1} + (\eta+\nu)\Delta\frac{\partial \omega_{model}}{\partial t} - \eta\nu\Delta^2\omega_{model}. \quad (4.12)$$

Next we look for plane-wave solutions of the form

$$\omega_{model} \sim \omega_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\theta t)}, \quad (4.13)$$

where $\mathbf{k}$ is the wavenumber. It follows from (4.13) that

$$\frac{\partial \omega_{model}}{\partial t} = -i\theta\omega_{model}, \qquad \frac{\partial^2 \omega_{model}}{\partial t^2} = -\theta^2\omega_{model},$$
$$\Delta\frac{\partial \omega_{model}}{\partial t} = i\theta k^2\omega_{model}, \qquad \Delta^2(\omega_{model}) = k^4\omega_{model}.$$

The substitution of (4.13) into the wave equation (4.12) gives

$$-\theta^2\omega_{model} = \frac{1}{\rho\mu}\overline{W_0 \cdot \nabla(\overline{W_0 \cdot \nabla\omega_{model}}^{\delta_2})}^{\delta_1} + (\eta+\nu)i\theta k^2\omega_{model} - \eta\nu k^4\omega_{model}. \quad (4.14)$$

Note that by (1.8) we have

$$\overline{W_0 \cdot \nabla\omega_{model}}^{\delta_2} = W_0 \cdot \nabla\omega_{model} + O(\delta_2^2),$$
$$\overline{W_0 \cdot \nabla(\overline{W_0 \cdot \nabla\omega_{model}}^{\delta_2})}^{\delta_1} = (W_0\cdot\nabla)^2\omega_{model} + O(\delta_1^2) + O(\delta_2^2),$$

therefore

$$\begin{aligned}-\theta^2\omega_{model} = &\frac{1}{\rho\mu}(W_0\cdot\nabla)^2\omega_{model} + (\eta+\nu)i\theta k^2\omega_{model} \\ &- \eta\nu k^4\omega_{model} + O(\delta_1^2+\delta_2^2).\end{aligned} \quad (4.15)$$

It follows from (4.13) that

$$(W_0\cdot\nabla)^2\omega_{model} = -W_0^2 k_{||}^2\omega_{model}, \quad (4.16)$$

where $k_{||}$ is the component of $\mathbf{k}$ parallel to W_0, which by (4.15) implies

$$-\theta^2 = -\frac{W_0^2 k_{||}^2}{\rho\mu} + (\eta+\nu)i\theta k^2 - \eta\nu k^4 + O(\delta_1^2+\delta_2^2).$$

Solving this quadratic equation for θ gives the dispersion relationship

$$\theta = -\frac{(\eta+\nu)k^2}{2}i \pm \Big(\sqrt{\frac{W_0^2 k_{||}^2}{\rho\mu} - \frac{(\nu-\eta)^2 k^4}{4}} + O(\delta_1^2+\delta_2^2)\Big).$$

Hence, for a perfect fluid ($\nu = \eta = 0$) we obtain

$$\theta = \pm \tilde{v}_a k_{||},$$
$$\tilde{v}_a = v_a + O(\delta_1^2 + \delta_2^2),$$

where v_a is the Alfvén velocity $W_0/\sqrt{\rho\mu}$.

When $\nu = 0$ and η is small (i.e. for high Re_m) we have

$$\theta = \pm \tilde{v}_a k_{||} - \frac{\eta k^2}{2} i,$$

which represents a transverse wave with a group velocity equal to $\pm v_a + O(\delta_1^2 + \delta_2^2)$.

In conclusion, model (1.7) preserves the Alfvén waves and the group velocity of the waves $\tilde{v}_a$ tends to the true Alfvén velocity v_a as the radii tend to zero.

5. Computational results. In this section we present computational results for the LES-MHD model. We look at the two-dimensional Chorin's model (circular motion in a square) of electrically conducting fluid with the magnetic field directed along the diagonal and increasing in time. We compare the solution obtained by the LES-MHD model to the average of the known true solution. The convergence rates are presented; we also compare the energy of the model to the energy of the averaged MHD. We take the filtering widths $\delta_1 = \delta_2 = h$ to verify the acclaimed second order accuracy of the model; this is also a typical choice of filtering widths in real life applications.

Consider the MHD flow in $\Omega = (0.5, 1.5) \times (0.5, 1.5)$. The Reynolds number and magnetic Reynolds number are $Re = 10^5, Re_m = 10^5$, the final time is $T = 1/4$, and the averaging radii are $\delta_1 = \delta_2 = h$.

Take

$$f = \begin{pmatrix} \frac{1}{2}\pi \sin(2\pi x) e^{-4\pi^2 t/Re} - x e^{2t} \\ \frac{1}{2}\pi \sin(2\pi y) e^{-4\pi^2 t/Re} - y e^{2t} \end{pmatrix},$$
$$\nabla \times g = \begin{pmatrix} e^t (x - (\cos \pi x \sin \pi y + \pi x \sin \pi x \sin \pi y + \pi y \cos \pi x \cos \pi y) e^{-2\pi^2 t/Re}) \\ e^t (-y - (\sin \pi x \cos \pi y + \pi x \cos \pi x \cos \pi y + \pi y \sin \pi x \sin \pi y) e^{-2\pi^2 t/Re}) \end{pmatrix}.$$

The solution to this problem is

$$u = \begin{pmatrix} -\cos(\pi x) \sin(\pi y) e^{-2\pi^2 t/Re} \\ \sin(\pi x) \cos(\pi y) e^{-2\pi^2 t/Re} \end{pmatrix},$$
$$p = -\frac{1}{2}(\cos(2\pi x) + \cos(2\pi y)) e^{-4\pi^2 t/Re},$$
$$B = \begin{pmatrix} xe \\ -ye \end{pmatrix}.$$

Hence, although the theoretical results were obtained only for the periodic boundary conditions, we apply the LES-MHD model to the problem with Dirichlet boundary conditions.

The results presented were obtained by using the software $FreeFEM++$. The velocity and magnetic field are sought in the finite element space of piecewise quadratic polynomials, and the pressure in the space of piecewise linears. In order to draw conclusions about the convergence rate, we take the time step $\Delta t = h^2$. We compare

Table 5.1
Approximating the average solution, $Re = 10^5$, $Re_m = 10^5$.

h	$\|w-\bar{u}\|_{L^2(0,T;L^2(\Omega))}$	*rate*	$\|W-\bar{B}\|_{L^2(0,T;L^2(\Omega))}$	*rate*
1/4	0.0247837		0.0253257	
1/8	0.0245241	0.0152	0.0268628	-0.085
1/16	0.0131042	0.9042	0.0132399	1.0207
1/32	0.00434599	1.5923	0.00412013	1.6841
1/64	0.00120907	1.8458	0.001116	1.8844

the solutions (w, W), obtained by the model, to the average of the true solution $(\bar{u}, \bar{B})$. The second order accuracy in approximating the the averaged solution $(\bar{u}, \bar{B})$ is expected, according to the Theorem 2.4 and Remark 2.3.

Hence, the computational results verify the claimed accuracy of the model.

Since the flow is not ideal (nonzero power input, nonzero viscosity/magnetic diffusivity, non-periodic boundary conditions), the energy is not conserved. But we expect the energy of the model to approximate the energy of the averaged MHD.

Indeed, Figure 5.1 shows that the graph of the model's energy is hardly distinguishable from that of the averaged MHD.

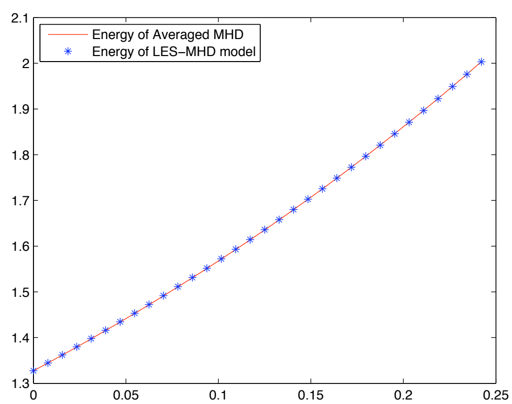

Fig. 5.1. *LES-MHD Energy vs. averaged MHD*

6. Appendix. We use the semigroup approach, based on the machinery of nonlinear differential equations of accretive type in Banach spaces.

We define the operator $\mathscr{A} \in \mathscr{L}(V, V')$ by setting

$$\langle \mathscr{A}(w_1, W_1), (w_2, W_2) \rangle = \int_\Omega \left(\frac{1}{\mathrm{Re}} \nabla w_1 \cdot \nabla w_2 + \frac{S}{\mathrm{Re_m}} \mathrm{curl}\, W_1 \mathrm{curl}\, W_2 \right) dx, \qquad (6.1)$$

for all $(w_i, W_i) \in V$. The operator $\mathscr{A}$ is an unbounded operator on H, with the domain $D(\mathscr{A}) = \{(w, W) \in V; (\Delta w, \Delta W) \in H\}$ and we denote again by $\mathscr{A}$ its restriction to H.

We define also a continuous tri-linear form $\mathscr{B}_0$ on $V \times V \times V$ by setting

$$\mathscr{B}_0((w_1, W_1), (w_2, W_2), (w_3, W_3)) = \int_\Omega \bigg(\nabla \cdot (\overline{w_2 w_1^T}^{\delta_1})\, w_3 \tag{6.2}$$
$$-S\nabla \cdot (\overline{W_2 W_1^T}^{\delta_1})\, w_3 + \nabla \cdot (\overline{W_2 w_1^T}^{\delta_2})\, W_3 - \nabla \cdot (\overline{w_2 W_1^T}^{\delta_2})\, W_3 \bigg)\, dx$$

and a continuous bilinear operator $\mathscr{B}(\cdot) : V \to V$ with

$$\langle \mathscr{B}(w_1, W_1), (w_2, W_2)\rangle = \mathscr{B}_0((w_1, W_1), (w_1, W_1), (w_2, W_2))$$

for all $(w_i, W_i) \in V$.
The following properties of the trilinear form $\mathscr{B}_0$ hold (see [33, 44, 20, 15])

$$\begin{aligned} &\mathscr{B}_0((w_1, W_1), (w_2, W_2), (A_{\delta_1} w_2, SA_{\delta_2} W_2)) = 0, \\ &\mathscr{B}_0((w_1, W_1), (w_2, W_2), (A_{\delta_1} w_3, SA_{\delta_2} W_3)) \\ &\qquad = -\mathscr{B}_0((w_1, W_1), (w_3, W_3), (A_{\delta_1} w_2, SA_{\delta_2} W_2)), \end{aligned} \tag{6.3}$$

for all $(w_i, W_i) \in V$. Also

$$\begin{aligned} &|\mathscr{B}_0((w_1, W_1), (w_2, W_2), (w_3, W_3))| \\ &\qquad \le C\|(w_1, W_1)\|_{m_1}\|(w_2, W_2)\|_{m_2+1}\|(\overline{w_3}^{\delta_1}, \overline{W_3}^{\delta_2})\|_{m_3} \end{aligned} \tag{6.4}$$

for all $(w_1, W_1) \in H^{m_1}(\Omega), (w_2, W_2) \in H^{m_2+1}(\Omega), (w_3, W_3) \in H^{m_3}(\Omega)$ and

$$\begin{aligned} &m_1 + m_2 + m_3 \ge \frac{d}{2}, \quad \text{if } m_i \ne \frac{d}{2} \text{ for all } i = 1, \dots, d, \\ &m_1 + m_2 + m_3 > \frac{d}{2}, \quad \text{if } m_i = \frac{d}{2} \text{ for any of } i = 1, \dots, d. \end{aligned}$$

In terms of $V, H, \mathscr{A}, \mathscr{B}(\cdot)$ we can rewrite (1.7) as

$$\begin{aligned} &\frac{d}{dt}(w, W) + \mathscr{A}(w, W)(t) + \mathscr{B}((w, W)(t)) = (\overline{\boldsymbol{f}}^{\delta_1}, \operatorname{curl} \overline{\boldsymbol{g}}^{\delta_2}), t \in (0, T), \\ &(w, W)(0) = (\overline{u}_0^{\delta_1}, \overline{B}_0^{\delta_2}), \end{aligned} \tag{6.5}$$

where $(\boldsymbol{f}, \operatorname{curl} \boldsymbol{g}) = P(f, \operatorname{curl} g)$, and $P : L^2(\Omega) \to H$ is the Hodge projection.

Let us define the modified nonlinearity $\mathscr{B}_N(\cdot) : V \to V$ by setting

$$\mathscr{B}_N(w, W) = \begin{cases} \mathscr{B}(w, W) & \text{if } \|(w, W)\|_1 \le N, \\ \left(\frac{N}{\|(w,W)\|_1}\right)^2 \mathscr{B}(w, W) & \text{if } \|(w, W)\|_1 > N. \end{cases} \tag{6.6}$$

By (6.4) we have for the case of $\|(w_1, W_1)\|_1, \|(w_2, W_2)\|_1 \le N$

$$\begin{aligned} &|\langle \mathscr{B}_N(w_1, W_1) - \mathscr{B}_N(w_2, W_2), (w_1 - w_2, W_1 - W_2)\rangle| \\ &= |\mathscr{B}_0((w_1 - w_2, W_1 - W_2), (w_1, W_1), (w_1 - w_2, W_1 - W_2)| \\ &\quad + \mathscr{B}_0((w_2, W_2), (w_1 - w_2, W_1 - W_2), (w_1 - w_2, W_1 - W_2)| \\ &\le C\|(w_1 - w_2, W_1 - W_2)\|_{1/2}\|(w_1, W_1)\|_1\|(\overline{w_1 - w_2}^{\delta_1}, \overline{W_1 - W_2}^{\delta_2})\|_1 \\ &\le \frac{\nu}{2}\|(w_1 - w_2, W_1 - W_2)\|_1^2 + C_N\|(w_1 - w_2, W_1 - W_2)\|_0^2, \end{aligned}$$

where $\nu = \inf\{1/\mathrm{Re}, S/\mathrm{Re}_m\}$.
In the case of $\|(w_i, W_i)\|_1 > N$ we have

$$
\begin{aligned}
&|\langle \mathscr{B}_N(w_1, W_1) - \mathscr{B}_N(w_2, W_2), (w_1 - w_2, W_1 - W_2)\rangle| \\
&= \frac{N^2}{\|(w_1, W_1)\|_1^2} \mathscr{B}_0((w_1 - w_2, W_1 - W_2), (w_1, W_1), (w_1 - w_2, W_1 - W_2)) \\
&\quad + \left(\frac{N^2}{\|(w_1, W_1)\|_1^2} - \frac{N^2}{\|(w_2, W_2)\|_1^2} \right) \mathscr{B}_0((w_2, W_2), (w_2, W_2), (w_1 - w_2, W_1 - W_2)) \\
&\leq CN\|(w_1 - w_2, W_1 - W_2)\|_1^{3/2} \|(w_1 - w_2, W_1 - W_2)\|_0^{1/2} \\
&\quad + CN\|(w_1 - w_2, W_1 - W_2)\|_1^2 \\
&\leq \frac{\nu}{2}\|(w_1 - w_2, W_1 - W_2)\|_1^2 + C_N\|(w_1 - w_2, W_1 - W_2)\|_0^2.
\end{aligned}
$$

For the case of $\|(w_1, W_1)\|_1 > N, \|(w_2, W_2)\|_1 \leq N$ (similar estimates are obtained when $\|(w_1, W_1)\|_1 \leq N, \|(w_2, W_2)\|_1 > N$) we have

$$
\begin{aligned}
&|\langle \mathscr{B}_N(w_1, W_1) - \mathscr{B}_N(w_2, W_2), (w_1 - w_2, W_1 - W_2)\rangle| \\
&= \frac{N^2}{\|(w_1, W_1)\|_1^2} \mathscr{B}_0((w_1 - w_2, W_1 - W_2), (w_1, W_1), (w_1 - w_2, W_1 - W_2)) \\
&\quad - \left(1 - \frac{N^2}{\|(w_1, W_1)\|_1^2} \right) \mathscr{B}_0((w_2, W_2), (w_2, W_2), (w_1 - w_2, W_1 - W_2)) \\
&\leq CN\|(w_1 - w_2, W_1 - W_2)\|_1^{3/2} \|(w_1 - w_2, W_1 - W_2)\|_0^{1/2} \\
&\quad + CN\|(w_1 - w_2, W_1 - W_2)\|_1 \|(w_1 - w_2, W_1 - W_2)\|_{1/2} \\
&\leq \frac{\nu}{2}\|(w_1 - w_2, W_1 - W_2)\|_1^2 + C_N\|(w_1 - w_2, W_1 - W_2)\|_0^2.
\end{aligned}
$$

Combining all the cases above we conclude that

$$
\begin{aligned}
&|\langle \mathscr{B}_N(w_1, W_1) - \mathscr{B}_N(w_2, W_2), (w_1 - w_2, W_1 - W_2)\rangle| \\
&\quad \leq \frac{\nu}{2}\|(w_1 - w_2, W_1 - W_2)\|_1^2 + C_N\|(w_1 - w_2, W_1 - W_2)\|_0^2.
\end{aligned}
\tag{6.7}
$$

The operator $\mathscr{B}_N$ is continuous from V to V'. Indeed, as above we have (using (6.4) with $m_1 = 1, m_2 = 0, m_3 = 1$)

$$
\begin{aligned}
&|\langle \mathscr{B}_N(w_1, W_1) - \mathscr{B}_N(w_2, W_2), (w_3, W_3)\rangle| \\
&\leq |\mathscr{B}_0\left((w_1 - w_2, W_1 - W_2), (w_1, W_1), (w_3, W_3)\right)| \\
&\quad + |\mathscr{B}_0\left((w_2, W_2), (w_1 - w_2, W_1 - W_2), (w_3, W_3)\right)| \\
&\leq C_N\|(w_1 - w_2, W_1 - W_2)\|_1 \|(w_3, W_3)\|_1.
\end{aligned}
\tag{6.8}
$$

Now consider the operator $\Gamma_N : D(\Gamma_N) \to H$ defined by

$$
\Gamma_N = \mathscr{A} + \mathscr{B}_N, \quad D(\Gamma_N) = D(\mathscr{A}).
$$

Here we used (6.4) with $m_1 = 1, m_2 = 1/2, m_3 = 0$ and interpolation results (see e.g. [18, 50, 15]) to show that

$$
\|\mathscr{B}_N(w, W)\|_0 \leq C\|(w, W)\|_1^{3/2} \|\mathscr{A}(w, W)\|_0^{1/2} \leq C_N \|\mathscr{A}(w, W)\|_0^{1/2}. \tag{6.9}
$$

LEMMA 6.1. *There exists $\alpha_N > 0$ such that $\Gamma_N + \alpha_N I$ is m-accretive (maximal monotone) in $H \times H$.*

Proof. By (6.7) we have that

$$((\Gamma_N + \lambda)(w_1, W_1) - (\Gamma_N + \lambda)(w_2, W_2), (w_1 - w_2, W_1 - W_2)) \\ \geq \frac{\nu}{2} \|(w_1 - w_2, W_1 - W_2)\|_1^2, \text{ for all } (w_i, W_i) \in D(\Gamma_N), \tag{6.10}$$

for $\lambda \geq C_N$. Next we consider the operator

$$\mathscr{F}_N(w, W) = \mathscr{A}(w, W) + \mathscr{B}_N(w, W) + \alpha_N(w, W), \quad \text{for all } (w, W) \in D(\mathscr{F}_N),$$

with

$$D(\mathscr{F}_N) = \{(w, W) \in V; \mathscr{A}(w, W) + \mathscr{B}_N(w, W) \in H\}.$$

By (6.8) and (6.10) we see that $\mathscr{F}_N$ is monotone, coercive and continuous from V to V'. We infer that $\mathscr{F}_N$ is maximal monotone from V to V' and the restriction to H is maximal monotone on H with the domain $D(\mathscr{F}_N) \supseteq D(\mathscr{A})$ (see e.g. [11, 4]).
Moreover, we have $D(\mathscr{F}_N) = D(\mathscr{A})$. For this we use the perturbation theorem for nonlinear m-accretive operators and split $\mathscr{F}_N$ into a continuous and a ω-m-accretive operator on H

$$\mathscr{F}_N^1 = (1 - \frac{\varepsilon}{2})\mathscr{A}, \quad D(\mathscr{F}_N^1) = D(\mathscr{A}),$$
$$\mathscr{F}_N^2 = \frac{\varepsilon}{2}\mathscr{A} + \mathscr{B}_N(\cdot) + \alpha_N I, \quad D(\mathscr{F}_N^2) = \{(w, W) \in V, \mathscr{F}_N^2(w, W) \in H\}.$$

As seen above by (6.9) we have

$$\left\|\mathscr{F}_N^2(w, W)\right\|_0 \leq \frac{\varepsilon}{2}\|\mathscr{A}(w, W)\|_0 + \|\mathscr{B}_N(w, W)\|_0 + \alpha_N\|(w, W)\|_0$$
$$\leq \varepsilon\|\mathscr{A}(w, W)\|_0 + \alpha_N\|(w, W)\|_0 + \frac{C_N^2}{2\varepsilon}, \quad \text{for all } (w, W) \in D(\mathscr{F}_N^1) = D(\mathscr{A}),$$

where $0 < \varepsilon < 1$.
Since $\mathscr{F}_N^1 + \mathscr{F}_N^2 = \Gamma_N + \alpha_N I$ we infer that $\Gamma_N + \alpha_N I$ with domain $D(\mathscr{A})$ is m-accretive in H as claimed. □

Proof. [Proof of Theorem 2.2] As a consequence of Lemma 6.1 (see, e.g., [4, 5]) we have that for $(\overline{u_0}^{\delta_1}, \overline{B_0}^{\delta_2}) \in D(\mathscr{A})$ and $(\overline{\boldsymbol{f}}^{\delta_1}, \operatorname{curl} \overline{\boldsymbol{g}}^{\delta_2}) \in W^{1,1}([0,T], H)$ the equation

$$\begin{aligned} &\frac{d}{dt}(w, W) + \mathscr{A}(w, W)(t) + \mathscr{B}_N((w, W)(t)) = (\overline{\boldsymbol{f}}^{\delta_1}, \operatorname{curl} \overline{\boldsymbol{g}}^{\delta_2}), \ t \in (0, T), \\ &(w, W)(0) = (\overline{u_0}^{\delta_1}, \overline{B_0}^{\delta_2}), \end{aligned} \tag{6.11}$$

has a unique strong solution $(w_N, W_N) \in W^{1,\infty}([0,T]; H) \cap L^\infty(0, T; D(\mathscr{A}))$.
By a density argument (see, e.g., [5, 33]) it can be shown that if $(\overline{u_0}^{\delta_1}, \overline{B_0}^{\delta_2}) \in H$ and $(\overline{\boldsymbol{f}}^{\delta_1}, \operatorname{curl} \overline{\boldsymbol{g}}^{\delta_2}) \in L^2(0, T, V')$ then there exist absolute continuous functions $(w_N, W_N) : [0, T] \to V'$ that satisfy $(w_N, W_N) \in C([0,T]; H) \cap L^2(0, T : V) \cap W^{1,2}([0,T], V')$ and (6.11) a.e. in $(0, T)$, where d/dt is considered in the strong topology of V'.

First, we show that $D(\mathscr{A})$ is dense in H. Indeed, if $(w, W) \in H$ we set $(w_\varepsilon, W\varepsilon) = (I + \varepsilon\Gamma_N)^{-1}(w, W)$, where I is the unity operator in H. Multiplying the equation

$$(w_\varepsilon, W_\varepsilon) + \varepsilon\Gamma_N(w_\varepsilon, W_\varepsilon) = (w, W)$$

by $(w_\varepsilon, W_\varepsilon)$ it follows by (6.3), (6.7) that

$$\|(w_\varepsilon, W_\varepsilon)\|_0^2 + 2\varepsilon\nu\|(w_\varepsilon, W_\varepsilon)\|_1^2 \leq \|(w, W)\|_0^2$$

and by (6.6)

$$\|(w_\varepsilon - w, W_\varepsilon - W)\|_{-1} = \varepsilon\|\Gamma_\varepsilon(w_\varepsilon, W_\varepsilon)\|_{-1} \leq \varepsilon N\|(w_\varepsilon, W_\varepsilon)\|_0^{1/2}\|(w_\varepsilon, W_\varepsilon)\|_1^{1/2}.$$

Hence, $\{(w_\varepsilon, W_\varepsilon)\}$ is bounded in H and $(w_\varepsilon, W_\varepsilon) \to (w, W)$ in V' as $\varepsilon \to 0$. Therefore, $(w_\varepsilon, W_\varepsilon) \rightharpoonup (w, W)$ in H as $\varepsilon \to 0$, which implies that $D(\Gamma_N)$ is dense in H.

Secondly, let $(\overline{u_0}^{\delta_1}, \overline{B_0}^{\delta_2}) \in H$ and $(\overline{\boldsymbol{f}}^{\delta_1}, \operatorname{curl}\overline{\boldsymbol{g}}^{\delta_2}) \in L^2(0, T, V')$. Then there are sequences $\{(\overline{u_0}_n^{\delta_1}, \overline{B_0}_n^{\delta_2})\} \subset D(\Gamma_N)$, $\{(\overline{\boldsymbol{f}}_n^{\delta_1}, \operatorname{curl}\overline{\boldsymbol{g}}_n^{\delta_2})\} \subset W^{1,1}([0, T]; H)$ such that

$$\begin{aligned}
&(\overline{u_0}_n^{\delta_1}, \overline{B_0}_n^{\delta_2}) \to (\overline{u_0}^{\delta_1}, \overline{B_0}^{\delta_2}) \quad \text{in } H,\\
&(\overline{\boldsymbol{f}}_n^{\delta_1}, \operatorname{curl}\overline{\boldsymbol{g}}_n^{\delta_2}) \to (\overline{\boldsymbol{f}}^{\delta_1}, \operatorname{curl}\overline{\boldsymbol{g}}^{\delta_2}) \quad \text{in } L^2(0, T; V'),
\end{aligned}$$

as $n \to \infty$. Let $(w_N^n, W_N^n) \in W^{1,\infty}([0, T]; H)$ be the solution to problem (6.11) where $(w, W)(0) = (\overline{u_0}_n^{\delta_1}, \overline{B_0}_n^{\delta_2})$ and $(\overline{\boldsymbol{f}}^{\delta_1}, \operatorname{curl}\overline{\boldsymbol{g}}^{\delta_2}) = (\overline{\boldsymbol{f}}_n^{\delta_1}, \operatorname{curl}\overline{\boldsymbol{g}}_n^{\delta_2})$. By (6.10) we have

$$\begin{aligned}
&\frac{d}{dt}\|(w_N^n - w_N^m, W_N^n - W_N^m)\|_0^2 + \frac{\nu}{2}\|(w_N^n - w_N^m, W_N^n - W_N^m)\|_1^2\\
&\quad\leq 2C_N\|(w_N^n - w_N^m, W_N^n - W_N^m)\|_0^2 + \frac{2}{\nu}\|(\overline{\boldsymbol{f}}_n^{\delta_1} - \overline{\boldsymbol{f}}_m^{\delta_1}, \operatorname{curl}(\overline{\boldsymbol{g}}_n^{\delta_2} - \overline{\boldsymbol{g}}_m^{\delta_2}))\|_{-1}^2,
\end{aligned}$$

for a.e. $t \in (0, T)$. By the Gronwall inequality we obtain

$$\begin{aligned}
&\|(w_N^n - w_N^m, W_N^n - W_N^m)(t)\|_0^2 \leq e^{2C_N t}\|(\overline{u_0}_n^{\delta_1} - \overline{u_0}_m^{\delta_1}, \overline{B_0}_n^{\delta_2} - \overline{B_0}_m^{\delta_2})\|_0^2\\
&\quad + \frac{2e^{2C_N t}}{\nu}\int_0^t \|(\overline{\boldsymbol{f}}_n^{\delta_1} - \overline{\boldsymbol{f}}_m^{\delta_1}, \operatorname{curl}(\overline{\boldsymbol{g}}_n^{\delta_2} - \overline{\boldsymbol{g}}_m^{\delta_2}))(\tau)\|_{-1}^2 d\tau.
\end{aligned}$$

Hence

$$(w_N(t), W_N(t)) = \lim_{n\to\infty}(w_N^n(t), W_N^n(t))$$

exists in H uniformly in t on $[0, T]$. Similarly we obtain

$$\begin{aligned}
&\|w_N^n(t)\|_0^2 + \|W_N^n(t)\|_0^2 + \int_0^t \left(\frac{1}{\mathrm{Re}}(\|\nabla w_N^n(s)\|_0^2 + \frac{S}{\mathrm{Re_m}}(\|\operatorname{curl} W_N^n(s)\|_0^2\right) ds\\
&\quad\leq C_N\left[\|\overline{u_0}_n^{\delta_1}\|_0^2 + \|\overline{B_0}_n^{\delta_2}\|_0^2 + \int_0^t \left(\|\overline{\boldsymbol{f}}_n^{\delta_1}(s)\|_{-1}^2 + \|\operatorname{curl}\overline{\boldsymbol{g}}_n^{\delta_2}(s)\|_{-1}^2\right) ds\right],
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T \left\|\frac{d}{dt}(w_N^n, W_N^n)(t)\right\|_{-1}^2 dt\\
&\quad\leq C_N\left[\|\overline{u_0}_n^{\delta_1}\|_0^2 + \|\overline{B_0}_n^{\delta_2}\|_0^2 + \int_0^t \left(\|\overline{\boldsymbol{f}}_n^{\delta_1}(s)\|_{-1}^2 + \|\operatorname{curl}\overline{\boldsymbol{g}}_n^{\delta_2}(s)\|_{-1}^2\right) ds\right].
\end{aligned}$$

Hence on a sequence we have

$$\begin{aligned}
&(w_N^n, W_N^n) \to (w_N, W_N) \qquad \text{weakly in } L^2(0, T; V),\\
&\frac{d}{dt}(w_N^n, W_N^n) \to \frac{d}{dt}(w_N, W_N) \qquad \text{weakly in } L^2(0, T; V'),
\end{aligned}$$

where $d(w_N, W_N)/dt$ is considered in the sense of V'-valued distributions on $(0, T)$. We proved that $(w_N, W_N) \in C([0,T]; H) \cap L^2(0,T;V) \cap W^{1,2}([0,T]; V')$.

It remains to prove that (w_N, W_N) satisfies the equation (6.11) a.e. on $(0, T)$. Let $(w, W) \in V$ be arbitrary but fixed. We multiply the equation

$$\frac{d}{dt}(w_N^n, W_N^n) + \Gamma_N(w_N^n, W_N^n) = (\overline{\boldsymbol{f}}_n^{\delta_1}, \operatorname{curl} \overline{\boldsymbol{g}}_n^{\delta_2}), \quad \text{a.e. } t \in (0,T),$$

by $(w_N^n - w, W_N^n - W)$, integrate on (s,t) and get

$$\begin{aligned}
&\frac{1}{2}\Big(\|(w_N^n(t), W_N^n(t)) - (w,W)\|_0^2 - \|(w_N^n(s), W_N^n(s)) - (w,W)\|_0^2\Big) \\
&\quad \le \int_s^t \langle (\overline{\boldsymbol{f}}_n^{\delta_1}(\tau), \operatorname{curl} \overline{\boldsymbol{g}}_n^{\delta_2}(\tau)) - \Gamma_N(w,W), (w_N^n(\tau), W_N^n(\tau)) - (w,W)\rangle d\tau.
\end{aligned}$$

After we let $n \to \infty$ we get

$$\begin{aligned}
&\left\langle \frac{(w_N(t), W_N(t)) - (w_N(s), W_N(s))}{t-s}, (w_N(s), W_N(s)) - (w,W) \right\rangle \\
&\quad \le \frac{1}{t-s}\int_s^t \langle (\overline{\boldsymbol{f}}^{\delta_1}(\tau), \operatorname{curl} \overline{\boldsymbol{g}}^{\delta_2}(\tau)) - \Gamma_N(w,W), (w_N(\tau), W_N(\tau)) - (w,W)\rangle d\tau.
\end{aligned} \tag{6.12}$$

Let t_0 denote a point at which (w_N, W_N) is differentiable and

$$(\overline{\boldsymbol{f}}^{\delta_1}(t_0), \operatorname{curl} \overline{\boldsymbol{g}}^{\delta_2}(t_0)) = \lim_{h\to 0} \frac{1}{h}\int_{t_0}^{t_0+h} (\overline{\boldsymbol{f}}^{\delta_1}(h), \operatorname{curl} \overline{\boldsymbol{g}}^{\delta_2}(h)) dh.$$

Then by (6.12) we have

$$\left\langle \frac{d(w_N, W_N)}{dt}(t_0) - (\overline{\boldsymbol{f}}^{\delta_1}, \operatorname{curl} \overline{\boldsymbol{g}}^{\delta_2})(t_0) + \Gamma_N(w,W), (w_N, W_N)(t_0) - (w,W) \right\rangle \le 0.$$

Since (w, W) is arbitrary in V and Γ_N is maximal monotone in $V \times V'$ we conclude that

$$\frac{d(w_N, W_N)}{dt}(t_0) + \Gamma_N(w_N, W_N)(t_0) = (\overline{\boldsymbol{f}}^{\delta_1}, \operatorname{curl} \overline{\boldsymbol{g}}^{\delta_2})(t_0).$$

If we multiply (6.11) by $(A_{\delta_1} w_N, S A_{\delta_2} W_N)$, use (6.3) and integrate in time we obtain

$$\begin{aligned}
&\frac{1}{2}\left(\|w_N(t)\|_0^2 + S\|W_N(t)\|_0^2\right) + \frac{{\delta_1}^2}{2}\|\nabla w_N(t)\|_0^2 + \frac{{\delta_2}^2 S}{2}\|\operatorname{curl} W_N(t)\|_0^2 \\
&\quad + \int_0^t \Bigg(\frac{1}{\mathrm{Re}}(\|\nabla w_N(s)\|_0^2 + {\delta_1}^2\|\Delta w_N(s)\|_0^2) \\
&\quad + \frac{S}{\mathrm{Re_m}}(\|\operatorname{curl} W_N(s)\|_0^2 + {\delta_2}^2\|\operatorname{curl}\operatorname{curl} W_N(s)\|_0^2)\Bigg) ds \\
&= \frac{1}{2}\left(\|\overline{u_0}^{\delta_1}\|_0^2 + S\|\overline{B_0}^{\delta_2}\|_0^2\right) + \frac{{\delta_1}^2}{2}\|\nabla \overline{u_0}^{\delta_1}\|_0^2 + \frac{{\delta_2}^2 S}{2}\|\operatorname{curl} \overline{B_0}^{\delta_2}\|_0^2 \\
&\quad + \int_0^t \Big(\|\overline{f}^{\delta_1}(s)\|_{-1}\|w_N(s)\|_1 + S\|\operatorname{curl} \overline{g}^{\delta_2}(s)\|_{-1}\|W_N(s)\|_1\Big) ds.
\end{aligned}$$

Using the Cauchy-Schwarz and Gronwall inequalities this implies

$$\|(w_N, W_N)(t)\|_1 \leq C_{\delta_1,\delta_2} \quad \text{for all } t \in (0,T),$$

where C_{δ_1,δ_2} is independent of N. In particular, for N sufficiently large it follows from (6.6) that $\mathscr{B}_N = \mathscr{B}$ and $(w_N, W_N) = (w, W)$ is a solution to (1.7).

In the following we prove the uniqueness of the weak solution. Let (w_1, W_1) and (w_2, W_2) be two solutions of the system (6.5) and set $\varphi = w_1 - w_2$, $\Phi = B_1 - B_2$. Thus (φ, Φ) is a solution to the problem

$$\begin{aligned}
&\frac{d}{dt}(\varphi,\Phi) + \mathscr{A}(\varphi,\Phi)(t) = -\mathscr{B}((w_1,W_1)(t)) + \mathscr{B}((w_2,W_2)(t)), \quad t \in (0,T),\\
&(\varphi,\Phi)(0) = (0,0).
\end{aligned}$$

We take $(A_{\delta_1}\varphi, SA_{\delta_2}\Phi)$ as test function, integrate in space, use the incompressibility condition (6.3) and the estimate (6.4) to get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\varphi\|_0^2 + {\delta_1}^2\|\nabla\varphi\|_0^2 + S\|\Phi\|_0^2 + S{\delta_2}^2\|\nabla\Phi\|_0^2\right)\\
&\qquad + \frac{1}{\mathrm{Re}}\left(\|\nabla\varphi\|_0^2 + \delta_1^2\|\Delta\varphi\|_0^2\right) + \frac{S}{\mathrm{Re_m}}\left(\|\nabla\Phi\|_0^2 + \delta_2^2\|\Delta\Phi\|_0^2\right)\\
&= \mathscr{B}_0((\varphi,\Phi),(w_1,W_1),(A_{\delta_1}\varphi, SA_{\delta_2}\Phi))\\
&\leq C\|(w_1,W_1)\|_0\|(\varphi,\Phi)\|_0^{1/2}\|(\nabla\varphi,\nabla\Phi)\|_0^{3/2}\\
&\leq C_{\delta_1,\delta_2}\|(w_1,W_1)\|_0\left(\|\varphi\|_0^2 + {\delta_1}^2\|\nabla\varphi\|_0^2 + S\|\Phi\|_0^2 + S{\delta_2}^2\|\nabla\Phi\|_0^2\right).
\end{aligned}$$

Applying the Gronwall's lemma we deduce that (φ, Φ) vanishes for all $t \in [0,T]$, and hence the uniqueness of the solution. ☐

6.1. Regularity.

THEOREM 6.2. *Let $m \in \mathbb{N}$, $(u_0, B_0) \in V \cap H^{m-1}(\Omega)$ and $(f, curl\, g) \in L^2(0,T; H^{m-1}(\Omega))$. Then there exists a unique solution w, W, q to the equation (1.7) such that*

$$\begin{aligned}
&(w,W) \in L^\infty(0,T;H^{m+1}(\Omega)) \cap L^2(0,T;H^{m+2}(\Omega)),\\
&q \in L^2(0,T;H^m(\Omega)).
\end{aligned}$$

Proof. The result is already proved when $m = 0$ in Theorem 2.2. For any $m \in \mathbb{N}^*$, we assume that

$$(w,W) \in L^\infty(0,T;H^m(\Omega)) \cap L^2(0,T;H^{m+1}(\Omega)) \tag{6.13}$$

so it remains to prove

$$(D^m w, D^m W) \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)),$$

where D^m denotes any partial derivative of total order m. We take the m^{th} derivative of (1.7) and have

$$\begin{aligned}
&(D^m w)_t - \frac{1}{\mathrm{Re}}\Delta(D^m w) + \overline{D^m(w\cdot\nabla w)}^{\delta_1} - S\overline{D^m(W\cdot\nabla W)}^{\delta_1} = \overline{D^m f}^{\delta_1},\\
&(D^m W)_t + \frac{1}{\mathrm{Re_m}}\nabla\times\nabla\times(D^m W) + \overline{D^m(w\cdot\nabla W)}^{\delta_2} - \overline{D^m(W\cdot\nabla w)}^{\delta_2} = \nabla\times\overline{D^m g}^{\delta_2},\\
&\nabla\cdot(D^m w) = 0, \nabla\cdot(D^m W) = 0,\\
&D^m w(0,\cdot) = D^m\overline{u_0}^{\delta_1}, D^m W(0,\cdot) = D^m\overline{B_0}^{\delta_2},
\end{aligned}$$

with periodic boundary conditions and zero mean, and the initial conditions with zero divergence and mean. Taking $A_{\delta_1} D^m w, A_{\delta_1} D^m W$ as test functions we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|D^m w\|_0^2 + {\delta_1}^2\|\nabla D^m w\|_0^2 + S\|D^m W\|_0^2 + S{\delta_2}^2\|\nabla D^m W\|_0^2\right) \tag{6.14}$$
$$+\frac{1}{\mathrm{Re}}\left(\|\nabla D^m w\|_0^2 + \delta_1^2\|\Delta D^m w\|_0^2\right) + \frac{1}{\mathrm{Re_m}}\left(\|\nabla D^m W\|_0^2 + \delta_2^2\|\Delta D^m W\|_0^2\right)$$
$$= \int_\Omega \left(D^m f D^m w + \nabla \times g D^m W\right) dx - \mathscr{X},$$

where

$$\mathscr{X} = \int_\Omega \Big(D^m(w\cdot\nabla w) - SD^m(W\cdot\nabla W)\Big) D^m w + \Big(D^m(w\cdot\nabla W) - D^m(W\cdot\nabla w)\Big) D^m W dx.$$

Now we apply (6.4) and use the induction assumption (6.13)

$$\mathscr{X} = \sum_{|\alpha|\le m}\begin{pmatrix} m \\ \alpha \end{pmatrix}\sum_{i,j=1}^{3}\int_\Omega D^\alpha w_i D^{m-\alpha} D_i w_j D^m w_j - SD^\alpha W_i D^{m-\alpha} D_i W_j D^m w_j$$
$$- D^\alpha w_i D^{m-\alpha} D_i W_j D^m W_j - D^\alpha W_i D^{m-\alpha} D_i w_j D^m W_j$$
$$\le \|w\|_{m+1}^{3/2}\|w\|_{m+2}^{1/2}\|w\|_m + \|W\|_{m+1}^{3/2}\|W\|_{m+2}^{1/2}\|w\|_m$$
$$+ \|w\|_{m+1}\|W\|_{m+1}^{1/2}\|W\|_{m+2}^{1/2}\|W\|_m + \|W\|_{m+1}^{3/2}\|W\|_{m+2}^{1/2}\|W\|_m.$$

Integrating (6.14) on $(0,T)$, using the Cauchy-Schwarz and Hölder inequalities, and the assumption (6.13) we obtain the desired result for w, W. We conclude the proof mentioning that the regularity of the pressure term q is obtained via classical methods, see e.g. [49, 3]. □

REFERENCES

[1] N.A. Adams and S. Stolz, *Deconvolution methods for subgrid-scale approximation in large-eddy simulation*, Modern Simulation Strategies for Turbulent Flow, R.T. Edwards, 2001.

[2] H. Alfvén, *Existence of electromagnetic-hydrodynamic waves*, Nature **150** (1942), 405.

[3] C. Amrouche and V. Girault, *Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension*, Czechoslovak Math. J. **44(119)** (1994), no. 1, 109–140.

[4] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff, Leyden, 1976.

[5] ———, *Analysis and control of nonlinear infinite-dimensional systems*, Mathematics in Science and Engineering, vol. 190, Academic Press Inc., Boston, MA, 1993.

[6] V. Barbu and S. S. Sritharan, *Flow invariance preserving feedback controllers for the Navier-Stokes equation*, J. Math. Anal. Appl. **255** (2001), no. 1, 281–307.

[7] M.A. Berger, *Inverse cascades in a periodic domain*, Astrophysical Letters and Communications **34** (1996), 225–230.

[8] T. Berger, J. Kim, C. Lee, and J. Lim, *Turbulent boundary layer control utilizing Lorentz force*, Phys. Fluids **12** (2000), 631649.

[9] L.C. Berselli, T. Iliescu, and W.J. Layton, *Mathematics of large eddy simulation of turbulent flows*, Scientific Computation, Springer-Verlag, Berlin, 2006.

[10] D. Biskamp, *Magnetohydrodynamic turbulence*, Cambridge: Cambridge University Press, 2003.

[11] H. Brezis, *Opérateurs Maximaux et Semigroupes de Contractions dans les Espaces de Hilbert*, North-Holland, New York, 1973.

[12] T.G. Cowling, *Magnetohydrodynamics*, Interscience Tracts on Physics and Astronomy, New York, 1957.

[13] P. A. Davidson, *An introduction to magnetohydrodynamics*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2001.

[14] U. Frisch, A. Pouquet, J. Leorat, and A. Mazure, *Possibility of an inverse cascade of magnetic helicity in magnetohydrodynamic turbulence*, J. Fluid Mech. **68** (1975), 769–778.

[15] A.V. Fursikov, *Optimal control of distributed systems. Theory and applications*, Translations of Mathematical Monographs, vol. 187, American Mathematical Society, Providence, RI, 2000, Translated from the 1999 Russian original by Tamara Rozhkovskaya.

[16] A. Gailitis and O. Lielausis, *On a possibility to reduce the hydrodynamical resistance of a plate in an electrolyte*, (in Russian), Applied Magnetohydrodynamics. Reports of the Physics Institute **12** (1961), 143–146.

[17] M. Germano, *Differential filters of elliptic type*, Phys. Fluids **29** (1986), no. 6, 1757–1758.

[18] V. Girault and P.-A. Raviart, *Finite element methods for Navier-Stokes equations*, Springer Series in Computational Mathematics, vol. 5, Springer-Verlag, Berlin, 1986, Theory and algorithms.

[19] J. Graham, P. Mininni, and A. Pouquet, *Lagrangian-averaged model for magnetohydrodynamic turbulence and the absence of bottlenecks*, Phys. Rev. E **80** (2009), 016313.

[20] P. Grisvard, *Boundary value problems in non-smooth domains*, Univ. of Maryland, Dept. of Math. Lecture Notes (1980), no. 19.

[21] M.D. Gunzburger, A.J. Meir, and J. Peterson, *On the existence, uniqueness, and finite element approximation of solutions of the equations of stationary, incompressible magnetohydrodynamics*, Math. Comp. **56** (1991), no. 194, 523–563.

[22] M.D. Gunzburger, J. Peterson, and C. Trenchea, *The velocity and magnetic field tracking problem for MHD flows with distributed controls*, submitted.

[23] C. Henoch and J. Stace, *Experimental investigation of a salt water turbulent boundary layer modified by an applied streamwise magnetohydrodynamic body force*, Physics of Fluids, **7** (1995), 1371–1383.

[24] ———, *Experimental investigations of a salt water turbulent boundary layer modified by an applied streamwise magnetohydrodynamic body force*, Phys. Fluids **7** (1995), 1371–1381.

[25] V. John, *Large eddy simulation of turbulent incompressible flows*, Lecture Notes in Computational Science and Engineering, vol. 34, Springer-Verlag, Berlin, 2004, Analytical and numerical results for a class of LES models.

[26] L. Landau and E. Lifschitz, *Électrodynamique des milieux continus*, Physique théorique, vol. tome VIII, MIR, Moscow, 1969.

[27] W. Layton and R. Lewandowski, *On a well-posed turbulence model*, Discrete Contin. Dyn. Syst. Ser. B **6** (2006), no. 1, 111–128.

[28] W. Layton, M. Sussman, and C. Trenchea, *Bounds on energy, magnetic helicity and cross helicity dissipation rates of approximate deconvolution models of turbulence for MHD flows*, Numer. Funct. Anal. Optim. (2010), accepted.

[29] J. Leray, *Sur le mouvement d'un fluide visqueux emplissant l'espace*, Acta Math. **63** (1934), no. 1, 193–248.

[30] M. Lesieur, *Turbulence in fluids*, Kluwer Academic Publishers, 1997.

[31] M. Lesieur, O. Metais, and P. Comte, *Large-eddy simulations of turbulence*, Cambridge, 2005.

[32] J.S. Linshiz and E.S. Titi, *Analytical study of certain magnetohydrodynamic-α models*, J. Math. Phys. **48** (2007), no. 6, 065504, 28.

[33] J.L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Études mathématiques, Dunod Gauthiers-Villars, 1969.

[34] W.H. Matthaeus and M.L. Goldstein, *Measurement of the rugged invariants of magnetohydrodynamic turbulence in the solar wind*, J. Geophys. Res. **87** (1982), 6011–6028.

[35] J.C.S. Meng, P. J. Hendricks, and J.D. Hrubes, *Superconducting electromagnetic thrusters*, Sea Technology **33** (1992), 29–39.

[36] J.C.S. Meng, C.W. Henoch, and J.D. Hrubes, *Seawater electromagnetohydrodynamics: A new frontier*, Magnetohydrodynamics **30** (1994), 401–418.

[37] J.-L. Meyer, F. Durand, R. Ricou, and C. Vives, *Steady flow of liquid aluminum in a rectangular-vertical ingot mold, thermally or electromagnetically activated*, Metall. Trans. B **15B** (1984), 471–478.

[38] D.C. Montgomery and J.W. Bates, *The geometry and symmetries of magnetohydrodynamic turbulence: anomalies of spatial periodicity*, Phys. Plasmas **6** (1999), no. 7, 2727–2733.

[39] S. Oughton and R. Prandi, *Kinetic helicity and MHD turbulence*, J. Plasma Phys. **64** (2000), 179–193.

[40] T.M. Özgökmen, T. Iliescu, and P.F. Fischer, *Large eddy simulation of stratified mixing in a three-dimensional lock-exchange system*, Ocean Modelling **26** (2009), 134–155.

[41] P.J. Prescott and F.P. Incropera, *Magnetically damped convection during solidification of a binary metal alloy*, J. Heat Transfer (Trans. ASME) **115** (1993), 302–310.

[42] P. Sagaut, *Large eddy simulation for incompressible flows*, third ed., Scientific Computation,

Springer-Verlag, Berlin, 2006, An introduction, Translated from the 1998 French original, With forewords by Marcel Lesieur and Massimo Germano, With a foreword by Charles Meneveau.

[43] J. Sakane, B.Q. Li, and J.W. Evans, *Mathematical modeling of meniscus profile and melt flow in electromagnetic casters*, Metall. Trans. B **19B** (1988), 397–408.

[44] M. Sermange and R. Temam, *Some mathematical questions related to the MHD equations*, Comm. Pure Appl. Math. **36** (1983), 635–664.

[45] J.A. Shercliff, *A Textbook of Magnetohydrodynamics*, Pergamon, Oxford, 1965.

[46] N.S. Singh and P.R. Bandyopandhyay, *Linear feedback control on boundary layer using electromagnetic microtiles*, Trans. ASME J. Fluid Engrg. **119** (1997), 852–858.

[47] T. Stribling, W.H. Matthaeus, and S. Ghosh, *Nonlinear decay of magnetic helicity in magnetohydrodynamics with a mean magnetic field*, J. Geophys. Res. **99** (1994), 2567–2576.

[48] T. Stribling, W.H. Matthaeus, and S. Oughton, *Magnetic helicity in magnetohydrodynamic turbulence with a mean magnetic field*, Phys. Plasmas **2** (1995), 1437–1452.

[49] L. Tartar, *Topics in nonlinear analysis*, Publications Mathématiques d'Orsay, vol. 13, Université de Paris-Sud, Départment de Mathématique, Orsay, 1978.

[50] R. Temam, *Navier-Stokes Equations. Theory and Numerical Analysis*, North-Holland, Amsterdam, 1979.

[51] ———, *Navier-Stokes equations and nonlinear functional analysis*, CBMS-NSF Regional Conference Series in Applied Mathematics, Society for Industrial and Applied Mathematics, Philadelphia, 1995.

[52] A. Tsinober, *MHD flow drag reduction*, Viscous Drag Reduction in Boundary Layers, (D.M. Busnell and J.N. Hefner, eds.), Progress in Astronautics and Aeronautics, vol. 123, American Institute of Astronautics and Aeronautics, Reston, VA, 1990, pp. 327–349.

[53] A. B. Tsinober and A. G. Shtern, *Possibility of increasing the flow stability in a boundary layer by means of crossed electric and magnetic fields*, Magnetohydrodynamics **3** (1967), no. 2, 152–154.

[54] Ch. Vivès and R. Ricou, *Fluid flow phenomena in a single phase coreless induction furnace*, Metall. Trans. B **16B** (1985), 227–235.