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Identification for a Nonlinear Periodic Wave Equation

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Abstract. This work is concerned with an approximation process for the identification of nonlinearities in the nonlinear periodic wave equation. It is based on the least-squares approach and on a splitting method. A numerical algorithm of gradient type and the numerical implementation are given.

Key Words. Nonlinear periodic wave equation, Convex function, Minimization problem, Splitting method, Optimality conditions, Gradient method.

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1. Introduction

In this work we study the identification of the nonlinearity g: $\mathbb{R} \to \mathbb{R}$ in the one-dimensional periodic wave equation

$$\begin{cases} y_{tt}(x,t) - y_{xx}(x,t) + g(y(x,t)) \ni f(x,t), & x \in (0,\pi), \quad t \in \mathbb{R}, \\ y(0,t) = y(\pi,t) = 0, & t \in \mathbb{R}, \\ y(x,t+2\pi) = y(x,t), & x \in (0,\pi), \quad t \in \mathbb{R}. \end{cases} \tag{1.1}$$

Here $f \in L^\infty((0,\pi)\times\mathbb{R})$, $f(x,t+2\pi) = f(x,t), \forall x \in (0,\pi), t \in \mathbb{R}$. The multivalued mapping g is to be chosen from certain classes of subpotential functions. For the direct problem, when g is given and (1.1) is solved for y, existence results are well known, see [3] for example. For the identification or inverse problem we are given a possibly perturbed observation y_0 corresponding to the state variable y and we must determine g in (1.1) such that $y(g)$ best approximates y_0.

By a solution to (1.1) we mean a *weak solution*, i.e., a function $y\colon (0,\pi)\times\mathbb{R}\to\mathbb{R}$ which is 2π-periodic in t and such that $y\in L^2(Q)$ and

$$\int_Q y(x,t)(\varphi_{tt}(x,t)-\varphi_{xx}(x,t))\,dx\,dt \\ = \int_Q (f(x,t)-g(y(x,t)))\varphi(x,t)\,dx\,dt \tag{1.2}$$

for all

$$\varphi\in X=\{\varphi\in C^2([0,\pi]\times[0,2\pi]); \\ \varphi(0,t)=\varphi(\pi,t)=0, \varphi(x,0)=\varphi(x,2\pi)=0, \\ \varphi_t(x,0)=\varphi_t(x,2\pi), \forall(x,t)\in[0,\pi]\times[0,2\pi]\}.$$

Let $\mathcal{W}\colon L^2(Q)\to L^2(Q)$ be the wave operator, i.e.,

$$\mathcal{W}y=f \qquad \text{for} \quad [y,f]\in D(\mathcal{W})\times R(\mathcal{W}) \quad \text{if and only if} \tag{1.3}$$

$$\int_Q y(\varphi_{tt}-\varphi_{xx})\,dx\,dt=\int_Q f\varphi\,dx\,dt, \qquad \forall\varphi\in X. \tag{1.4}$$

In terms of $\mathcal{W}$ the weak solution to (1.1) is the solution to the operator equation $\mathcal{W}y+g(y)\ni f$.

In Proposition 1.1 below we recall for later use some properties of $\mathcal{W}$ (see [3] and [2]).

Proposition 1.1. *$\mathcal{W}$ is self-adjoint, $R(\mathcal{W})$ (the range of $\mathcal{W}$) is closed, and $\mathcal{W}^{-1}\in L(R(\mathcal{W}),R(\mathcal{W}))$. Moreover, $\mathcal{W}^{-1}$ is compact on $R(\mathcal{W})$ and in addition*

$$\|\mathcal{W}^{-1}f\|_{L^\infty(Q)}\le C\|f\|_{L^1(Q)}, \qquad \forall f\in R(\mathcal{W}),$$
$$\|\mathcal{W}^{-1}f\|_{H^1(Q)}\le C\|f\|_{L^2(Q)}, \qquad \forall f\in R(\mathcal{W}).$$

In particular, it follows from Proposition 1.1 that the space $L^2(Q)$ admits the orthogonal decomposition $L^2(Q)=R(\mathcal{W})\oplus N(\mathcal{W})$ and $\mathcal{W}^{-1}$ is continuous from $R(\mathcal{W})$ to itself, where $N(\mathcal{W})=\{y\in L^2(Q);\mathcal{W}y=0\}$.

The identification problem is formulated as a nonlinear least-squares problem, which is approximated and analyzed in the framework of convex analysis. In Section 2 we discuss existence for the identification problem associated with (1.1) and prove convergence of an approximating scheme. In Section 3 we present an algorithm for the estimation of g in (1.1), based on a nonlinear control formulation of the modified least-squares approach of Section 2. A numerical algorithm and numerical results are given in Section 4.

2. An Approximating Identification Scheme for System (1.3)

Here we study the identification of the mapping g in the class

$$\mathcal{A}=\{g=\partial j\colon j\in\mathcal{K}\}, \tag{2.1}$$

where

$$\mathcal{K} = \{j\colon \mathbb{R} \to \mathbb{R} \text{ with } j \text{ convex, continuous and} \\ \beta_1|r|^2 + \gamma_1 \le j(r) \le \beta_2|r|^2 + \gamma_2,\ \forall r \in \mathbb{R}\} \tag{2.2}$$

for some $0 < \beta_1 < \beta_2 < \frac{3}{4}$ and $\gamma_1 \le \gamma_2$. Here $\partial j\colon \mathbb{R} \to \mathbb{R}$ denotes the subdifferential of j.

The identification problem consists in determining, for given y_0, a function $g \in \mathcal{A}$ such that $y(g) = y_0$. Of course, such a g may not exist. The least-squares approach leads us to the minimization problem:

$$\min\{\|y - y_0\|^2_{L^2(Q)}\colon \mathcal{W}y + g(y) \ni f,\ y \in L^2(Q),\ g = \partial j,\ j \in \mathcal{K}\} \tag{P}$$

for given $y_0 \in L^2(Q)$.

Using the approach taken in [5] and [1], we approximate (P) by the following family of minimization problems:

$$\min\left\{\|y - y_0\|^2_{L^2(Q)} + \frac{1}{\varepsilon}\int_Q (j(y) + j^*(v) - yv)\,dx\,dt : \right. \\ \left. y \in L^2(Q),\ j \in \mathcal{K},\ v \in L^2(Q),\ \mathcal{W}y + v = f\right\}, \tag{P_ε}$$

where $\varepsilon > 0$ and j^* stands for the conjugate function of j defined by

$$j^*(\tilde{r}) = \sup_{r \in \mathbb{R}}\{r\tilde{r} - j(r)\}.$$

The motivation for the cost functional of problem (P_ε) comes from the conjugacy formula,

$$j(r) + j^*(\tilde{r}) = r\tilde{r}, \qquad \text{if and only if} \quad \tilde{r} \in \partial j(r),$$

while $j(r) + j^*(\tilde{r}) \ge r\tilde{r}$ for all $(r, \tilde{r}) \in \mathbb{R} \times \mathbb{R}$.

The existence of solutions to (P_ε) is established next.

Theorem 2.1. *For every $\varepsilon > 0$, problem (P_ε) has at least one solution $(y_\varepsilon, v_\varepsilon, j_\varepsilon) \in L^2(Q) \times L^2(Q) \times \mathcal{K}$.*

Proof. Let $d = \inf(P_\varepsilon)$, and let $(y_n, v_n, j_n) \in L^2(Q) \times L^2(Q) \times \mathcal{K}$ be such that $\mathcal{W}y_n + v_n = f$ and

$$d \le \|y_n - y_0\|^2_{L^2(Q)} + \frac{1}{\varepsilon}\int_Q (j_n(y_n) + j_n^*(v_n) - y_n v_n)\,dx\,dt \le d + \frac{1}{n}. \tag{2.3}$$

Since -3 is the first negative eigenvalue of $\mathcal{W}$, we have

$$(\mathcal{W}y, y) \ge -\tfrac{1}{3}\|\mathcal{W}y\|^2_{L^2(Q)} \qquad \text{for all} \quad y \in D(\mathcal{W}), \tag{2.4}$$

and so by (2.2) and (2.3) $C_1 > 0$ exists such that

$$\|y_n\|_{L^2(Q)} + \|v_n\|_{L^2(Q)} \le C_1 \qquad \text{for all } n.$$

Therefore on a further subsequence we have

$$\begin{cases} y_n \to y & \text{weakly in } L^2(Q), \\ v_n \to v & \text{weakly in } L^2(Q). \end{cases} \tag{2.5}$$

Note also that by convexity of j_n we have

$$|\gamma_n(r)| \le 2\beta_2|r|, \qquad |\gamma_n^*(r)| \le (2\beta_2)^{-1}|r| \qquad \text{for all} \quad r \in \mathbb{R},$$

where $|\gamma_n(r)| = \sup\{|w|\colon\ w \in \partial j_n(r)\}$, $|\gamma_n^*(r)| = \sup\{|w|\colon\ w \in \partial j_n^*(r)\}$. Then by the Arzela–Ascoli lemma there exists $j \in \mathcal{K}$ such that

$$j_n(r) \to j(r) \quad \text{and} \quad j_n^*(r) \to j^*(r) \tag{2.6}$$

uniformly on compact subsets of $\mathbb{R}$. By definition of ∂j_n we have

$$\begin{aligned} &\liminf_{n\to\infty} \int_Q j_n(y_n)\,dx\,dt \\ &\quad \ge \liminf_{n\to\infty} \int_Q j_n((1+\lambda\partial j_n)^{-1}y)\,dx\,dt \\ &\qquad - \limsup_{n\to\infty} \int_Q \partial j_n((1+\lambda\partial j_n)^{-1}y)((1+\lambda\partial j_n)^{-1}y - y_n)\,dx\,dt. \end{aligned} \tag{2.7}$$

To conclude the proof, we need the following lemma (see [5] and [1]):

Lemma 2.2. *Let $h \in L^p(Q)$, where $1 \le p \le \infty$ and let $j_n,\ j \in \mathcal{K}$ with $j_n \to j$ uniformly on bounded sets of $\mathbb{R}$. Then $\lim_{n\to\infty}(1+\lambda\partial j_n)^{-1}h = (1+\lambda\partial j)^{-1}h$ in $L^p(Q)$, and consequently $\lim_{n\to\infty}(1+\lambda\partial j_{n_k})^{-1}h(x) = (1+\lambda\partial j)^{-1}h(x)$ a.e. in Q, for a subsequence $\{n_k\}$ of $\{n\}$ and for every $\lambda > 0$.*

Now, from (2.7) and Lemma 2.2 it follows that

$$\lim_{n\to\infty}(1+\lambda\partial j_n)^{-1}(y) = (1+\lambda\partial j)^{-1}(y) \qquad \text{in } L^2(Q)$$

and almost everywhere in Q on some subsequence. By Fatou's lemma and (2.7) we get

$$\liminf_{n\to\infty} \int_Q j_n(y_n)\,dx\,dt \ge \int_Q j((1+\lambda\partial j)^{-1}y)\,dx\,dt + \lambda \int_Q |(\partial j)_\lambda(y)|^2 dx\,dt,$$

where $(\partial j)_\lambda = (1/\lambda)(1-(1+\lambda\,\partial j)^{-1})$ for all $\lambda > 0$.

Since $\lim_{\lambda\to 0} j((1+\lambda\partial j)^{-1}y) = j(y)$ a.e. in Q, another application of Fatou's lemma to the latter inequality implies

$$\liminf_{n\to\infty} \int_Q j_n(y_n)\,dx\,dt \ge \int_Q j(y)\,dx\,dt. \tag{2.8}$$

A similar argument applied to the subsequences $\{j_n^*\}$ and $\{v_n\}$ gives

$$\liminf_{n\to\infty} \int_Q j_n^*(v_n)\,dx\,dt \ge \int_Q j^*(v)\,dx\,dt. \tag{2.9}$$

Now taking limit $n \to \infty$ in (2.3) and using Proposition 1.1, (2.5), (2.8), and (2.9) we obtain

$$d = \|y - y_0\|^2 + \frac{1}{\varepsilon} \int_Q (j(y) + j^*(v) - yv)\, dx\, dt,$$

which gives the claim with $(y_\varepsilon, v_\varepsilon, j_\varepsilon) = (y, v, j)$. □

Theorem 2.3. *The family of solutions* $\{(y_\varepsilon, v_\varepsilon, j_\varepsilon)\}_{\varepsilon>0}$ *of* (P_ε) *has at least one cluster point* $(\bar{y}, \bar{v}, \bar{j})$ *in* $L^2(Q)_w \times L^2(Q)_w \times \mathcal{K}$ *for* $\varepsilon \to 0^+$. *For every such cluster point* $\mathcal{W}\bar{y} + \partial\bar{j}(\bar{y}) \ni f$, $(\bar{y}, \bar{j})$ *is a solution to problem* (P) *and*

$$\lim_{\varepsilon\to 0^+} \inf(P_\varepsilon) = \inf(P). \tag{2.10}$$

Moreover, if $\tilde{y}_\varepsilon \in L^2(Q)$ *denotes the solution to problem* (1.1) *with* $g = \partial j_\varepsilon$, *then*

$$\lim_{\varepsilon\to 0} \|\tilde{y}_\varepsilon - y_0\|_{L^2(Q)} = \inf(P). \tag{2.11}$$

Proof. For every $\varepsilon > 0$ we have

$$\|y_\varepsilon - y_0\|^2_{L^2(Q)} + \varepsilon^{-1}\int_Q (j_\varepsilon(y_\varepsilon) + j^*_\varepsilon v_\varepsilon) - y_\varepsilon v_\varepsilon)\, dx\, dt \le \|y_j - y_0\|^2_{L^2(Q)}, \tag{2.12}$$

where y_j denotes a solution to (1.1) with $g = \partial j$, $j \in \mathcal{K}$.

This yields

$$\limsup_{\varepsilon\to 0^+} \|y_\varepsilon - y_0\|^2_{L^2(Q)} \le \inf\{\|y_j - y_0\|^2_{L^2=(Q)}\colon\ y \in L^2(Q),\ j \in \mathcal{K},\ \mathcal{W}y + \partial j(y) \ni f\} \tag{2.13}$$

and

$$\lim_{\varepsilon\to 0^+} \int_Q (j_\varepsilon(y_\varepsilon) + j^*_\varepsilon(v_\varepsilon) - y_\varepsilon v_\varepsilon)\, dx\, dt = 0. \tag{2.14}$$

By the same arguments as in the proof of Theorem 2.1 there is a subsequence, again denoted by ε, and an element $(\bar{y}, \bar{v}, \bar{j}) \in L^2(Q) \times L^2(Q) \times \mathcal{K}$ such that

$$\begin{aligned} &y_\varepsilon \to \bar{y} && \text{weakly in } L^2(Q),\\ &v_\varepsilon \to \bar{v} && \text{weakly in } L^2(Q),\\ &j_\varepsilon(r) \to \bar{j}(r) \quad \text{and} \quad j^*_\varepsilon(r) \to \bar{j}^*(r) && \text{uniformly on compact sets.} \end{aligned}$$

Moreover, we find that

$$\liminf_{\varepsilon\to 0^+} \int_Q (j_\varepsilon(y_\varepsilon) + j^*_\varepsilon(v_\varepsilon))\, dx\, dt \ge \int_Q (\bar{j}(\bar{y}) + \bar{j}^*(\bar{v}))\, dx\, dt \tag{2.15}$$

and

$$\lim_{\varepsilon\to 0^+} \int_Q y_\varepsilon v_\varepsilon\, dx\, dt = \int_Q \bar{y}\, \bar{v}\, dx\, dt. \tag{2.16}$$

Then by (2.14)–(2.16) we have

$$\int_Q (\bar{j}(\bar{y}) + \bar{j}^*(\bar{v}) - \bar{y}\,\bar{v})\,dx\,dt \le 0,$$

and so

$$\mathcal{W}\bar{y} + \partial\bar{j}(\bar{y}) \ni f. \tag{2.17}$$

Hence by (2.13) the latter yields

$$\|\bar{y} - y_0\|_{L^2(Q)} = \inf\{\|y - y_0\|_{L^2(Q)};\ j \in \mathcal{K},\ \mathcal{W}y + \partial j(y) \ni f\}$$

and $(\bar{y}, \bar{j})$ is a solution to (P). To verify (2.11), we observe that due to $\mathcal{W}\tilde{y}+\partial j_\varepsilon(\tilde{y}_\varepsilon) \ni f$, we have

$$\int_Q (j_\varepsilon(\tilde{y}_\varepsilon) + j_\varepsilon^*(v_\varepsilon) - \tilde{y}_\varepsilon v_\varepsilon)\,dx\,dt = 0 \tag{2.18}$$

with $v_\varepsilon = f - \mathcal{W}\tilde{y}_\varepsilon$. Arguing as above, it follows that on a subsequence, again denoted by ε, and for some $(\tilde{y}, \tilde{v}) \in L^2(Q)_w \times L^2(Q)_w$ we have

$$\tilde{y}_\varepsilon \to \tilde{y} \qquad \text{weakly in } L^2(Q),$$
$$v_\varepsilon \to \tilde{v} \qquad \text{weakly in } L^2(Q).$$

Moreover, we find that

$$\mathcal{W}\tilde{y} + \tilde{v} = f$$

and

$$\int_Q (\bar{j}(\tilde{y}) + \bar{j}^*(\tilde{v}) - \tilde{y}\tilde{v})\,dx\,dt = 0.$$

Finally, by (2.13) with $y_\varepsilon := \tilde{y}_\varepsilon$ the latter yields (2.11) and the proof is complete. □

3. The Approximate Identification Processes Revised

Due to the fact that j is defined on an unbounded domain and satisfies a convexity constraint, problems (P_ε) are not amenable for computer implementation. So we reformulate (P_ε) as a nonlinear optimal control problem in Hilbert space. The first difficulty can be avoided by an a priori estimate on the range of functions on which ∂j needs to operate. The convexity condition is replaced by a control constraint.

The parameters will be chosen from the following subset of $\mathcal{K}$:

$$\tilde{\mathcal{K}} = \{j = \varphi + \varphi_0\colon\ \varphi \in H_0^1(I) \cap H^2(I), \alpha \le \varphi'' \le \beta \text{ a.e. in } I\}, \tag{3.1}$$

where $I = (-2\delta, 2\delta)$ is an interval with $\delta > 0$ to be determined below, $0 < \alpha < \beta < \frac{3}{2}$ and $\varphi_0 \in H^2(I)$ a given convex function. The interval I is a priori determined by the condition $y(x,t) \in I$ for almost all $(x,t) \in Q$ and for all solutions y to $\mathcal{W}y + \partial j(y) \ni f$, where $\alpha \le \varphi'' \le \beta$. In fact, we have the following:

Lemma 3.1. *Assume that $f \in L^\infty(Q)$. Then there is $\delta = \delta(\alpha, \beta, f) > 0$ such that for every cluster point $(\bar{y}, \bar{v}, \bar{j})$ from Theorem 2.3 we have $\bar{y} \in L^\infty(Q)$ and*

$$\|\bar{y}(x,t)\|_{L^\infty(Q)} \le \delta. \tag{3.2}$$

Proof. We use some arguments from [4] and [3]. We consider the family of minimization problems

$$\min \left\{ \|y - y_0\|^2_{L^2(Q)} + \frac{1}{\varepsilon} \int_Q (j^\lambda(y) + (j^\lambda)^*(v) - yv)\, dx\, dt : \right. \\ \left. y \in L^2(Q),\ j \in \mathcal{K},\ v \in L^2(Q),\ \mathcal{W}y + v = f \right\}, \tag{P_ε^λ}$$

where $\varepsilon, \lambda > 0$ and j^λ is the Moreau–Yosida approximation of j. Arguing as above, we see that there is $(y_\varepsilon^\lambda, v_\varepsilon^\lambda, j_\varepsilon^\lambda)$ such that

$$\|y_\varepsilon^\lambda - y_0\| + \frac{1}{\varepsilon} \int_Q (j_\varepsilon^\lambda(y_\varepsilon^\lambda) + (j_\varepsilon^\lambda)^*(v_\varepsilon^\lambda) - y_\varepsilon^\lambda v_\varepsilon^\lambda)\, dx\, dt = \inf(P_\varepsilon^\lambda)$$

and for $\varepsilon \to 0^+$ there is at least one cluster point $(\bar{y}^\lambda, \bar{v}^\lambda, \bar{j}^\lambda)$, with

$$\mathcal{W}\bar{y}^\lambda + \partial \bar{j}^\lambda(\bar{y}^\lambda) \ni f \tag{3.3}$$

and $(\bar{y}^\lambda, \bar{j}^\lambda)$ is a solution to problem (P) for every $\lambda > 0$. Also for $\lambda \to 0^+$ there is $(\bar{y}, \bar{j}) \in L^2(Q \times L^2(Q)$ such that $\bar{y}^\lambda \to \bar{y}$, $\bar{j}^\lambda \to \bar{j}$, $\mathcal{W}\bar{y} + \partial \bar{j}(\bar{y}) \ni f$ and $\|\bar{y} - y_0\|^2_{L^2(Q)} = \inf(P)$.

We write $\bar{y}^\lambda = (\bar{y}^\lambda)^1 + (\bar{y}^\lambda)^2$ where $(\bar{y}^\lambda)^1 \in R(\mathcal{W})$ and $(\bar{y}^\lambda)^2 \in N(\mathcal{W})$. Since $\{\bar{y}^\lambda\}$ is bounded in $L^2(Q)$ we have by Proposition 1.1 that

$$\|(\bar{y}^\lambda)^1\|_{L^\infty(Q)} \le C_1, \qquad \forall \varepsilon > 0. \tag{3.4}$$

It is readily seen that

$$N(\mathcal{W}) = \left\{ y \in L^2(Q);\ y(x,t) = q(t+x) - q(t-x); \right. \\ \left. q \text{ is } 2\pi\text{-periodic and } \int_0^{2\pi} q(s)\, ds = 0 \right\}. \tag{3.5}$$

We may therefore write

$$(\bar{y}^\lambda)^2(x,t) = q^\lambda(t+x) - q^\lambda(t-x), \qquad \forall (x,t) \in Q, \tag{3.6}$$

where q^λ is 2π-periodic. This yields

$$q^\lambda(t) = \frac{1}{2\pi} \int_0^{2\pi} ((\bar{y}^\lambda)^2(x, t-x) - (\bar{y}^\lambda)^2(x, t+x))\, dx, \qquad \forall t \in (0, 2\pi), \tag{3.7}$$

and therefore

$$\|q^\lambda\|_{L^2(0,2\pi)} \le C_2, \qquad \forall \lambda > 0. \tag{3.8}$$

Note also that by virtue of (3.5) a function $\eta \in L^2(Q)$ belongs to $R(\mathcal{W}) = N(\mathcal{W})^\perp$ if and only if

$$\int_0^\pi (\eta(x, t-x) - \eta(x, t+x))\, dx = 0 \qquad \text{a.e.} \quad t \in (0, 2\pi). \tag{3.9}$$

Then by (2.17) and (3.3) we have

$$\begin{aligned}
&\int_0^\pi ((\bar{j}^\lambda)'((\bar{y}^\lambda)^1(x, t-x) + (\bar{y}^\lambda)^2(x, t-x)) \\
&\quad - (\bar{j}^\lambda)'((\bar{y}^\lambda)^1(x, t+x) + (\bar{y}^\lambda)^2(x, t+x)))\, dx \\
&\quad = \int_0^\pi (f(x, t-x) - f(x, t+x))\, dx, \quad \text{a.e.} \quad t \in (0, 2\pi).
\end{aligned}$$

Since $(\bar{j}^\lambda)'$ is not decreasing, by (3.3) we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^\pi (\bar{j}^\lambda)'(-C_1 + (\bar{y}^\lambda)^2(x, t-x)) - (\bar{j}^\lambda)'(C_1 + (\bar{y}^\lambda)^2(x, t+x))\, dx \\
&\quad \le \|f\|_{L^\infty(Q)}, \qquad \text{a.e.} \quad t \in (0, 2\pi).
\end{aligned} \tag{3.10}$$

Then by (3.5) we see that

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^\pi ((\bar{j}^\lambda)'(-C_1 + q^\lambda(t) - q^\lambda(t-2x)) - (\bar{j}^\lambda)'(C_1 - q^\lambda(t) + q^\lambda(t+2x)))\, dx \\
&\quad \le \|f\|_{L^\infty(Q)}, \qquad \text{a.e.} \quad t \in (0, 2\pi).
\end{aligned}$$

Finally, since q is 2π-periodic, we have

$$\begin{aligned}
&\frac{1}{4\pi}\int_0^{2\pi} ((\bar{j}^\lambda)'(-C_1 + q^\lambda(t) - q^\lambda(s)) - (\bar{j}^\lambda)'(C_1 - q^\lambda(t) + q^\lambda(s)))\, ds \\
&\quad \le \|f\|_{L^\infty(Q)}, \qquad \text{a.e.} \quad t \in (0, 2\pi).
\end{aligned}$$

Let $\bar{M} = \operatorname{ess\,sup}\{q^\lambda(t); t \in (0, 2\pi)\}$ and $\bar{E} = \{s \in (0, 2\pi); q(t) > 2^{-1}\bar{M}\}$. Hence

$$\begin{aligned}
&\frac{1}{4\pi}(2\pi - m(\bar{E}))((\bar{j}^\lambda)'(-C_1 + q(t) - \tfrac{1}{2}\bar{M}) - (\bar{j}^\lambda)'(C_1 + \tfrac{1}{2}\bar{M} - q(t))) \\
&\qquad + \frac{1}{4\pi} m(\bar{E})((\bar{j}^\lambda)'(-C_1 + q(t) - \bar{M}) - (\bar{j}^\lambda)'(C_1 + \bar{M} - q(t))) \\
&\quad \le \|f\|_{L^\infty(Q)}, \qquad \text{a.e.} \quad t \in (0, 2\pi),
\end{aligned}$$

where $m(E)$ is the measure of the set E. Since $(\bar{j}^\lambda)'$ is continuous, we have

$$\begin{aligned}
&\frac{1}{4\pi}(2\pi - m(\bar{E}))((\bar{j}^\lambda)'(-C_1 + \tfrac{1}{2}\bar{M}) - (\bar{j}^\lambda)'(C_1 - \tfrac{1}{2}\bar{M})) \\
&\quad + \frac{1}{4\pi} m(\bar{E})((\bar{j}^\lambda)'(-C_1) - (\bar{j}^\lambda)'(C_1)) \le \|f\|_{L^\infty(Q)}
\end{aligned}$$

and then by (3.1) we deduce that

$$4\pi \| f\|_{L^\infty(Q)} + 2\pi(\alpha+\beta+\varphi_0'')C_1 \\ \geq (2\pi - m(\bar{E}))(\alpha+\beta+\varphi_0'')(\tfrac{1}{2}\bar{M} - C_1). \tag{3.11}$$

By (3.7) we have

$$m(\bar{E}) \leq C_2 \bar{M}^{-2}, \qquad \forall\lambda > 0,$$

and so (3.11) implies

$$4\pi \| f\|_{L^\infty(Q)} + 4\pi C_1(\alpha+\beta+\varphi_0'') \geq (\alpha+\beta+\varphi_0'')\left(\pi\bar{M} - \frac{C_2}{2\bar{M}}\right).$$

Since $\theta(r) = \pi r - C_2/2r$ is bijective and nondecreasing on $\mathbb{R}_+^*$, the latter yields

$$\bar{M} \leq \theta^{-1}(4\pi(C_1 + \| f\|_{L^\infty(Q)}/(\alpha+\beta+\varphi_0''))). \tag{3.12}$$

After a similar evaluation for $N = \operatorname{ess\,inf}\{q^\lambda(t), t \in (0, 2\pi)\}$, we can define $\delta = \max\{M, -N\}$ and the proof is complete. □

Now we consider the penalized problems:

$$\inf\left\{\|y - y_0\|^2_{L^2(Q)} + \frac{1}{\varepsilon}\int_Q (j(y) + j^*(v) - yv)\,dx\,dt; \right.\\ \left. y \in L^2(Q),\ j \in \tilde{\mathcal{K}},\ |y(x,t)| \leq 2\delta \text{ a.e. } (x,t) \in Q,\ \mathcal{W}y + v = f\right\}. \tag{$\tilde{P}_\varepsilon$}$$

As in the general case of Section 2 we have

Theorem 3.2. *For every $\varepsilon > 0$ problem $(\tilde{P}_\varepsilon)$ has at least one cluster solution $(y_\varepsilon, v_\varepsilon, j_\varepsilon)$ in $L^2(Q) \times L^2(Q) \times \tilde{\mathcal{K}}$. Moreover, the family $\{(y_\varepsilon, v_\varepsilon, j_\varepsilon)\colon \varepsilon > 0\}$ has at least one cluster point $(\bar{y}, \bar{v}, \bar{j})$ in $L^2(Q) \times L^2(Q) \times \tilde{\mathcal{K}}$ and each point $(\bar{y}, \bar{j})$ is a solution of*

$$\inf\{\|y - y_0\|^2_{L^2(Q)};\ j \in \tilde{\mathcal{K}},\ y \in L^2(Q),\ \mathcal{W}y + \partial j(y) \ni f\}.$$

Problems $(\tilde{P}_\varepsilon)$ can be approached by iterating the following two problems:

SP1. For fixed $y \in L^2(Q)$, with $|y(x,t)| \leq 2\delta$, $\mathcal{W}y + v = f$ a.e. in Q, solve

$$\inf\left\{\int_Q (j(y) + j^*(v))\,dx\,dt\colon\ j \in \tilde{\mathcal{K}}\right\}, \tag{3.13}$$

where $j^*(p) = \sup\{pr - j(r)\colon |r| \leq 2\delta\}$, for $p \in \mathbb{R}$.

SP2. For fixed $j \in \tilde{\mathcal{K}}$, solve

$$\inf\left\{ J_2(y,v) = \|y - y_0\|^2_{L^2(Q)} + \varepsilon^{-1} \int_Q (j(y) + j^*(v) - yv)\, dx\, dt; \right.$$
$$\left. y, v \in L^2(Q), |y(x,t)| \le 2\delta,\ \mathcal{W}y + v = f \right\}. \tag{3.14}$$

Using the same approach as in [5] and [1], problem (3.13) can be equivalently written as a constrained optimal control problem in $L^2(Q)$, namely

$$\inf\left\{ \int_Q (\varphi(y) + (\varphi + \varphi_0)^*(v))\, dx\, dt : \right.$$
$$\left. \varphi'' = \psi \text{ in } I, \varphi \in H^1_0(I), \psi \in L^2(I), \alpha \le \psi \le \beta \text{ a.e. in } I \right\}. \tag{3.15}$$

If we denote by $\Phi_0\colon\ H^1_0(I) \to \mathbb{R}$ the function

$$\Phi_0(\varphi) = \int_Q (\varphi(y) + (\varphi + \varphi_0)^*(v))\, dx\, dt,$$

then (3.15) reduces to

$$\inf\{\Phi_0(\varphi); \varphi'' = \psi \text{ in } I, \psi \in L^2(I), \varphi \in H^1_0(I), \alpha \le \psi \le \beta\}. \tag{3.16}$$

The function Φ_0 is continuous and convex on $H^1_0(I)$. Next we define $\Phi\colon\ L^2(I) \to \bar{\mathbb{R}}$ by

$$\Phi(\psi) = \begin{cases} \Phi_0(\varphi), & \text{if } \quad \varphi'' = \psi \quad \text{in } I \text{ with } \quad \varphi \in H^1_0(I), \quad \alpha \le \psi \le \beta, \\ +\infty, & \text{otherwise.} \end{cases}$$

It is known that the subdifferential of Φ is given by (see [5])

$$\partial\Phi(\psi) = N_U(\psi) - p, \tag{3.17}$$

where $N_U(\psi)$ is the normal cone to $U = \{\psi \in L^2(I); \alpha \le \psi \le \beta \text{ a.e. in } I\}$ and $p \in H^1_0(I)$ is the solution to

$$\begin{cases} -p'' = \mu_1 + \mu_2 & \text{in } I \\ p \quad = 0 & \text{on } \partial I, \end{cases} \tag{3.18}$$

with $\mu_1, \mu_2 \in H^{-1}(I)$ defined by

$$\mu_1(k) = \int_Q k(y)\, dx\, dt,$$
$$\mu_2(k) = -\int_Q k(j' + N_I)^{-1}(v)\, dx\, dt \tag{3.19}$$

for all $k \in H_0^1(I)$, where

$$(j' + N_I)^{-1}(v) = \begin{cases} (j')^{-1}(v) & \text{if} \quad |(j')^{-1}(v)| < 2\delta, \\ 2\delta & \text{if} \quad (j')^{-1}(v) \geq 2\delta, \\ -2\delta & \text{if} \quad (j')^{-1}(v) \leq -2\delta, \end{cases}$$

and $j = \varphi + \varphi_0$.

Since problem (3.16) can be written as

$$\inf\{\Phi(\psi) \colon \ \psi \in L^2(I)\},$$

we infer by (3.17) that a solution ψ must satisfy

$$\begin{cases} \psi = \alpha & \text{if} \quad p < 0, \\ \psi = \beta & \text{if} \quad p > 0, \\ \psi \in (\alpha, \beta) & \text{if} \quad p = 0, \end{cases} \tag{3.20}$$

where p satisfy (3.18).

As regards **SP2**, by variational arguments it follows that the solution (y, v) to (3.14) satisfies

$$\int_Q (2\varepsilon(y - y_0) + j'(y) - v)z\,dx\,dt + \int_Q (j^{*\prime}(v) - y)w\,dx\,dt = 0 \tag{3.21}$$

for all $(z, w) \in L^2(Q) \times L^2(Q)$ such that $\mathcal{W}z + w = 0$. In particular, for $w = 0$, (3.21) yields

$$2\varepsilon(y - y_0) + j'(y) - v \in N^{\perp}(\mathcal{W}) = R(\mathcal{W}).$$

Hence, there is $q \in L^2(Q)$ such that

$$\mathcal{W}q = 2\varepsilon(y - y_0) + j'(y) - v. \tag{3.22}$$

By substituting the latter in (3.21), we get

$$\int_Q (-q + j^{*\prime}(v) - y)w\,dx\,dt = 0, \qquad \forall w \in R(\mathcal{W}),$$

i.e., $-q + j^{*\prime}(v) - y \in R(\mathcal{W})^{\perp} = N(\mathcal{W})$. Let $\eta \in N(\mathcal{W})$. If we denote again by q the function $q + \eta$, we get

$$v = j'(q + y), \qquad \text{a.e.} \quad (x, t) \in Q. \tag{3.23}$$

Then, taking into account (3.22) and (3.23), we infer that the solution of (3.14) must satisfy the maximum principle

$$\begin{cases} \mathcal{W}q = 2\varepsilon(y - y_0) + j'(y) - v, \\ v = j'(q + y), \\ \mathcal{W}y + v = f. \end{cases} \tag{3.24}$$

4. A Numerical Algorithm and Numerical Experiments

In this section we present a conceptual algorithm of gradient type, in order to calculate the approximating cluster point $(\bar{y}, \bar{v}, \bar{j}) \in L^2(Q) \times L^2(Q) \times \tilde{\mathcal{K}}$, using the iterative processes **SP1** and **SP2** established in the above section (see Theorem 3.2).

Firstly, we construct the discrete problems corresponding to **SP1** and **SP2**. For this we divide the time-interval $[0, 2\pi]$ into M equal parts

$$0 = t_0 < t_1 < \cdots < t_M = 2\pi$$

with $h_1 = 2\pi/M$, and the space-interval $[0, \pi]$ into N equal parts

$$0 = x_0 < x_1 < \cdots < x_N = \pi$$

with $h_2 = \pi/N$. For the interval $I = [-2\delta, 2\delta]$ we consider the grid

$$\left\{\frac{2\delta}{L}\ell - 2\delta\right\}, \qquad \ell = \overline{0, 2L},$$

with $h_3 = 2\delta/L$.

Denote by y_k^i, v_k^i, f_k^i, and q_k^i the approximate matrix for the unknowns $y(x,t)$, $v(x,t)$, $f(x,t)$, and $q(x,t)$, respectively, that is

$$\begin{cases} y_k^i = y(x_k, t_i), \\ v_k^i = v(x_k, t_i), \\ f_k^i = f(x_k, t_i), \\ q_k^i = q(x_k, t_i), \end{cases} \qquad k = \overline{0, N}, \quad i = \overline{0, M}.$$

The unknowns $(p, \psi, \varphi) \in H_0^1(I) \times L^2(I) \times H_0^1(I)$ are approximated by the vectors $(p^\ell)^T_{\ell=\overline{0,2L}}$, $(\psi)^T_{\ell=\overline{0,2L}}$, $(\varphi^\ell)^T_{\ell=\overline{0,2L}}$, where

$$p^\ell = p(\ell h_3 - 2\delta), \quad \psi^\ell = \psi(l h_3 - 2\delta), \quad \varphi^\ell = \varphi(\ell h_3 - 2\delta), \quad \ell = \overline{0, 2L}.$$

For numerical experiments we choose $\varphi_0 = 0$ such that $j = \varphi \in \tilde{\mathcal{K}}$, i.e.,

$$\tilde{\mathcal{K}} = \{j = \varphi,\ \varphi \in H_0^1(I) \times H^2(I),\ \alpha \le \varphi'' \le \beta \text{ a.e. in } I\}.$$

To approximate the strictly convex functions $j \in \tilde{\mathcal{K}}$ we follow the same process as in [5] and [1]. So, let $\{E_\ell^L\}_{\ell=1}^{2L-1}$ be a basis to approximate elements in $\tilde{\mathcal{K}}$ given by

$$E_\ell^L(s) = \begin{cases} s^2/2 + a_\ell s + b_\ell, & \text{if} \quad s \in [s_{\ell-1}, s_{\ell+1}], \\ 0 & \text{otherwise}, \end{cases}$$

where $s_\ell = (2\delta/L)\ell - 2\delta$, $\ell = \overline{0, 2L}$. Every element E_ℓ^L could be computed analytically as a solution to

$$\begin{cases} \Delta E_\ell^L = e_\ell^L, \\ E_\ell^L(-2\delta) = E_\ell^L(2\delta) = 0, \end{cases}$$

where

$$e_\ell^L = \begin{cases} 1 & \text{for} \quad s \in [(2\delta/L)(\ell-1) - 2\delta, (2\delta/L)(\ell+1) - 2\delta], \\ 0 & \text{otherwise}. \end{cases}$$

Thus, for numerical computations we consider elements $j^L \in K^L$ to approximate elements $j \in \tilde{\mathcal{K}}$, where

$$K^L = \left\{ j^L = \sum_{\ell=1}^{2L-1} j_\ell^L E_\ell^L, \alpha \leq j_\ell^L \leq \beta, \ell = \overline{1, 2L-1}. \right\}$$

We note that the functions E_ℓ^L, $\ell = \overline{1, 2L-1}$, are convex. So the condition $\alpha \leq j_\ell^L \leq \beta$, $\ell = \overline{1, 2L-1}$, guarantees that $j^L \in \tilde{\mathcal{K}}$.

Next, we give the discretized forms of problems **SP1** and **SP2**. On the interval I, relation (3.18) becomes

$$\begin{cases} -\int_I p''(s)k(s)\,ds = \int_Q k(y)\,dx\,dt - \int_Q k((j' + N_I)^{-1}(v))\,dx\,dt, \\ p(-2\delta) = p(2\delta) = 0. \end{cases} \tag{4.1}$$

So, on every subinterval $[s_{\ell-1}, s_{\ell+1}]$, $\ell = \overline{1, 2L-1}$, we consider

$$k_\ell(s) = \begin{cases} as + b & \text{if} \quad s \in [s_{\ell-1}, s_\ell], \\ cs + d & \text{if} \quad s \in [s_\ell, s_{\ell+1}], \end{cases}$$

such that

$$k_\ell(s_{\ell-1}) = k_\ell(s_{\ell+1}) = 0 \quad \text{and} \quad \int_{s_{\ell-1}}^{s_{\ell+1}} k_\ell(s)\,ds = 1.$$

Then, using a standard implicit scheme, (4.1) can be approximated as follows:

$$-p^{\ell-1} + 2p^\ell - p^{\ell+1} = h_3^2(\mu_1(k_\ell(s_\ell)) + \mu_2(k_\ell(s_\ell))), \qquad \ell = \overline{1, 2L-1}, \tag{4.2}$$

where, for suitable weights $\xi_k^i > 0$,

$$\begin{aligned} \mu_1(k_\ell(s)) &= \int_Q k_\ell(y(x,t))\,dx\,dt \\ &\approx \begin{cases} \sum_{i,k} (ay_k^i + b)\xi_k^i, & \text{if} \quad y_k^i \in [s_{\ell-1}, s_\ell], \\ \sum_{i,k} (cy_k^i + d)\xi_k^i, & \text{if} \quad y_k^i \in [s_\ell, s_{\ell+1}], \end{cases} \end{aligned} \tag{4.3}$$

$$\begin{aligned} \mu_2(k_\ell(s)) &= -\int_Q k_\ell((j' + N_I)^{-1}(v(x,t)))\,dx\,dt \\ &\approx \begin{cases} \sum_{i,k} (a\theta_k^i + b)\xi_k^i, & \text{if} \quad \theta_k^i \in [s_{\ell-1}, s_\ell], \\ \sum_{i,k} (c\theta_k^i + d)\xi_k^i, & \text{if} \quad \theta_k^i \in [s_\ell, s_{\ell+1}], \end{cases} \end{aligned} \tag{4.4}$$

with

$$\theta_k^i = \begin{cases} w_k^i & \text{if} \quad |w_k^i| < 2\delta, \\ -2\delta & \text{if} \quad w_k^i \leq -2\delta, \\ 2\delta & \text{if} \quad w_k^i \geq 2\delta, \end{cases}$$

$$w_k^i = \frac{v_k^i - \sum_{\ell=0}^{2L} j_\ell^L a_\ell}{\sum_{\ell=0}^{2L} j_\ell^L}, \quad i = \overline{0, M}, \quad k = \overline{0, N}.$$

From (3.20) we derive that

$$\psi^\ell = \begin{cases} \alpha & \text{if} \quad p^\ell < 0, \\ \beta & \text{if} \quad p^\ell > 0, \qquad \ell = \overline{0, 2L}. \\ \psi^\ell \in (\alpha, \beta) & \text{if} \quad p^\ell = 0. \end{cases} \tag{4.5}$$

Using again a standard implicit scheme, the equation $\varphi'' = \psi$ in (3.15) can be approximated by

$$A\varphi = d_1, \tag{4.6}$$

where $\varphi = (\varphi^\ell)^T_{\ell=\overline{1,2L-1}}$, $d_1 = h_3^2(\psi^\ell)^T_{\ell=\overline{1,2L-1}}$ and

$$A = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 2 \end{pmatrix}.$$

Now, for fixed y and v, $(3.24)_1$ takes the discretized form (we also used a standard implicit scheme to approximate q_{tt} and q_{xx})

$$Bq = d_2, \tag{4.7}$$

where B is the $(N-1)M \times (N-1)M$ matrix given by

$$\begin{aligned} &B[k][k] = c_7, \qquad k = \overline{1, (N-1)M}; \\ &B[k][k+N-1] = B[k+N-1][k] = c_5, \qquad k = \overline{1, (N-1)(M-1)}; \\ &B[k][(M-1)(N-1)+k] = B[(M-1)(N-1)+k][k] = c_5, \\ &\qquad k = \overline{1, N-2}; \\ &B[k+1+(N-1)i][k+(N-1)i] = B[k+(N-1)i][k+1+(N-1)i] \\ &\qquad = -c_6, \qquad i = \overline{0, M-1}, \quad k = \overline{1, N-2}, \end{aligned}$$

with

$$c_5 = \frac{1}{h_1^2}, \qquad c_6 = \frac{1}{h_2^2}, \qquad c_7 = 2(c_6 - c_5),$$

and, for $i = 0, 1, \dots, M-1$,

$$\begin{cases} d_2[i(N-1)+1] = 2\varepsilon(y_1^{i+1} - y_1^{0,i+1}) + j'(y_1^{i+1}) - v_1^{i+1} + c_6 y_0^{i+1}, \\ d_2[i(N-1)+N-1] = 2\varepsilon(y_{N-1}^{i+1} - y_{N-1}^{0,i+1}) + j'(y_{N-1}^{i+1}) - v_{N-1}^{i+1} + c_6 y_N^{i+1}, \\ d_2[i(N-1)+k] = 2\varepsilon(y_k^{i+1} - y_k^{0,i+1}) + j'(y_k^{i+1}) - v_k^{i+1}, \qquad k = \overline{2, N-2}, \end{cases}$$

while $q = (q_k^i)_{i=\overline{0,M}, k=\overline{1,N-1}}$ is the unknown matrix.

Relation $(3.24)_2$ leads to the discrete form

$$v_k^i = j'(q_k^i + y_k^i), \qquad i = \overline{0, M}, \quad k = \overline{0, N}. \tag{4.8}$$

Finally, making use of (4.8), $(3.24)_3$ is discretized as follows:

$$By = d_3, \tag{4.9}$$

where

$$\begin{cases} d_3[i(N-1)+1] = f_1^{i+1} - v_1^{i+1} + c_6 y_0^{i+1}, \\ d_3[i(N-1)+N-1] = f_{N-1}^{i+1} - v_{N-1}^{i+1} + c_6 y_N^{i+1}, \\ d_3[i(N-1)+k] = f_k^{i+1} - v_k^{i+1}, \qquad \text{for} \quad i = \overline{0, M-1}, \quad k = \overline{2, N-2}, \end{cases}$$

and $y = (y_k^i)_{i=\overline{0,M}, k=\overline{1,N-1}}$ is the unknown matrix.

To calculate the approximate solution in problem $(\tilde{P}_\varepsilon)$ we use the following numerical algorithm of gradient type (itk denotes the iteration number):

Step 0: *Initializations*.
Set $itk = 1$;
Choose $\alpha, \beta \in (0, \frac{3}{2})$ and $\delta > 0$;
Choose $y_{\exp}(x, t)$ and $j_{\exp}(s)$, $(x, t) \in Q$, $s \in I$;
Compute $j^{itk,L}(s_\ell) = j_{\exp}(s_\ell)$, for all $s_\ell \in \{(2\delta/L)\kappa - 2\delta,\ \kappa = \overline{0, 2L}\}$;
Compute

$$\begin{cases} y_k^{itk,i} = y_0(x_k, t_i), & i = \overline{0, M-1}, \quad k = \overline{1, N-1}, \\ y_0^{itk,i} = y_N^{itk,i} = 0, & i = \overline{0, M}, \\ y_k^{itk,M} = y_k^{itk,0}, & k = \overline{1, N}; \end{cases}$$

ritk:
Compute $v_k^{itk,i} = j^{itk,L}(y_k^{itk,i})$, $i = \overline{0, M}$, $k = \overline{0, N}$;
if *itk* $= 1$ then compute $f_k^i = \mathcal{W} y_{\exp}(x_k, t_i) + v_k^{itk,i}$, $i = \overline{0, M}$, $j = \overline{0, N}$;

Step 1: *Solve* **SP1**.
Let $\varepsilon j > 0$ be a prescribed precision;
if *itk* $= 1$ then
  Compute $J_2(y^{itk}, v^{itk})$;
  Test: if $J_2(y^{itk}, v^{itk}) < \varepsilon j$ then STOP;
Compute $p = (p^\ell)^T_{\ell=\overline{1,2L-1}}$ solving (4.2);
Compute $\psi = (\psi^\ell)^T_{\ell=\overline{1,2L-1}}$ from (4.5);
Establish a new j (denoted $j_{\text{new}}^{itk,L}$) solving (4.6);

Step 2: *Solve* **SP2**.
  Let $0 < dd \le \frac{1}{2}$ be prescribed;

Step 2.1.
Compute $\lambda = 1 - dd$;
Compute $j_{\text{work}}^{itk,L}$ by the following formula:

$$j_{\text{work}}^{itk+1,L}(s_\ell) = \lambda j^{itk,L}(s_\ell) + (1-\lambda) j_{\text{new}}^{itk,L}(s_\ell), \qquad \ell = \overline{1, 2L-1}; \tag{4.10}$$

Compute $q^{itk} = (q_k^{itk,i})_{i=\overline{0,M-1},k=\overline{1,N-1}}$ solving (4.7) where j is substituted by $j_{\text{work}}^{itk,L}$;
Compute $v_k^{itk+1,i}$ from (4.8) with $j_{\text{work}}^{itk,L}$ in place of j;
Compute $y^{itk+1} = (y_k^{itk+1,i})_{i=\overline{0,M-1},k=\overline{1,N-1}}$ solving (4.9);
Compute

$$y_k^{itk+1,i} = \begin{cases} y_k^{itk+1,i} & \text{if} \quad |y_k^{itk+1,i}| < 2\delta, \\ -2\delta & \text{if} \quad y_k^{itk+1,i} < -2\delta, \quad i = \overline{0, M-1}, \\ 2\delta & \text{if} \quad y_k^{itk+1,i} > 2\delta, \quad k = \overline{1, N-1}; \end{cases}$$

Compute y_{work}^{itk+1} by the following formula:

$$y_{\text{work},k}^{itk+1,i} = \lambda y_{\exp}(x_k, t_i) + (1-\lambda) y_k^{itk+1,i}, \qquad i = \overline{0, M}, \quad k = \overline{0, N};$$

Compute $J_2(y_{\text{work}}^{itk+1}, v^{itk+1})$ from (3.14);
Test: $J_2(y_{\text{work}}^{itk+1}, v^{itk+1}) < J_2(y^{itk}, v^{itk})$?
YES $\to$ there is a decrease; mark it and *Go to* Step 2.1;
NO $\to$ *Go to* Step 2.2.

Step 2.2.
Let *itkmax* > 0 be a prescribed number of iterations;
Test: is there a decrease?
YES $\to$ Compute $valj2 = |J_2(y_{\text{work}}^{itk+1}, v^{itk+1}) - J_2(y^{itk}, v^{itk})|$;
Test: if $(valj2 < \varepsilon j)$ then STOP.
$itk = itk + 1$;
Test: $itk \leq itkmax$?
YES $\to$ Retain the best values for $j_{\text{work}}^{itk,L}$ and $y_{\text{work},k}^{itk,i}$ in $j^{itk,L}$ and $y_k^{itk,i}$, respectively;
Go to ritk.
NO $\to$ STOP.
Go to Step 2.3.

Step 2.3.
Compute $dd = dd/2$;
Let $\varepsilon d > 0$ be a prescribed precision;
Test: $dd < \varepsilon d$?
YES $\to$ STOP.
NO $\to$ *Go to* Step 2.1.

The value λ from (4.10) is chosen from the following sequence (see also [6]):

$$1 - dd, 1 - 2 * dd, \ldots, 0. \tag{4.11}$$

Since we have a finite number of options for λ we choose λ_{itk} to be the value from the sequence (4.11) which minimizes $J_2(y, v)$ for all $J_{\text{work}}^{itk+1,L}$ in (4.10).

For numerical tests we choose

$$\alpha = 0.1, \quad \beta = 0.74, \quad \delta = 0.5, \quad M = L = 8, \quad N = 6, \quad itkmax = 5,$$

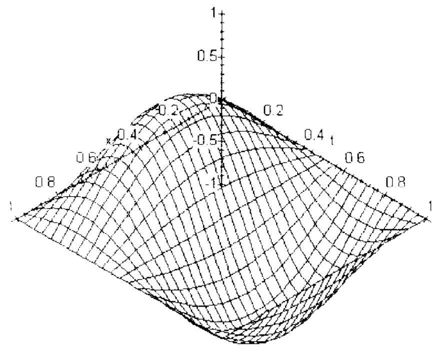

Figure 1

$$\varepsilon = 0.00001, \quad \varepsilon j = 0.0001, \quad \varepsilon d = 0.001,$$
$$y_{\exp}(x,t) = \sin x \cdot \sin t, \qquad t \in [0, 2\pi], \quad x \in [0, \pi] \text{ (see Figure 1)},$$
$$j_{\exp}(s) = \sum_{\ell=1}^{2L-1} j_\ell^L E_\ell^L \text{ with}$$
$$\begin{cases} j_\ell^L = \dfrac{\alpha+\beta}{2L^2}\ell(2L-\ell), \\ a_\ell = 2\delta\left(1 - \dfrac{\ell}{L}\right), & \ell = \overline{1, 2L-1}, \\ b_\ell = 2\delta^2\left(\dfrac{\ell+1}{L} - 1\right)\left(\dfrac{\ell-1}{L} - 1\right), \end{cases}$$
$$y0_k^i = y_k^i + \frac{2\delta}{L}, \qquad i = \overline{0, M}, \quad k = \overline{0, N}.$$

The optimal value for J_2 is obtained in three iterations, with the precision εj. The numerical solution for y is given in Figure 2, while the numerical solutions for j are given in Figure 3.

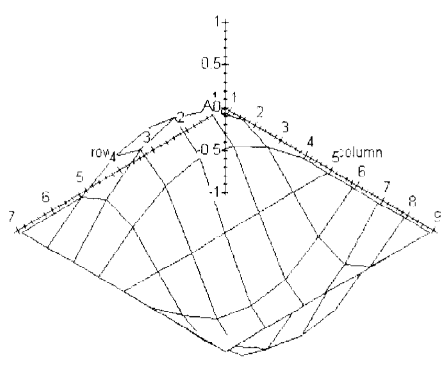

Figure 2. The numerical solution of y.

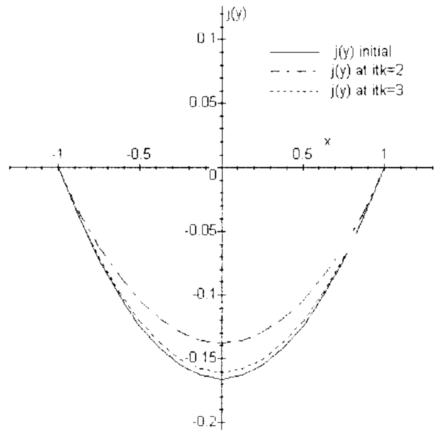

Figure 3. The numerical solution of $j(y)$.

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