Hamilton-Jacobi Equation and Optimality conditions for control systems governed by semilinear parabolic equations with boundary control

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January 7, 2004

[1]**Abstract** : We characterize the value function by an appropriate Hamilton Jacobi Bellman equation (in viscosity sense) and derive optimality conditions from the knowledge of the value function.

Keywords : value function, viscosity solutions.

AMS Classification. 49K20, 49L25, 49L20, 35B37, 93C20, 35K20.

1 Introduction

Let Ω be an open bounded subset in $\mathbf{R}^N$ $(N \geq 2)$ of class $C^{2,\beta}$ for some $\beta > 0$ and $Q = \Omega \times (0,T)$, $\Sigma = \Gamma \times (0,T)$, Γ is the boundary of Ω, $T > 0$.

We consider a control system described by the following parabolic equation

$$\begin{aligned}
&\frac{\partial y}{\partial t} + Ay + f(x,t,y) = 0 \text{ in } Q,\\
&\frac{\partial y}{\partial n_A} + g(s,t,y,v) = 0 \text{ on } \Sigma,\\
&y(\cdot,t_0) = y_0 \text{ in } \Omega,
\end{aligned} \tag{1.1}$$

where A is a second-order differential operator defined by

$$Ay(x) = -\sum_{i,j=1}^{N} D_i(a_{ij}(x)D_j y(x))$$

with coefficients a_{ij} belonging to $C^{1,\beta}(\overline{\Omega})$ and satisfying the conditions

$$a_{ij}(x) = a_{ji}(x) \quad \text{for every } i,j \in \{1,...,N\},$$

[1]Work supported by the TMR No. HPRN-CT-2002-00281 Evolution Equations and Deterministic and Stochastic Systems

$$m_0|\xi|^2 \le \sum_{i,j=1}^{N} a_{ij}(x)\xi_j\xi_i \le M|\xi|^2$$

for all $x \in \overline{\Omega}$ and all $\xi \in \mathbf{R}^N$, with $0 < m_0 < M$ (D_i denotes the partial derivative with respect to x_i).

Let $(\partial y/\partial n_A)(s,t) = \sum_{i,j} a_{ij}(s) D_j\, y(s,t) n_i(s)$ denote the normal derivative of y associated with A, and $n = (n_1, ..., n_N)$ is the outward unit normal to Γ.

In the present paper we will investigate the following optimal control problem

$$\inf \left\{ \int_\Omega L(x, y(T))dx \mid (y,v) \in L^\infty(Q) \times L^\sigma(\Sigma),\ (y,v) \text{ satisfies } (1.1) \right\}. \qquad (P)$$

With the above control problem we associate a value function

$$V(t_0, y_0) = \inf \left\{ \int_\Omega L(x, y(T; t_0, y_0, v))dx \,\middle|\, y \text{ is a solution to } (1.1),\ v \in L^\sigma(\Sigma) \right\} \tag{1.2}$$

letting (t_0, y_0) range over $[0,T] \times Y$, with $Y = L^\infty(\Omega) \cap H^1(\Omega)$.

Our main aim in this article is twofold: we wish first to characterize the value function by a appropriate Hamilton Jacobi Bellman equation (in viscosity sense), second to derive optimality conditions from the knowledge of the value function.

This problem has been considered in the early literature in the case of smooth value function [1] and for distributed control problems, namely the control does not act only on the boundary but on the set Ω (cf [2], [3], [13], [14]). Since the control acts on the system through a boundary condition, one cannot apply the existing results of characterization of the value function by viscosity solutions of Hamilton Jacobi equations ([4], [5], [7], [8], [9], [12]) but one has to provide a new proof. We derive necessary and sufficient optimality conditions using the -Lipschitz- value function. Note again that because this Mayer control problem has a boundary control, our technique slightly differ from those employed in [2] and [4].

Let us explain how our paper is organized. In the first section we state our assumptions and recall some basic estimations on the control dynamics. The second section is devoted to study the value function: we prove a dynamic programming principle and we obtain the Lipschitz regularity and the semi-concavity of the value-function. Section 3 establishes a characterization of the value function in terms of the unique solution (in viscosity sense) of an Hamilton Jacobi Bellman equation. The last section is intended to prove necessary conditions of optimality; sufficient conditions are also obtained.

2 Preliminaries

We denote by q, σ positive numbers satisfying

$$q > \frac{N}{2} + 1 \quad \text{and} \quad \sigma > N + 1.$$

We make the following assumptions:

(A1) For every $y \in \mathbf{R}$, $f(\cdot, y)$ is continuous on Q. For almost every $(x,t) \in Q$, $f(x,t,\cdot)$ is of class C^1 on $\mathbf{R}$. The following estimates hold:

$$|f(x,t,0)| \le M_1(x,t), \qquad |f'_y(x,t,y) - f'_y(x,t,z)| \le M_1(x,t)\eta(|y|)\eta(|z|)\omega(|y-z|),$$

$$C_0 \le f'_y(x,t,y) \le M_1(x,t)\eta(|y|), \qquad |f'_t(x,t,y)| \le M_1(x)\eta(t)\eta(|y|),$$

where M_1 belongs to $L^q(Q)$, η is a nondecreasing function from $\mathbf{R}$ to $\mathbf{R}$, and $C_0 \in \mathbf{R}$.

(A2) For every $(y,v) \in \mathbf{R}^2$, $g(\cdot, y, v)$ is continuous on Σ. For almost every $(s,t) \in \Sigma$ and every $v \in \mathbf{R}$, $g(s,t,\cdot,v)$ is of class C^1 on $\mathbf{R}$. For almost every $(s,t) \in \Sigma$, $g(s,t,\cdot)$ and $g'(s,t,\cdot)$ are continuous on $\mathbf{R} \times \mathbf{R}$. The following estimates hold:

$$|g(s,t,0,v)| \le M_2(s,t) \qquad C_0 \le g'_y(s,t,y,v) \le (M_2(s,t) + m_1|v|)\eta(|y|),$$

$$|g'_y(s,t,y,v) - g'_y(s,t,z,v)| \le M_2(s,t)\eta(|y|)\eta(|z|)\omega(|y-z|),$$

where M_2 belongs to $L^\sigma(\Sigma)$, $m_1 > 0$, and C_0 and η are as in (A1).

(A3) For every $y \in \mathbf{R}$, $L(\cdot, y)$ is measurable on Ω. For almost every $x \in \Omega$, $L(x,\cdot)$ is of class C^1 on $\mathbf{R}$. The following estimates are verified:

$$|L'_y(x,y)| \le M_3\eta(|y|), \qquad |L'_y(x,y) - L'_y(x,z)| \le M_3\eta(|y|)\eta(|z|)\omega(|y-z|),$$

where $M_3 \in \mathbf{R}_+$, η is as in (A1) and ω is an increasing continuous function from $\mathbf{R}^+$ to $\mathbf{R}^+$ such that $\omega(0) = 0$.

We recall the following results (see [13]) on the existence, uniqueness and regularity of the state variable.

Proposition 2.1 *Under assumptions* (A1) *and* (A2)*, if* $v \in L^\sigma(\Sigma)$ *and* $y_0 \in L^\infty(\Omega)$*, then* (1.1) *admits a unique weak solution* y_v *in* $W(0,T) \cap L^\infty(Q)$*. This solution belongs to* $C(\overline{Q}_{\varepsilon,T})$ *and satisfies the estimates*

$$\begin{aligned} \|y_v\|_{L^\infty(Q)} + \|y_v\|_{L^\infty(\Sigma)} &\le C_1\left(\|g(\cdot,0,v)\|_{L^\sigma(\Sigma)} + \|y_0\|_{L^\infty(\Omega)} + 1\right), \\ \|y_v\|_{C(\overline{Q}_{\varepsilon,T})} &\le C_2(\varepsilon)\left(\|g(\cdot,0,v)\|_{L^\sigma(\Sigma)} + \|y_0\|_{L^2(\Omega)} + 1\right) \end{aligned} \tag{2.1}$$

where $C_1 = C_1(T,\Omega,N,q,\sigma,C_0)$ *and* $C_2(\varepsilon) = C_2(\varepsilon,T,\Omega,N,q,\sigma,C_0)$*. Moreover, the mapping* $v \to y_v$ *is continuous from* $L^\sigma(\Sigma)$ *into* $L^\infty(Q)$*.*

Here $W(0,T) = \{y \in L^2(0,T;H^1(\Omega)) \mid \frac{\partial y}{\partial t} \in L^2(0,T;H^1(\Omega)')\}$ and $C(\overline{Q}_{\varepsilon,T}) = \{y \in L^2(Q) \mid y \text{ is continuous on } \bar{\Omega} \times [\varepsilon, T]\}$.

Proposition 2.2 *For every* $k > 0$ *and every* $\varepsilon > 0$*, there exist* $C_3 = C_3(T,\Omega,N,q,\sigma,C_0,k)$*,* $C_4 = C_4(T,\Omega,N,q,\sigma,C_0,k,\varepsilon)$*, and* $\alpha > 0$ *such that, for every* $(v,y_0) \in L^\sigma(\Sigma) \times C(\overline{\Omega})$ *satisfying* $\|v\|_{L^\sigma(\Sigma)} + \|y_0\|_{L^\infty(\Omega)} \le k$*, the weak solution* y_v *of* (1.1) *corresponding to* (v,y_0) *is Hölder continuous on* $[\varepsilon,T] \times \overline{\Omega}$ *and obeys*

$$\|y_v\|_{C(\overline{Q})} \le C_3, \qquad \|y_v\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}\times[\varepsilon,T])} \le C_4.$$

Moreover, if y_0 *is Hölder continuous on* Ω*, then* y_v *is Hölder continuous on* $\overline{Q}$*.*

3 Value function

In this section we establish some basic property of the value function.

Proposition 3.1 *V is continuous in $[0,T] \times L^\infty(\Omega)$ and bounded on bounded subsets of $[0,T] \times L^\infty(\Omega)$. Moreover, $y_0 \to V(t, y_0)$ is Lipschitz on all bounded subsets of $L^\infty(\Omega)$, uniformly in $t \in [0,T]$. Furthermore, for every $v \in L^\sigma(\Sigma)$ the function $V(\cdot, y(\cdot; t_0, y_0, v))$ is nondecreasing in $[t_0, T]$ and the dynamic programming principle hold*

$$V(t_0, y_0) = \inf \Big\{ V(t, y(t; t_0, y_0, v)) \mid v \in L^\sigma(\Sigma) \Big\} \text{ for all } t \in [t_0, T]. \tag{3.1}$$

Finally, $t \mapsto V(t, y(t; t_0, y_0, v))$ is constant if and only if v is optimal for problem (P).

Proof. These basic properties are well known in the Hilbert space case (see [1]) and can be proved by similar arguments in the present context. We will sketch the proof for reader's convenience.
Let denote by rB, $r > 0$, the closed ball in $L^\infty(\Omega)$ with radius r and center at 0. We note first that from (2.1) there exists $K_r > 0$ such that

$$|y(t; t_0, \xi, v)| \leq K_r, \qquad \forall \xi \in rB, \ \forall t \in [t_0, T],$$

and for all controls $v \in L^\sigma(\Sigma)$. Furthermore, there exists a constant $C_r > 0$ such that

$$\|y(t; t_0, \xi_1, v) - y(t; t_0, \xi_2, v)\|_{L^\infty(\Omega)} \leq C_r \|\xi_1 - \xi_2\|_{L^\infty(\Omega)}, \tag{3.2}$$

$\forall \xi_1, \xi_2 \in rB$, $\forall t \in [t_0, T]$ and for all controls $v \in L^\sigma(\Sigma)$. Indeed $y(t; t_0, \xi_1, v) - y(t; t_0, \xi_2, v)$ is also a solution to the following equation

$$\begin{aligned}
&\frac{\partial z}{\partial t} + Az + az = 0 \text{ in } Q, \\
&\frac{\partial z}{\partial n_A} + b_v z = 0 \text{ on } \Sigma, \\
&z(\cdot, t_0) = \xi_1 - \xi_2 \text{ in } \Omega,
\end{aligned}$$

where $a(x,t) = \int_0^1 f_y'(x, t, y_2 + \theta(y_1 - y_2)) d\theta \geq C_0$, $b_v(s,t) = \int_0^1 g_y'(s, t, y_2 + \theta(y_1 - y_2), v) d\theta \geq C_0$ and the estimate yields from Proposition 2.1.
Let $0 \leq t_0 \leq t_1 \leq T$ and let $\tilde{v} \in L^\sigma([t_0, T] \times \partial\Omega)$ be defined by

$$\tilde{v}(\theta) = v_0 \text{ for } t_0 \leq \theta < t_1; \qquad \tilde{v} = v_1^* \text{ on } [t_1, T]$$

where v_0 is an element of $L^\sigma([t_0, t_1] \times \partial\Omega)$ and v_1^* is optimal in $(P|_{[t_1, T]})$ with the initial datum $y(t_1) = y_0$. Then we have

$$\begin{aligned}
V(t_0, y_0) - V(t_1, y_0) &\leq \int_\Omega L(x, y(T; t_0, y_0, \tilde{v})) dx - \int_\Omega L(x, y(T; t_1, y_0, v_1^*)) dx \\
&\leq M_3 \eta(K_r) \int_\Omega |y(T; t_1, y(t_1; t_0, y_0, \tilde{v}), v_1^*) - y(T; t_1, y_0, v_1^*)| dx \\
&\leq M_3 \eta(K_r) m(\Omega) C_r \|y(t_1; t_0, y_0, \tilde{v}) - y_0\|_{L^\infty(\Omega)}
\end{aligned}$$

and the t-continuity of V yields from Proposition 2.1.

Now consider y_0, y_1 two initial data for (P) and, for a fixed $\varepsilon > 0$, a control v_ε such that

$$\int_\Omega L(x, y(T; t_0, y_1, v_\varepsilon))dx < V(t_0, y_1) + \varepsilon.$$

We have

$$V(t_0, y_0) - V(t_0, y_1) \leq \int_\Omega L(x, y(T; t_0, y_0, v_\varepsilon))dx - \int_\Omega L(x, y(T; t_0, y_1, v_\varepsilon))dx + \varepsilon$$

$$\leq M_3 \eta(K_r) \int_\Omega |y(T; t_0, y_0, v_\varepsilon) - y(T; t_0, y_1, v_\varepsilon)|dx + \varepsilon$$

$$\leq M_3 \eta(K_r) C_r m(\Omega) \|y_0 - y_1\|_{L^\infty(\Omega)} + \varepsilon$$

so $V(t, \cdot)$ is Lipschitz continuous, uniformly in $t \in [t_0, T]$.

Now we fix $v \in L^\sigma(\Sigma)$ and prove that $V(\cdot, y(\cdot, t_0, y_0, v))$ is nondecreasing in $[t_0, T]$. These yields from the fact that $\{u \in L^\sigma(\Sigma);\ u_{|_{[t_0,t_2]}} = v\} \subset \{u \in L^\sigma(\Sigma);\ u_{|_{[t_0,t_1]}} = v\}$ for $t_0 \leq t_1 \leq t_2 \leq T$, and the definition (1.2).

$$V(t_1, y(t_1; t_0, y_0, v)) = \inf_u \int_\Omega L(x, y(T; t_1, y(t_1; t_0, y_0, v), u))dx$$

$$\leq \inf_u \int_\Omega L(x, y(T; t_2, y(t_2; t_0, y_0, v), u))dx = V(t_2, y(t_2; t_0, y_0, v)).$$

Let prove first that

$$V(t_0, y_0) \leq \inf_{v \in L^\sigma(\Sigma)} V(t, y(t; t_0, y_0, v))$$

and consider an arbitrary control $v \in L^\sigma(\Sigma)$. Then for $V(t, y(t; t_0, y_0, v))$ there exists an control ε-optimal, $v^\varepsilon \in L^\sigma(\Sigma)$, such that

$$V(t, y(t; t_0, y_0, v)) + \varepsilon \geq \int_\Omega L(x, y^\varepsilon(T; t, y(t; t_0, y_0, v), v^\varepsilon))dx$$

where y^ε is the trajectory starting at $y(t; t_0, y_0, v)$, associated to the control v^ε. We define now for $V(t_0, y_0)$ a control $\tilde{v}$ in the following manner

$$\tilde{v}(s, \theta) = \begin{cases} v(s, \theta) & \text{if } t_0 \leq \theta < t, \\ v^\varepsilon(s, \theta) & \text{if } t \leq \theta \leq T \end{cases}$$

for all $s \in \partial\Omega$. If $\tilde{y}$ denote the trajectory associated to $\tilde{v}$, we have

$$V(t_0, y_0) \leq \int_\Omega L(x, \tilde{y}(T; t_0, y_0, \tilde{v}))dx = \int_\Omega L(x, y^\varepsilon(T; t, y(t; t_0, y_0, v), v^\varepsilon))dx.$$

Hence

$$V(t_0, y_0) \leq \varepsilon + V(t, y(t; t_0, y_0, v))$$

and we conclude by letting ε tend to 0 and taking the infimum with respect to v, which is arbitrary in $L^\sigma(\Sigma)$.

For the opposed inequality we consider an control v^ε, ε-optimal for $V(t_0, y_0)$, and denote by y^ε the associated trajectory. We have then

$$V(t_0, y_0) + \varepsilon \geq \int_\Omega L(x, y^\varepsilon(T; t_0, y_0, v^\varepsilon))dx = \int_\Omega L(x, y^\varepsilon(T; t, y^\varepsilon(t; t_0, y_0, v^\varepsilon), v^\varepsilon))dx$$
$$\geq \ V(t, y^\varepsilon(t; t_0, y_0, v^\varepsilon)) \geq \inf_{v \in L^\sigma(\Sigma)} V(t, y(t; t_0, y_0, v)).$$

We get the second inequality by letting $\varepsilon \to 0$ and therefore the desired (3.1).

To prove the final assertion let denote by (y^*, v^*) the optimal pair for problem (P). Then by (3.1) and (1.2) we have

$$V(t_0, y_0) = \inf_{v \in L^\sigma(\Sigma)} \{V(t, y(t; t_0, y_0, v))\} \leq V(t, y^*(t; t_0, y_0, v^*))$$

$$= \inf_{v \in L^\sigma(\Sigma)} \int_\Omega L(x, y(T; t, y^*(t; t_0, y_0, v^*), v\big|_{[t,T]}))dx$$

$$\leq \int_\Omega L(x, y^*(T; t, y^*(t; t_0, y_0, v^*), v^*\big|_{[t,T]}))dx = \int_\Omega L(x, y^*(T; t_0, y_0, v^*))dx$$
$$= V(t_0, y_0) \ \text{ for all } t \in [0, T]$$

and therefore $V(\cdot, y^*(\cdot; t_0, y_0, v^*))$ is constant. Conversely, let $\overline{v}$ be a control such that $t \to V(t, \overline{y}(t; t_0, y_0, \bar{v}))$ is constant, i.e.,

$$V(t, \overline{y}(t; t_0, y_0, \overline{v})) = V(t_0, y_0), \quad \forall t \in [0, T].$$

Using the definition of the value function and the boundary condition for V we get

$$\inf_v \left\{ \int_\Omega L(x, y(T; t_0, y_0, v))dx \right\} = V(t_0, y_0) = V(T, \overline{y}(T; t_0, y_0, \bar{v}))$$

$$= \int_\Omega L(x, \overline{y}(T))dx$$

and therefore $\overline{v} = v^*$. □

We recall next some generalizations of the notion of gradient for nonsmooth functions (see [2], [10] for more details). Let K be an open subset of Y and $\varphi : K \to \mathbf{R}$. For any $y_0 \in K$, the superdifferential $D^+\varphi(y_0)$ is defined as follows

$$D^+\varphi(y_0) = \left\{ p \in Y^* \middle| \limsup_{y \to y_0} \frac{\varphi(y) - \varphi(y_0) - \int_\Omega p(y - y_0)}{|y - y_0|} \leq 0 \right\}.$$

If φ is Fréchet differentiable at y_0, then $D^+\varphi(y_0)$ is a singleton and coincides with the gradient $\nabla\varphi(y_0)$.

We denote by $D^*\varphi(y_0)$ the set of all $p \in Y^*$ for which there exists a sequence $\{y_n\}_{n \in \mathbf{N}}$ of points in Y such that

(i) y_n converges to y_0 as $n \to \infty$,
(ii) φ is Fréchet differentiable at y_n for all $n \in \mathbf{N}$,
(iii) $\nabla\varphi(y_n)$ weakly-* converges to p as $n \to \infty$.

Let $K \subset Y$ be convex and $\varphi : K \to \mathbf{R}$. We say that φ is semiconcave if there exists a function

$$\omega : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+ \qquad \text{satisfying} \left\{ \begin{array}{ll} \forall r \leq R, s \leq S, & \omega(r,s) \leq \omega(R,S), \\ \forall r \geq 0, & \lim\limits_{s \to 0} \omega(r,s) = 0, \end{array} \right.$$

such that, for every $r > 0, \lambda \in [0,1]$, and $\zeta_1, \zeta_2 \in K \cap rB$,

$$\lambda\varphi(\zeta_1) + (1-\lambda)\varphi(\zeta_2) - \varphi(\lambda\zeta_1 + (1-\lambda)\zeta_2) \leq \lambda(1-\lambda)|\zeta_1 - \zeta_2|\omega(r, |\zeta_1 - \zeta_2|).$$

(For instance, by (A1)-(A3), we have that $f(x,t,\cdot), g(s,t,\cdot)$, and $L(x,\cdot)$ are semiconcave.)

Superdifferentials of semiconcave functions enjoy the following regularity property. If φ is semiconcave and Lipschitz in $y_0 + rB$ for some $r > 0$ then

$$D^+\varphi(y_0) = \overline{\text{co}}\, D^*\varphi(y_0),$$

where $\overline{\text{co}}$ denotes the closed convex hull.

Next we prove a semiconcavity result for the value function V.

Theorem 3.1 *Assume* (A1)-(A3). *Then* $V(t,\cdot)$ *is semiconcave for all* $t \in [0,T]$.

Proof. Fix $r > 0$, $t_0 \in [0,T]$, and $y_0^1, y_0^2 \in rB$. Now let $\lambda \in (0,1)$ and define $y_\lambda = \lambda y_0^2 + (1-\lambda)y_0^1$. Fix $\varepsilon > 0$ and consider a control v_ε such that

$$\int_\Omega L(x, y(T; t_0, y_\lambda, v_\varepsilon))dx < V(t_0, y_\lambda) + \varepsilon.$$

Set $\overline{y}_i(\cdot) = y(\cdot; t_0, y_0^i, v_\varepsilon)$, $i = 1,2$. Then using (A3) we get

$$\lambda V(t_0, y_0^2) + (1-\lambda)V(t_0, y_0^1) - V(t_0, y_\lambda)$$

$$\leq \lambda \int_\Omega L(x, \overline{y}_2(T))dx + (1-\lambda)\int_\Omega L(x, \bar{y}_1(T))dx - \int_\Omega L(x, y(T; t_0, y_\lambda, v_\varepsilon))dx + \varepsilon$$

$$\leq \int_\Omega \lambda(1-\lambda)|\overline{y}_2(T) - \bar{y}_1(T)|\omega(|\bar{y}_2(T) - \bar{y}_1(T)|)dx$$

$$+ \int_\Omega \left[L(x, \lambda\overline{y}_2(T) + (1-\lambda)\overline{y}_1(T)) - L(x, y(T; t_0, y_\lambda, v_\varepsilon))\right] dx + \varepsilon$$

$$\leq \int_\Omega \lambda(1-\lambda)|\overline{y}_2(T) - \bar{y}_1(T)|\omega(|\bar{y}_2(T) - \bar{y}_1(T)|)dx + \varepsilon$$

$$+ M_3\eta(K_r)\big|\lambda\bar{y}_2(T) + (1-\lambda)\overline{y}_1(T) - y(T; t_0, y_\lambda, v_\varepsilon)\big|.$$

It remains to estimate the last term in the above inequality. Set

$$\overline{y}_\lambda(t) = \lambda\bar{y}_2(t) + (1-\lambda)\bar{y}_1(t) - y(t; t_0, y_\lambda, v_\varepsilon).$$

As previously we have that $\overline{y}_\lambda$ satisfies the equation

$$\frac{\partial \overline{z}}{\partial t} + A\bar{z} + \widetilde{a}\bar{z} = f(x,t,\lambda\bar{y}_2 + (1-\lambda)\overline{y}_1) - \lambda f(x,t,\bar{y}_2) - (1-\lambda)f(x,t,\bar{y}_1) \text{ in } Q,$$

$$\frac{\partial \bar{z}}{\partial n_A} + \widetilde{b}\overline{z} = g(s,t,\lambda\bar{y}_2 + (1-\lambda)\bar{y}_1, v_\varepsilon) - \lambda g(s,t,\overline{y}_2, v_\varepsilon) - (1-\lambda)g(s,t,\bar{y}_1, v_\varepsilon) \text{ on } \Sigma,$$

$$\overline{z}(\cdot, t_0) = 0 \text{ in } \Omega,$$

where $\widetilde{a}(x,t)=\int_0^1 f'_y\big(x,t,y(t;t_0,y_\lambda,v_\varepsilon)+\theta(\lambda\overline{y}_2(t)+(1-\lambda)\bar{y}_1(t)-y(t;t_0,y_\lambda,v_\varepsilon))\big)d\theta\geq C_0$, and $\widetilde{b}(s,t)=\int_0^1 g'_y\big(s,t,y(t;t_0,y_\lambda,v_\varepsilon)+\theta(\lambda\overline{y}_2(t)+(1-\lambda)\bar{y}_1(t)-y(t;t_0,y_\lambda,v_\varepsilon))\big)d\theta\geq C_0$. Therefore

$$\|\overline{y}_\lambda\|_{C(\bar{Q})}\leq C\Big(\|f(\cdot,\lambda\overline{y}_2+(1-\lambda)\bar{y}_1)-\lambda f(\cdot,\bar{y}_2)-(1-\lambda)f(\cdot,\bar{y}_1)\|_{L^q(Q)}$$

$$+\|g(\cdot,\lambda\overline{y}_2+(1-\lambda)\bar{y}_1,v_\varepsilon)-\lambda g(\cdot,\bar{y}_2,v_\varepsilon)-(1-\lambda)g(\cdot,\bar{y}_1,v_\varepsilon)\|_{L^\sigma(\Sigma)}\Big)$$

$$\leq\lambda(1-\lambda)C_r\|y_0^1-y_0^2\|_{L^\infty(\Omega)}(m(Q)^{1/q}+m(\Sigma)^{1/\sigma})\eta^2(K_r)\omega(C_r\|\bar{y}_0^1-\bar{y}_0^2\|_{L^\infty(\Omega)})$$

which completes the proof. □

4 Hamilton-Jacobi-Bellman Equation

We define the Hamiltonian function for the Mayer problem (P) by

$$\mathcal{H}(t,y,p)=\min_{v\in L^\sigma(\Gamma)}\Bigg\{-\int_\Omega f(x,t,y)p\ dx-\int_\Omega\sum_{i,j}a_{ij}(x)D_jyD_ip\ dx \\ -\int_{\partial\Omega}g(s,t,y(s),v(s))p(s)\ ds\Bigg\} \tag{4.1}$$

for all $(t,y,p)\in[t_0,T]\times Y\times Y$.

Theorem 4.1 *The value function V is the viscosity solution to the Hamilton-Jacobi-Bellman equation*

$$\begin{cases}\dfrac{\partial V}{\partial t}+\mathcal{H}(t,y,\dfrac{\partial V}{\partial y})=0\ \ on\ [0,T)\times Y\\ V(T,y)=\displaystyle\int_\Omega L(x,y(T))dx.\end{cases} \tag{4.2}$$

We recall (see [9]) that $V\in C((0,T)\times Y)$ is a viscosity subsolution of (4.2) on $[0,T)\times Y$ if and only if for every $\phi\in C((0,T)\times Y)$

$$\frac{\partial\phi}{\partial t}+\mathcal{H}(t,y,\frac{\partial\phi}{\partial y})\geq 0$$

at each local maximum point $(t,y)\in(0,T)\times Y$ of $V-\phi$ at which ϕ is differentiable. Similarly, V is a viscosity supersolution of (4.2) when

$$\frac{\partial\phi}{\partial t}+\mathcal{H}(t,y,\frac{\partial\phi}{\partial y})\leq 0$$

at each local minimum point $(t,y)\in(0,T)\times Y$ of $V-\phi$ at which ϕ is differentiable. Finally, V is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

Proof. Let (t_0,y_0) be a local maximum point of $V-\phi$ and assume without loss of generality that $V(t_0,y_0)=\phi(t_0,y_0)$, i.e.,

$$V(t_0+h,y(t_0+h))\leq\phi(t_0+h,y(t_0+h)).$$

By the Dynamic Programming Principle (3.1) with a constant control $v(\cdot,t) \equiv v(\cdot) \in L^\sigma(\partial\Omega)$ we have

$$V(t_0, y_0) \leq V(t_0 + h, y(t_0 + h))$$

and therefore

$$0 \leq \phi(t_0 + h, y(t_0 + h)) - \phi(t_0, y_0).$$

Hence

$$0 \leq \frac{\partial \phi}{\partial t}(t_0, y_0) \cdot h + \int_\Omega \frac{\partial \phi}{\partial y}(t_0, y_0) \cdot (y(t_0 + h) - y(t_0))\, dx + o(h)$$

Dividing by h and letting $h \to 0$ we get

$$0 \leq \frac{\partial \phi}{\partial t}(t_0, y_0) + \lim_{h\to 0} \frac{1}{h} \int_{t_0}^{t_0+h} \mathcal{H}(t_0, y(t), \frac{\partial \phi}{\partial y}(t_0, y_0))dt$$

so V is a viscosity supersolution of (4.2).

Assume now that (t_0, y_0) is a local minimum point of $V-\phi$. If (y^*, v^*) is the optimal pair for problem (P), then again by Proposition 3.1 we have

$$V(t_0, y_0) = V(t_0 + h, y(t_0 + h))$$

which implies

$$\phi(t_0 + h, y(t_0 + h)) - \phi(t_0, y_0) \leq 0.$$

Since ϕ is differentiable we conclude that

$$0 \geq \frac{\partial \phi}{\partial t}(t_0, y_0) + \lim_{h\to 0} \left\langle \frac{\partial \phi}{\partial y}(t_0, y_0), \frac{y^*(t_0 + h) - y_0}{h} \right\rangle$$

and therefore V is a viscosity supersolution of (4.2). $\square$

Let assume through the last part of Section 4 that $g(t, x, y, v) = C(x)y - v$, $C(x) > 0$. We define the unbounded operator $\mathcal{A}$ in $L^2(\Omega)$ by

$$D(\mathcal{A}) = \left\{ y \in H^2(\Omega);\ \frac{\partial y}{\partial n_A} + C(x)y = 0 \right\}$$
$$\mathcal{A}y = -\sum_{i,j} \partial_j(a_{ij}\partial_i y) + y,$$

and the lifting map $M : L^2(\partial\Omega) \to L^2(\Omega)$ by

$$Mv = y \Leftrightarrow \begin{cases} \sum_{i,j} \partial_j(a_{ij}\partial_i y) = y \text{ in } \Omega \\ \dfrac{\partial y}{\partial n_A} + C(x)y = v \text{ on } \partial\Omega. \end{cases}$$

Then we may rewrite the equation (1.1) as

$$\begin{cases} y'(t) + \mathcal{A}y + F(t, y) = \mathcal{A}Mv(t) \\ y(t_0) = y_0 \end{cases}$$

where

$$F(t,y) = f(t,y) - y.$$

We note that the right-hand side of the above equation is not well defined, since the range of M is not contained in $D(\mathcal{A})$. Therefore we shall proceed as in [5]-[7] and use the fact that M has a regularizing effect. Indeed, $M : L^2(\partial\Omega) \to H^{3/2}(\Omega)$, which may be expressed in abstract form using the fractional powers of $\mathcal{A}$. Actually

$$D(\mathcal{A}^\theta) = \begin{cases} H^{2\theta} & \text{if } 0 \le \theta < \frac{3}{4} \\ \left\{ y \in H^2(\Omega);\ \frac{\partial y}{\partial n_A} + C(x)y = 0 \right\} & \text{if } \frac{3}{4} < \theta \le 1. \end{cases}$$

By classical results $M : L^2(\partial\Omega) \to D(\mathcal{A}^\alpha)$ for all $\alpha \in (0, 3/4)$. Consequently equation (1.1) can be written as

$$\begin{cases} y' + \mathcal{A}y + F(y) = \mathcal{A}^\beta M_\beta v \\ y(t_0) = y_0, \end{cases}$$

where $\beta \in (3/4, 1]$ and $M_\beta = \mathcal{A}^{1-\beta} M$.
Now the value function satisfies the Hamilton Jacobi Bellman equation

$$\begin{cases} \dfrac{\partial V}{\partial t} + \mathcal{H}(t, y, D_y V) = 0 \\ V(T, y) = \displaystyle\int_\Omega L(x, y(T))dx \end{cases}$$

where

$$\mathcal{H}(t,y,p) = \inf_v \left\{ -\int_\Omega f(t,y)p - \int_{\partial\Omega} C(\sigma)yp d\sigma - \int_\Omega a_{ij}\partial_i y \partial_j p + \int_\Omega M_\beta v \mathcal{A}^\beta p \right\}.$$

We give now a comparison result.

Theorem 4.2 *Assume that (A1)-(A3) hold true. Let U,$\mathcal{U}$ be respectively a viscosity subsolution, and a viscosity supersolution of (4.2). Let*

$$U(t,y), -\mathcal{U}(t,y) \le C(1 + \|y\|_{L^2(Q)})$$

and moreover

$$\begin{cases} \displaystyle\lim_{t \uparrow T} \left(U(t,y) - \int_\Omega L(x, y(T))dx \right)^+ = 0 \\ \displaystyle\lim_{t \uparrow T} \left(\mathcal{U}(t,y) - \int_\Omega L(x, y(T))dx \right)^- = 0 \end{cases}$$

uniformly on bounded subsets of Y. Then $U \le \mathcal{U}$.

Proof. Given $\sigma > 0$, define

$$U_\sigma(t,y) = U(t,y) - \frac{\sigma}{t}, \qquad \mathcal{U}_\sigma(t,y) = \mathcal{U}(t,y) + \frac{\sigma}{t}.$$

It is easy to see that U_σ and $\mathcal{U}_\sigma$ satisfy respectively

$$\begin{cases} \dfrac{\partial U_\sigma(t,y)}{\partial t} + \mathcal{H}(t, y, D_y U_\sigma(t,y)) \ge \dfrac{\sigma}{T^2} \\ U_\sigma(T,y) = \displaystyle\int_\Omega L(x, y(T))dx - \frac{\sigma}{T} \end{cases} \tag{4.3}$$

and

$$\begin{cases} \dfrac{\partial \mathcal{U}_\sigma(t,y)}{\partial t} + \mathcal{H}(t,y,D_y\mathcal{U}_\sigma(t,y)) \leq -\dfrac{\sigma}{T^2} \\ \mathcal{U}_\sigma(T,y) = \displaystyle\int_\Omega L(x,y(T))dx + \dfrac{\sigma}{T}. \end{cases} \tag{4.4}$$

For $\varepsilon, \delta, \gamma > 0$ we define the function

$$\Phi(t,\theta,y,z,\varepsilon,\delta,\gamma) = U_\sigma(t,y) - \mathcal{U}_\sigma(\theta,z) - \frac{1}{2\varepsilon}\left\langle \mathcal{A}^{-1}(y-z), y-z\right\rangle$$

$$-\frac{\delta}{2}(\|y\|^2_{L^2(\Omega)} + \|z\|^2_{L^2(\Omega)}) - \frac{t-\theta}{2\gamma}$$

$\forall (t,y),(\theta,z) \in (t_0,T)\times L^2(\Omega)$. Because of the continuity assumptions on $U, \mathcal{U}$ and the fact that $\mathcal{A}^{-1}$ is compact, the function above is weakly sequentially upper-semicontinuous and so it attains a global maximum at some points $(\bar t, \bar\theta, \bar y, \bar z)$, where $0 < \bar t, \bar\theta$ and $\bar y, \bar z$ are bounded independently of ε for a fixed δ. Moreover we have (as for instance in [4],[11],[12])

$$\limsup_{\delta\to 0}\limsup_{\varepsilon\to 0}\limsup_{\gamma\to 0}\delta(\|\bar y\|^2_{L^2(\Omega)} + \|\bar z\|^2_{L^2(\Omega)}) = 0, \tag{4.5}$$

$$\limsup_{\varepsilon\to 0}\limsup_{\gamma\to 0}\frac{\|\mathcal{A}^{-1/2}(\bar y-\bar z)\|^2_{L^2(\Omega)}}{2\varepsilon} = 0 \qquad \text{for fixed } \delta, \tag{4.6}$$

and

$$\limsup_{\gamma\to 0}\frac{(\bar t-\overline{\theta})^2}{2\gamma} = 0 \qquad \text{for fixed } \delta, \varepsilon. \tag{4.7}$$

To see this we set

$$m_1(\varepsilon,\delta,\gamma) = \sup_{y,z}\ \Phi(t,\theta,y,z,\varepsilon,\delta,\gamma)$$

$$m_2(\varepsilon,\delta) = \sup_{y,z}\Big\{U_\sigma(t,y) - \mathcal{U}_\sigma(\theta,z) - \frac{1}{2\varepsilon}\|\mathcal{A}^{-1/2}(y-z)\|^2_{L^2(\Omega)}$$

$$-\frac{\delta}{2}(\|y\|^2_{L^2(\Omega)} + \|z\|^2_{L^2(\Omega)})\Big\}$$

$$m_3(\delta) = \sup_{y,z}\left\{U_\sigma(t,y) - \mathcal{U}_\sigma(\theta,z) - \frac{\delta}{2}(\|y\|^2_{L^2(\Omega)} + \|z\|^2_{L^2(\Omega)})\right\}.$$

We note that we have

$$\lim_{\gamma\to 0} m_1(\varepsilon,\delta,\gamma) = m_2(\varepsilon,\delta), \lim_{\varepsilon\to 0} m_2(\varepsilon,\delta) = m_3(\delta), \lim_{\delta\to 0} m_3(\delta) = m.$$

Now

$$m_1(\varepsilon,\delta,\gamma) = \Phi(\overline{t},\overline{\theta},\overline{y},\overline{z},\varepsilon,\delta,\gamma) = U_\sigma(\overline{t},\overline{y}) - \mathcal{U}_\sigma(\overline{\theta},\overline{z}) - \frac{1}{2\varepsilon}|\mathcal{A}^{-1/2}(\overline{y}-\overline{z})|^2_{L^2(\Omega)}$$

$$-\frac{\delta}{2}(\|\overline{y}\|^2_{L^2(\Omega)} + \|\overline{z}\|^2_{L^2(\Omega)}) - \frac{(\overline{t}-\overline{\theta})^2}{2\gamma}$$

and for δ, γ fixed

$$m_1(\varepsilon,\delta,\gamma) + \frac{1}{4\varepsilon}|\mathcal{A}^{-1}(\overline{y}-\overline{z})|^2_{L^2(\Omega)} = U_\sigma(\overline{t},\overline{y}) - \mathcal{U}_\sigma(\overline{\theta},\overline{z}) - \frac{1}{4\varepsilon}|\mathcal{A}^{-1/2}(\overline{y}-\overline{z})|^2_{L^2(\Omega)}$$

$$-\frac{\delta}{2}(\|\overline{y}\|^2_{L^2(\Omega)} + \|\overline{z}\|^2_{L^2(\Omega)}) - \frac{(\overline{t}-\overline{\theta})^2}{2\gamma} \le m_1(2\varepsilon,\delta,\gamma).$$

Thus

$$\frac{1}{4\varepsilon}|\mathcal{A}^{-1}(\overline{y}-\overline{z})|^2_{L^2(\Omega)} \le m_1(2\varepsilon,\delta,\gamma) - m_1(\varepsilon,\delta,\gamma).$$

This gives (4.5). Similarly we have

$$m_1(\varepsilon,\delta,\gamma)+\frac{\delta}{2}(\|\overline{y}\|^2_{L^2(\Omega)}+\|\overline{z}\|^2_{L^2(\Omega)}) = U_\sigma(\overline{t},\overline{y})-\mathcal{U}_\sigma(\overline{\theta},\overline{z})-\frac{1}{2\varepsilon}|\mathcal{A}^{-1/2}(\overline{y}-\overline{z})|^2_{L^2(\Omega)}$$

$$-\frac{\delta}{2}\left(\|\overline{y}\|^2_{L^2(\Omega)} + \|\overline{z}\|^2_{L^2(\Omega)}\right) - \frac{(\overline{t}-\overline{\theta})^2}{2\gamma} \le m_1(\varepsilon,\frac{\delta}{2},\gamma)$$

which gives

$$\frac{\delta}{2}\left(\|\overline{y}\|^2_{L^2(\Omega)} + \|\overline{z}\|^2_{L^2(\Omega)}\right) \le m_1(\varepsilon,\frac{\delta}{2},\gamma) - m_1(\varepsilon,\delta,\gamma)$$

from which we obtain (4.6). Finally we have

$$m_1(\varepsilon,\delta,\gamma) + \frac{\overline{t}-\overline{\theta}}{4\gamma} = U_\sigma(\overline{t},\overline{y}) - \mathcal{U}_\sigma(\overline{\theta},\overline{z}) - \frac{1}{2\varepsilon}|\mathcal{A}^{-1/2}(\overline{y}-\overline{z})|^2_{L^2(\Omega)}$$

$$-\frac{\delta}{2}\left(\|\overline{y}\|^2_{L^2(\Omega)} + \|\overline{z}\|^2_{L^2(\Omega)}\right) - \frac{(\overline{t}-\overline{\theta})^2}{4\gamma} \le m_1(\varepsilon,\delta,2\gamma),$$

then

$$\frac{(\overline{t}-\overline{\theta})^2}{4\gamma} \le m_1(\varepsilon,\delta,\gamma) - m_1(\varepsilon,\delta,\gamma)$$

and (4.7).
If U is not less than or equal to $\mathcal{U}$ it then follows from the above that for small σ and δ, and for T_δ sufficiently close to T we have $\bar{t}, \bar{\theta} < T_\delta$ if γ and ε are sufficiently small. Now define the following functions

$$\varphi(t,y) = \mathcal{U}_\sigma(\overline{\theta},\overline{z})+\frac{1}{2\varepsilon}\left\langle \mathcal{A}^{-1}(y-\overline{z}), y-\overline{z}\right\rangle+\frac{\delta}{2}\left(\|y\|^2_{L^2(\Omega)} + \|\overline{z}\|^2_{L^2(\Omega)}\right)+\frac{(t-\overline{\theta})^2}{2\gamma}$$

and

$$\psi(\theta,z) = U_\sigma(\overline{t},\overline{y})-\frac{1}{2\varepsilon}\left\langle \mathcal{A}^{-1}(\overline{y}-z), \overline{y}-z\right\rangle-\frac{\delta}{2}\left(\|\overline{y}\|^2_{L^2(\Omega)} + \|z\|^2_{L^2(\Omega)}\right)-\frac{(\overline{t}-\theta)^2}{2\gamma}$$

Since $(\overline{t},\overline{y}) \in \operatorname{argmax}(U_\sigma - \varphi)$ and $(\overline{\theta},\overline{z}) \in \operatorname{argmin}(\mathcal{U}_\sigma - \psi)$, we can conclude that

$$D_y\varphi(\overline{t},\overline{y}),\ D_y\psi(\overline{\theta},\overline{z}) \in D(\mathcal{A}^\theta),\ \forall\theta\in[0,1).$$

Therefore recalling that U_σ is a viscosity subsolution of (4.3) we obtain

$$\frac{\sigma}{T^2} \le \frac{\bar{t}-\bar{\theta}}{\gamma} + \mathcal{H}(\bar{t},\bar{y},D_yU_\sigma(\bar{t},\bar{y})).$$

Similarly, recalling that $\mathcal{U}_\sigma$ is a viscosity supersolution of (4.4) we have

$$\frac{\bar t-\bar\theta}{\gamma}+\mathcal{H}(\bar\theta,\bar z,D_y\mathcal{U}_\sigma(\bar\theta,\bar z))\le -\frac{\sigma}{T^2}.$$

Combining these two inequalities we find

$$-\frac{2\sigma}{T^2}\ge \mathcal{H}(\bar\theta,\bar z,D_y\mathcal{U}_\sigma(\bar\theta,\bar z))-\mathcal{H}(\bar t,\bar y,D_yU_\sigma(\bar t,\bar y))$$

$$=-\int_\Omega F(\bar\theta,\bar z)\left(\frac{1}{\varepsilon}\mathcal{A}^{-1}(\bar y-\bar z)-\delta\bar z\right)-\int_\Omega \mathcal{A}\bar z\left(\frac{1}{\varepsilon}\mathcal{A}^{-1}(\bar y-\bar z)-\delta\bar z\right)$$
$$+\inf_v\int_\Omega M_\beta v\mathcal{A}^\beta\left(\frac{1}{\varepsilon}\mathcal{A}^{-1}(\bar y-\bar z)-\delta\bar z\right)+\int_\Omega F(\bar t,\bar y)\left(\frac{1}{\varepsilon}\mathcal{A}^{-1}(\bar y-\bar z)+\delta\bar y\right)$$
$$+\int_\Omega \mathcal{A}\bar y\left(\frac{1}{\varepsilon}\mathcal{A}^{-1}(\bar y-\bar z)+\delta\bar y\right)-\inf_v\int_\Omega M_\beta v\mathcal{A}^\beta\left(\frac{1}{\varepsilon}\mathcal{A}^{-1}(\bar y-\bar z)+\delta\bar y\right)$$
$$=\int_\Omega\frac{1}{\varepsilon}\mathcal{A}^{-1}(\bar y-\bar z)\left(F(\bar t,\bar y)-F(\bar\theta,\bar z)\right)+\delta\int_\Omega\left(\bar yF(\bar t,\bar y)+\bar zF(\bar\theta,\bar z)\right)$$
$$+\int_\Omega\frac{1}{\varepsilon}\mathcal{A}(\bar y-\bar z)\mathcal{A}^{-1}(\bar y-\bar z)+\delta\int_\Omega(\mathcal{A}(\bar y)\bar y+\mathcal{A}(\bar z)\bar z)$$
$$-\inf_v\int_\Omega M_\beta v\mathcal{A}^\beta\left(\frac{1}{\varepsilon}\mathcal{A}^{-1}(\bar y-\bar z)+\delta\bar y\right)+\inf_v\int_\Omega M_\beta v\mathcal{A}^\beta\left(\frac{1}{\varepsilon}\mathcal{A}^{-1}(\bar y-\bar z)-\delta\bar z\right).$$

The first term writes

$$\frac{1}{\varepsilon}\int_\Omega\mathcal{A}^{-1}(\bar y-\bar z)\Big(F(\bar t,\bar y)-F(\bar\theta,\bar z)\Big)=\frac{1}{\varepsilon}\int_\Omega\mathcal{A}^{-1}(\bar y-\bar z)\Big(f(\bar t,\bar y)-\bar y-f(\bar\theta,\bar z)+\bar z\Big)$$
$$=\frac{1}{\varepsilon}\int_\Omega\mathcal{A}^{-1}(\bar y-\bar z)(\bar y-\bar z)+\frac{1}{\varepsilon}\int_\Omega\mathcal{A}^{-1}(\bar y-\bar z)\Big(f(\bar t,\bar y)-f(\bar t,\bar z)+f(\bar t,\bar z)-f(\bar\theta,\bar z)\Big)$$
$$=\frac{1}{\varepsilon}\left|\mathcal{A}^{-1/2}(\bar y-\bar z)\right|^2_{L^2(\Omega)}+\frac{1}{\varepsilon}\int_\Omega\mathcal{A}^{-1}(\bar y-\bar z)\times$$
$$\times\Big(f'_y(\bar t,\lambda\bar y+(1-\lambda)\bar z)(\bar y-\bar z)+f'_t(\lambda\bar t+(1-\lambda)\bar\theta,\bar z)(\bar t-\bar\theta)\Big)(\bar t-\bar\theta)$$
$$\ge\frac{1}{\varepsilon}\left|\mathcal{A}^{-1/2}(\bar y-\bar z)\right|^2_{L^2(\Omega)}-\frac{1}{\varepsilon}\|\mathcal{A}^{-1/2}\|\|\mathcal{A}^{-1/2}(\bar y-\bar z)\|_{L^2(\Omega)}\times$$
$$\times\left[\left(\int_\Omega|f'_y(\bar t,\lambda\bar y+(1-\lambda)\bar z)|^2|\bar y-\bar z|^2\right)^{1/2}+\left(\int_\Omega|f'_t(\lambda\bar t+(1-\lambda)\bar\theta,\bar z)|^2|\bar t-\bar\theta|^2\right)^{1/2}\right].$$

The second term gives

$$\delta\int_\Omega\bar yF(\bar t,\bar y)=\delta\int_\Omega\bar y\Big(f(\bar t,\bar y)-\bar y\Big)=-\,\delta\int_\Omega\bar y^2+\delta\int_\Omega\bar y\Big(f(\bar t,0)+f'_y(\bar t,\lambda\bar y)\bar y\Big)$$
$$\ge-\,\delta\|\bar y\|^2_{L^2(\Omega)}-\delta\|\bar z\|_{L^2(\Omega)}$$

while the third one yields

$$-\,\delta\int_\Omega M_\beta v_*\mathcal{A}^\beta(\bar y+\bar z)=-\,\delta\int_\Omega\mathcal{A}^\beta M_\beta v_*(\bar y+\bar z)\ge-\,\delta C(\|\bar y\|_{L^2(\Omega)}+\|\bar z\|_{L^2(\Omega)}).$$

So finally we get

$$-\frac{2\sigma}{T^2}\ge\frac{1}{\varepsilon}\|\bar y-\bar z\|^2_{L^2(\Omega)}-C\frac{1}{\varepsilon}\|\mathcal{A}^{-1/2}(\bar y-\bar z)\|^2-\,\delta(\|\bar y\|^2+\|\bar z\|^2).$$

This inequality yields a contradiction if we let $\varepsilon\to0$ and $\delta\to0$. $\square$

5 Optimality Conditions

5.1 Necessary Conditions for Optimality

In order to derive the optimality conditions for problem (P), we introduce the function

$$H(s,t,y,v,p) = -pg(s,t,y,v)$$

for all $(s,t,y,v,p) \in \partial\Omega \times [0,T] \times \mathbf{R}^3$.

Theorem 5.1 *If* (A1)-(A3) *holds true and* (y^*, v^*) *is an optimal trajectory-control pair for problem* (P)*, then there exists* $p^* \in W(0,T) \cap C_b(\overline{Q} \setminus (\overline{\Omega} \times \{T\}))$ *satisfying the equation*

$$\begin{aligned}
&-\frac{\partial p^*}{\partial t} + Ap^* + f'_y(x,t,y^*)p^* = 0 \quad \text{in } Q,\\
&\frac{\partial p^*}{\partial n_A} + g'_y(s,t,y^*,v^*)p^* = 0 \quad \text{on } \Sigma,\\
&p^*(T) = L'_y(x,y^*(T)) \quad \text{in } \Omega,
\end{aligned} \tag{5.1}$$

and such that

$$H(s,t,y^*(s,t),v^*(s,t),p^*(s,t)) = \min_{v \in L^\sigma(\Sigma)} H(s,t,y^*(s,t),v,p^*(s,t)) \tag{5.2}$$

for a.e. $(s,t) \in \Sigma$*. Moreover, we have*

$$p^*(t) \in D_y^+ V(t, y^*(t)), \quad \forall t \in [t_0, T] \tag{5.3}$$

and there exists a subset $\mathcal{L} \subset [t_0,T]$*, of full measure, such that, for all* $t \in \mathcal{L}$*,*

$$\Big(-\mathcal{H}(t,y^*(t),p^*(t)), -p^*(t) \Big) \in D^+V(t,y^*(t)). \tag{5.4}$$

Proof. The equality (5.2) is a well-known result (see for instance [14]-[15]).
Let (y^*, v^*) be a solution of (P) and let p^* be its associated adjoint state. To show (5.3), fix $y_1 \in Y$ and $t \in [t_0, T]$. If $y(\tau)$ denote $y(\tau; t, y_1, v^*)$ for all $t \le \tau \le T$, then by (1.1) we have

$$\begin{aligned}
&\frac{\partial}{\partial t}(y-y^*) + A(y-y^*) + f(x,\tau,y) - f(x,\tau,y^*) = 0 \quad \text{in } \Omega \times [t,T],\\
&\frac{\partial}{\partial n_A}(y-y^*) + g(s,\tau,y,v^*) - g(s,\tau,y^*,v^*) = 0 \quad \text{on } \partial\Omega \times [t,T],\\
&(y-y^*)(t) = y_1 - y^*(t) \quad \text{in } \Omega.
\end{aligned}$$

Multiplying this equation by p^*, the solution of (5.1), integrating over $Q_t = \Omega \times [t,T]$ and applying the Green formula, we get subsequently that

$$\int_\Omega \frac{\partial L}{\partial y}(y^*(T))(y(T) - y^*(T))dx = \int_\Omega (y_1 - y^*(t))p^*(t)dx$$

$$+ \int_{\Sigma_t} p^* \big((y-y^*)g'_y - (g(y,v^*) - g(y^*,v^*)) \big) ds d\tau + \int_{\Omega_t} p^* \big((y-y^*)f'_y - (f(y) - f(y^*)) \big) dx d\tau.$$

We have

$$\limsup_{y_1\to y^*(t)} \frac{V(t,y_1)-V(t,y^*)-<p^*,y_1-y^*(t)>}{|y_1-y^*(t)|}$$

$$\leq \limsup_{y_1\to y^*(t)} \frac{\displaystyle\int_\Omega \frac{\partial L}{\partial y}(x,y^\theta(T))\big(y(T;t,y_1,v^*))-y(T;t,y^*(t),v^*)\big)dx-<p^*,y_1-y^*(t)>}{|y_1-y^*(t)|}$$

(where y^θ denote $y^*+\theta(y-y^*)$, for all $\theta\in(0,1)$)

$$\leq \limsup_{y_1\to y^*(t)} \frac{\displaystyle\int_\Omega p^*(T)\big(y(T;t,y_1,v^*))-y(T;t,y^*(t),v^*)\big)dx-\int_\Omega p^*(t)(y_1-y^*(t))dx}{|y_1-y^*(t)|}$$

$$+\limsup_{y_1\to y^*(t)} \frac{\displaystyle\int_\Omega\left(\frac{\partial L}{\partial y}(x,y^\theta(T))-\frac{\partial L}{\partial y}(x,y^*(T))\right)\big(y(T;t,y_1,v^*))-y(T;t,y^*(t),v^*)\big)dx}{|y_1-y^*(t)|}$$

$$\overset{(A3)}{\leq} \limsup_{y_1\to y^*(t)} \frac{1}{|y_1-y^*(t)|}\Bigg\{\int_{Q_t} p^*(\tau)\Big(\big(y(\tau;t,y_1,v^*)-y^*(\tau;t,y^*(t),v^*)\big)f'_y$$

$$-\big(f(y(\tau;t,y_1,v^*))-f(y^*(\tau;t,y^*(t),v^*))\big)\Big)dxd\tau$$

$$+\int_{\Sigma_t} p^*(\tau)\Big(\big(y(\tau;t,y_1,v^*)-y^*(\tau;t,y^*(t),v^*)\big)g'_y$$

$$-\big(g(y(\tau;t,y_1,v^*),v^*)-g(y^*(\tau;t,y^*(t),v^*),v^*)\big)\Big)dsd\tau\Bigg\}$$

$$+\limsup_{y_1\to y^*(t)} \frac{\displaystyle\int_\Omega |y^\theta(T)-y^*(T)||y(T;t,y_1,v^*))-y(T;t,y^*(t),v^*)|dx}{|y_1-y^*(t)|}$$

$$=\limsup_{y_1\to y^*(t)} \frac{1}{|y_1-y^*(t)|}\Bigg\{\int_{Q_t} p^*(\tau)\big(y(\tau)-y^*(\tau)\big)\times$$

$$\times\Big(\int_0^1 f'_y(x,t,y^\theta(\tau))d\theta-\int_0^1 f'_y(x,t,y^\zeta(\tau))d\zeta\Big)dxd\tau$$

$$+\int_{\Sigma_t} p^*(\tau)\big(y(\tau)-y^*(\tau)\big)\Big(\int_0^1 g'_y\big(x,t,y^\theta(\tau),v^*\big)d\theta-\int_0^1 g'_y\big(x,t,y^\zeta(\tau),v^*\big)d\zeta\Big)dsd\tau\Bigg\}$$

$$\leq \limsup_{y_1\to y^*(t)} \frac{1}{|y_1-y^*(t)|}\Bigg\{\int_{Q_t} p^*(\tau)\big(y(\tau)-y^*(\tau)\big)\times$$

$$\times\left(\int_0^1\int_0^1 M_1(x,t)\eta(|y^\theta|)\eta(|y^\zeta|)\omega(|(\theta-\zeta)(y(\tau)-y^*(\tau))|)d\theta d\zeta\right)dxd\tau$$

$$+\int_{\Sigma_t} p^*(\tau)\big(y(\tau)-y^*(\tau)\big)\times$$

$$\times\left(\int_0^1\int_0^1 M_2(s,t)\eta(|y^\theta|)\eta(|y^\zeta|)\omega(|(\theta-\zeta)(y(\tau)-y^*(\tau))|)d\theta d\zeta\right)dsd\tau\Bigg\}$$

$$\leq \limsup_{y_1 \to y^*(t)} \frac{C(M_1, M_2)}{|y_1 - y^*(t)|} \left\{ \int_{Q_t} |p^*| \, |y(\tau) - y^*(\tau)| \, \omega(|y(\tau) - y^*(\tau)|) dx d\tau \right.$$

$$\left. + \int_{\Sigma_t} |p^*| \, |y(\tau) - y^*(\tau)| \, \omega(|y(\tau) - y^*(\tau)|) ds d\tau \right\}$$

$$\leq \limsup_{y_1 \to y^*(t)} \frac{C_5 C(M_1, M_2)}{|y_1 - y^*(t)|} \|p^*\|_{L^\infty(Q)} \left\{ \int_{Q_t} |y(\tau) - y^*(\tau)| \, \omega(|y(\tau) - y^*(\tau)|) dx d\tau \right.$$

$$\left. + \int_{\Sigma_t} |y(\tau) - y^*(\tau)| \, \omega(|y(\tau) - y^*(\tau)|) ds d\tau \right\}$$

$$\overset{(3.2)}{\leq} \limsup_{y_1 \to y^*(t)} \frac{C_5 C(M_1, M_2)}{|y_1 - y^*(t)|} \|y_1 - y^*(t)\|_{L^\infty(\Omega)} \omega(C \|y_1 - y^*(t)\|_{L^\infty(\Omega)})$$

and (5.3) follows.

Finally, we prove (5.4). We denote by $\mathcal{L}$ the set of Lebesgue points of ${y^*}'(\cdot)$ and recall that $\mathcal{L}$ has a full measure in $[t_0, T]$. Let $t \in (t_0, T) \cap \mathcal{L}$. Fix $r \in (t_0, T)$ and $y \in Y$. As previously we have

$$\begin{aligned}
&\frac{\partial}{\partial t}(y - y^*) + A(y - y^*) + f(x, \tau, y) - f(x, \tau, y^*) = 0 \quad \text{in } \Omega \times (r, T), \\
&\frac{\partial}{\partial n_A}(y - y^*) + g(s, \tau, y, v^*) - g(s, \tau, y^*, v^*) = 0 \quad \text{on } \partial\Omega \times (r, T), \\
&(y - y^*)(r) = y_1 - y^*(r) \quad \text{in } \Omega.
\end{aligned}$$

Thus multiplying by p^* and integrating over $\Omega \times (r, T)$ we get

$$\int_\Omega p^*(T)(y(T; r) - y^*(T; t)) dx = \int_\Omega p^*(r)(y_1 - y^*(r)) dx + \int_{Q_r} f'_y p^*(y - y^*) dx d\tau$$

$$- \int_{Q_r} (f(y) - f(y^*)) p^* dx d\tau + \int_{Q_r} A p^*(y - y^*) dx d\tau - \int_{Q_r} A(y - y^*) p^* dx d\tau. \quad (5.5)$$

Therefore

$$\limsup_{y_1 \to y^*(t), r \to t} \left\{ \frac{V(r, y_1) - V(t, y^*(t))}{|r - t| + |y_1 - y^*(t)|} - \frac{-\mathcal{H}(t, y^*(t), p^*(t))(r - t) + < p^*(t), y_1 - y^*(t) >}{|r - t| + |y_1 - y^*(t)|} \right\}$$

$$= \limsup_{y_1 \to y^*(t), \; r \to t} \left\{ \frac{1}{|r - t| + |y_1 - y^*(t)|} \times \right.$$

$$\times \int_\Omega L'_y(x, y^*(T; t, y^*(t), v^*), y(T; r, y_1, v^*))) \big(y(T; r, y_1, v^*) - y^*(T; t, y^*(t), v^*) \big) dx$$

$$+ \frac{\int_\Omega \big(L'_y(x, y^\theta(T)) - L'_y(x, y^*(T)) \big) \big(y(T; r, y_1, v^*) - y^*(T; t, y^*(t), v^*) \big) dx}{|r - t| + |y_1 - y^*(t)|}$$

$$\left. - \frac{-\mathcal{H}(t, y^*(t), p^*(t))(r - t) + < p^*(t), y_1 - y^*(t) >}{|r - t| + |y_1 - y^*(t)|} \right\}$$

(where $y^\theta(T) = y^*(T;t,y^*(t),v^*) + \theta(y(T;r,y_1,v^*) - y^*(T;t,y^*(t),v^*))$; we also use that $y^*(T;t,y^*(t),v^*) = y^*(T;r,y^*(r),v^*)$)

$$\overset{(A3),(5.1)}{\leq} \limsup_{y_1 \to y^*(t),\ r \to t} \left\{ \frac{\int_\Omega p^*(T)\big(y(T;r,y_1,v^*) - y^*(T;t,y^*(t),v^*)\big)dx}{|r-t| + |y_1 - y^*(t)|} \right.$$

$$\left. - \frac{-\mathcal{H}(t,y^*(t),p^*(t))(r-t) + < p^*(t), y_1 - y^*(t) >}{|r-t| + |y_1 - y^*(t)|} \right\}$$

and using (5.5) we get

$$= \limsup_{y_1 \to y^*(t),\ r \to t} \left\{ \frac{\int_{Q_r} \Big(f'_y p^*(y-y^*) - (f(y) - f(y^*))p^*\Big) dx d\tau}{|r-t| + |y_1 - y^*(t)|} \right.$$

$$+ \frac{\int_{\Sigma_r} \Big(g'_y p^*(y-y^*) - (g(y,v^*) - g(y^*,v^*))p^*\Big) ds d\tau}{|r-t| + |y_1 - y^*(t)|}$$

$$\left. + \frac{\int_\Omega p^*(r)\big(y_1 - y^*(r)\big)dx - \int_\Omega p^*(t)\big(y_1 - y^*(t)\big)dx}{|r-t| + |y_1 - y^*(t)|} - \frac{-\mathcal{H}(t,y^*(t),p^*(t))(r-t)}{|r-t| + |y_1 - y^*(t)|} \right\}$$

$$= \limsup_{y_1 \to y^*(t),\ r \to t} \left\{ \frac{\int_\Omega (p^*(r) - p^*(t))\big(y_1 - y^*(t)\big)dx + \int_\Omega p^*(r)\big(y^*(t) - y^*(r)\big)dx}{|r-t| + |y_1 - y^*(t)|} \right.$$

$$\left. + \frac{\mathcal{H}(t,y^*(t),p^*(t))(r-t)}{|r-t| + |y_1 - y^*(t)|} \right\}$$

$$\leq \limsup_{y_1 \to y^*(t),\ r \to t} \left\{ \int_\Omega |p^*(r) - p^*(t)| dx + \frac{\int_\Omega p^*(r)\big(y^*(t) - y^*(r)\big)dx}{|r-t| + |y_1 - y^*(t)|} \right.$$

$$\left. + \frac{\mathcal{H}(t,y^*(t),p^*(t))(r-t)}{|r-t| + |y_1 - y^*(t)|} \right\}$$

$$\leq \limsup_{y_1 \to y^*(t),\ r \to t} \frac{r-t}{|r-t| + |y_1 - y^*(t)|} \int_\Omega \left(p^*(r) \frac{y^*(t) - y^*(r)}{r-t} + \mathcal{H}(t,y^*(t),p^*(t)) \right) dx = 0.$$

5.2 Sufficient Conditions for Optimality

Theorem 5.2 *Assume* $(A1)-(A3)$ *and consider a trajectory-control pair* (y,v) *for the system* (1.1)*. If there is* $p(t)$ *such that*

$$\Big(-\mathcal{H}(t,y(t),p(t)), p(t) \Big) \in D^+V(t,y(t)) \tag{5.6}$$

for a.e. $t \in [t_0,T]$*, then* u *is optimal for problem* (P)*. In particular,* v *is optimal if for a.e.* $t \in [t_0,T]$ *there is* $p(t)$ *satisfying* (5.2) *and* (5.4)*.*

Proof. Consider the function $\psi(t) = V(t, y(t))$, which is continuous and nondecreasing in $[t_0, T]$ by Proposition 3.1. We will show that ψ is also nonincreasing and the conclusion will follow.
We obtain

$$\limsup_{h\to 0} \frac{V(t+h, y(t+h)) - V(t, y(t))}{h} = \limsup_{h\to 0} \left\{ \frac{V(t+h, y(t+h)) - V(t, y(t))}{h} \right.$$

$$\left. - \left(-\mathcal{H}(t, y(t), p(t)) + \left\langle p, \frac{y(t+h) - y(t)}{h} \right\rangle \right) \right\} \leq 0.$$

Consequently,

$$\limsup_{h\to 0} \frac{\psi(t+h) - \psi(t)}{h} \leq 0$$

for a.e. $t \in [0, T]$.

Next we claim that ψ is absolutely continuous on $[t_0, T]$.
Recalling (3.1), there exists v_t such that

$$0 \leq V(\theta, y(\theta)) - V(t, y(t)) \leq V(\theta, y(\theta)) - V(\theta, y(\theta; t, y(t), v_t)) + \theta - t$$

for all $t_0 \leq t \leq \theta \leq T$. Now set $y_t = y(\cdot; t, y(t), v_t)$ and note that, in view of Proposition 2.2,

$$\|y_t(\theta)\|_{L^\infty(\Omega)} \leq C_2$$

for some constant $C_2 > 0$ and all $t_0 < t \leq \theta \leq T$. Moreover, by Proposition 2.2 there exists $C_3 > 0$ such that

$$|V(\theta, y(\theta)) - V(\theta, y_t(\theta))| \leq C_3 \|y(\theta) - y_t(\theta)\|_{L^2(\Omega)}$$

for all $t_0 < t \leq \theta \leq T$. Therefore, again by Proposition 2.1 we get

$$0 \leq \psi(\theta) - \psi(t) \leq C_3 \|y(\theta) - y_t(\theta)\|_{L^2(\Omega)} + |\theta - t| \leq \int_t^\theta (\mathcal{C} + 1).$$

Indeed, we have

$$\begin{aligned}
&\frac{\partial}{\partial t}(y - y_t) + A(y - y_t) + f(x, t, y) - f(x, t, y_t) = 0 \text{ in } \Omega \times (t, T),\\
&\frac{\partial}{\partial n_A}(y - y_t) + g(s, t, y, v) - g(s, t, y_t, v_t) = 0 \text{ on } \Gamma \times (t, T),\\
&(y - y_t)(t) = 0 \text{ in } \Omega
\end{aligned}$$

and as above, by multiplying with $y - y_t$ and integrating over $\Omega \times (t, \theta)$ we get

$$\frac{1}{2}|(y(\theta) - y_t(\theta)|^2_{L^2(\Omega)} \leq \int_t^\theta \int_\Omega |y - y_t|^2 |f'_y(x, \tau, y^\zeta)| dx d\tau$$

$$+ \int_t^\theta \int_{\partial\Omega} |y - y_t|^2 |g'_y(s, \tau, y^\xi, v)| ds d\tau$$

$$- \int_t^\theta \int_{\partial\Omega} (g(s, \tau, y_t, v) - g(s, \tau, y_t, v_t))(y - y_t) ds d\tau \leq C(\Omega, m_1, M_1, M_2)(\theta - t)$$

and the proof is complete. ∎

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