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Periodic optimal control of the Boussinesq equation☆

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Abstract

This paper is concerned with the existence and the maximum principle for the optimal control problem governed by the Boussinesq equation. The case of internal controllers supported on $\omega \subset \Omega$ is examined.

Keywords: Boussinesq equation; The maximum principle; Convex analysis; Periodic conditions

1. Introduction

We study the periodic optimal control for an incompressible fluid flow coupled with thermal dynamics in two dimensions:

$$\text{Minimize} \quad \int_Q (2^{-1}|v(x,t) - v^0(x,t)|^2 + |\operatorname{rot} v(x,t)|^2 + 2^{-1}|\theta(x,t) - \theta^0(x,t)|^2 + h(u(x,t)) + 2^{-1}|u_1(x,t) - u_1^0(x,t)|^2)\, dx\, dt \tag{1.1}$$

over $u, u_1, v, \theta \in L^2(Q)$ subject to the Boussinesq system

$$\frac{\partial v}{\partial t} - \nu\Delta v + (v,\nabla)v - \sigma\theta + \nabla p = m(x)u \quad \text{in } \Omega \times \mathbb{R}, \tag{1.2}$$

$$\operatorname{div} v = 0 \text{ in } \Omega \times \mathbb{R},$$

$$v = 0 \text{ in } \partial\Omega \times \mathbb{R}, \; v(x,t) = v(x,t+T) \quad \forall (x,t) \in \Omega \times \mathbb{R},$$

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$$\frac{\partial\theta}{\partial t} - \chi\Delta\theta + (v, \nabla\theta) = u_1 \text{ in } \Omega \times \mathbb{R}, \tag{1.3}$$

$$\theta = 0 \text{ on } \partial\Omega \times \mathbb{R}, \;\; \theta(x,t) = \theta(x, t+T) \quad \forall (x,t) \in \Omega \times \mathbb{R}.$$

Here $Q = \Omega \times (0,T)$, Ω is a open and bounded subset of $\mathbb{R}^2$ with smooth boundary $\partial\Omega$, $m \in L^\infty(\Omega)$, $\operatorname{supp} m \subset \omega$ a open subset of Ω, u is a T-periodic source field, u_1 is a T-periodic heat source, $v = (v_1, v_2)$ is the velocity vector, θ is the temperature of the fluid, p stands for the pressure, $v^0 \in L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega))$ is a T-periodic reference velocity, $\theta^0 \in L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega))$ is a T-periodic reference temperature, while χ = thermal diffusivity, ν = kinematic viscosity and σ = constant two component vector. Finally, $h: \mathbb{R} \to \bar{\mathbb{R}} =]-\infty, +\infty]$ is a lower semicontinuous convex function.

The functions u, u_1 are the control variables of the problem. Our objective is to determine u and u_1 in such a way that the velocity vector and the temperature distribution to be as close as possible, in the sense of (1.1), to the desired velocity v^0 and temperature distribution θ^0, and the turbulence to be minimal.

The outline of this paper is as follows. In Section 2, the classical weak formulation of the problem is recalled and the existence of an optimal pair is established. In Section 3 first order necessary conditions of optimality (the maximal principle) in terms of Euler–Lagrange system are obtained.

We shall use the standard notation for vectorial and real-valued spaces of functions on $(0,T)$ and Ω. For other literature on optimal control problems governed by Boussinesq equations and related to this paper we cite [1,5,7–9].

2. The weak formulation and existence

Let us introduce the functional spaces to represent the Navier–Stokes equation (1.1) as an infinite dimensional differential equation (see [10,11]). Let V be the free divergence subspace of $(H^1_0(\Omega))^2$, i.e.,

$$V = \{v \in (H^1_0(\Omega))^2;\; \operatorname{div} v = 0\}$$

and

$$H = \{v \in (L^2(\Omega))^2;\; \operatorname{div} v = 0 \text{ in } \Omega;\; n \cdot v = 0 \text{ in } \partial\Omega\}.$$

The space H is endowed with the usual $(L^2(\Omega))^2$-norm denoted $|\cdot|$ and V with the $(H^1_0(\Omega))^2$ norm $\|\cdot\|$ defined by

$$\|v\|^2 = \sum_{1 \leqslant i \leqslant 2} \int_\Omega |\nabla v_i|^2 \, dx, v = (v_1, v_2).$$

We shall denote by $(\cdot,\cdot)$ the scalar product of H and the pairing between V and the dual space V^*. Identifying H with its own dual we have $V \subset H \subset V^*$. Let $A \in L(V, V^*)$ and $b: V \times V \times V \to \mathbb{R}$ be defined by

$$(Av, w) = \int_\Omega \nabla v_i \cdot \nabla w_i \, dx \quad \forall v, w \in V, \tag{2.1}$$

$$b(u,v,w) = \sum_{1\leqslant i,j\leqslant 2} \int_\Omega u_i D_i v_j w_j \,\mathrm{d}x \quad \forall u,v,w \in V \tag{2.2}$$

and $B : V \to V^*$ given by

$$(Bv,w) = b(v,v,w) \quad \forall v,w \in V. \tag{2.3}$$

Let $A_1 \in L(H_0^1(\Omega), H^{-1}(\Omega))$, $A_1 = -\Delta$ and let denote again by $\sigma\theta(t) = P\sigma\theta(t)$, $mu(t) = Pmu(t)$, where $P : (L^2(\Omega))^2 \to H$ is the projection on H. Then we may rewrite the state equations (1.2)–(1.3) as

$$\frac{\mathrm{d}v}{\mathrm{d}t}(t) + vAv(t) + Bv(t) = \sigma\theta(t) + mu(t), \quad t \in (0,T), \tag{2.4}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}(t) + \chi A_1\theta(t) + v(t)\cdot\nabla\theta(t) = u_1(t), \quad t \in (0,T), \tag{2.5}$$

$$v(0) = v(T), \quad \theta(0) = \theta(T), \tag{2.6}$$

where $u \in L^2(0,T;H)$, $u_1 \in L^2(0,T;L^2(\Omega))$. Using a fixed point argument it can be easily proved the following:

Proposition 2.1. *Let $u \in L^2(0,T;H)$, $u_1 \in L^2(0,T;L^2(\Omega))$. Then there is a solution (v,θ) to (2.4)–(2.6) satisfying*

$$\begin{aligned} &v \in W^{1,2}([0,T];H) \cap C([0,T],V), \quad Av \in L^2(0,T;H), \quad Bv \in L^2(0,T;H),\\ &\theta \in W^{1,2}([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)\cap H^2(\Omega)). \end{aligned} \tag{2.7}$$

Now we may reformulate problem (1.1) as

$$\begin{aligned} \text{Minimize}\, J(v,\theta,u,u_1) = \int_0^T &(2^{-1}|v(t)-v^0(t)|^2 + |\mathrm{rot}\, v(t)| + 2^{-1}|\theta(t)-\theta^0(t)|^2\\ &+ h(u(t)) + 2^{-1}|u_1(t)-u_1^0(t)|^2)\,\mathrm{d}t \end{aligned} \tag{P}$$

over $(v,\theta,u,u_1) \in W^{1,2}([0,T];H)\cap L^2(0,T;D(A)) \times W^{1,2}(0,T;L^2(\Omega))\cap L^2(0,T;D(A_1)) \times L^2(0,T;H) \times L^2(0,T;L^2(\Omega))$ subject to (2.4)–(2.6).

The functional framework is precised by the hypotheses given below

(i) V and H are real Hilbert spaces with the norm denoted by $\|\cdot\|, |\cdot|$ and the scalar products denoted by $(\cdot,\cdot)_V, (\cdot,\cdot)$. Moreover, $V \subset H \subset V^*$ algebraically and topologically with compact injection. Denote again $(\cdot,\cdot)$ the pairing between V and its dual space V^*; the space H is identified with its own dual. The norm of V^* will be denoted $\|\cdot\|_*$.

(ii) $A \in L(V,V^*)$ is symmetric, $(Av,v) = \|v\|^2$ $\forall v \in V$ and $(A_1\theta,\theta) = \|\theta\|^2$ $\forall \theta \in H_0^1(\Omega)$.

We set $D(A) = \{v \in V;\ Av \in H\}$, $D(A_1) = \{\theta \in H_0^1(\Omega);\ A_1\theta \in L^2(\Omega)\}$ and denote again by A, A_1 the restriction to H, $L^2(\Omega)$, respectively.

(iii) The operator $B : V \to V^*$ is defined by (2.3) where $b : V \times V \times V \to \mathbb{R}$ is a trilinear continuous functional satisfying

$$b(u,v,w) = -b(u,w,v) \quad \forall u,v,w \in V, \tag{2.8}$$

$$|b(u,v,w)| \leqslant C(|u|\,\|u\|\,|w|\,\|w\|)^{1/2}\|v\| \quad \forall u,v,w \in V, \tag{2.9}$$

$$|b(u,v,w)| \leqslant C(|u|\,\|u\|\,|v||Av|)^{1/2}|w| \quad \forall u \in V,\ v \in D(A),\ w \in H. \tag{2.10}$$

(iv) $v^0 \in L^2(0,T;\,H)$, $\theta^0 \in L^2(0,T;\,L^2(\Omega))$.

(v) The function $h : H \to \bar{\mathbb{R}}$ is lower, semicontinuous and satisfies the coercivity condition

$$h(u) \geqslant \alpha|u|^2 + \beta \quad \forall u \in H \tag{2.11}$$

for some $\alpha > 0$, $\beta \in \mathbb{R}$.

Recall that the assumption (iii) is satisfied for the two-dimensional system (1.2) (see [11]). We have denoted by $W^{1,2}([0,T];\,H)$ the space of all absolutely continuous functions $v : [0,T] \to H$ such that $v' = \mathrm{d}v/\mathrm{d}t \in L^2(0,T;\,H)$. We have

$$W^{1,2}([0,T];\,H) \cap L^2(0,T;\,D(A)) \subset C([0,T];\,V).$$

Theorem 2.1. *Assume that hypotheses* (i)–(v) *hold. Then problem* (P) *has at least one solution* $(v^*, \theta^*, u^*, u_1^*)$.

Proof. Let $(v_n, \theta_n, u_n, u_{1n})$ be a minimizing subsequence in problem (P), i.e.,

$$\inf(\mathrm{P}) \leqslant J(v_n, \theta_n, u_n, u_{1n}) \leqslant \inf(\mathrm{P}) + n^{-1}, \tag{2.12}$$

$$v_n' + \nu A v_n + B v_n = \sigma\theta_n + m u_n, \quad \text{a.e. } t \in (0,T), \tag{2.13}$$

$$\theta_n' + \chi A_1 \theta_n + v_n \cdot \nabla\theta_n = u_{1n}, \quad \text{a.e. } t \in (0,T), \tag{2.14}$$

$$v_n(0) = v_n(T), \quad \theta_n(0) = \theta_n(T). \tag{2.15}$$

By (2.11) it follows the boundedness of $\{u_n\}, \{\mathrm{rot}(v_n)\}, \{u_{1n}\}$ in $L^2(0,T;\,H)$, $L^2(0,T;\,L^2(\Omega))$, $L^2(0,T;\,L^2(\Omega))$, respectively. Therefore, on subsequences, again denoted n, we have

$$u_n \to u^* \quad \text{weakly in } L^2(0,T;\,H),$$
$$u_{1n} \to u_1^* \quad \text{weakly in } L^2(0,T;\,L^2(\Omega)). \tag{2.16}$$

Recalling that $(v \cdot \nabla\theta, \theta) = 0\ \forall v \in V$, $\theta \in L^2(\Omega)$, by (2.14) we get that

$$2^{-1}\frac{\mathrm{d}}{\mathrm{d}t}|\theta_n(t)|^2 + \chi\|\theta_n(t)\|^2 = (u_{1n}(t), \theta_n(t)), \quad \text{a.e. } t \in (0,T).$$

Therefore,

$$|\theta_n(t)|^2 + \int_0^T \|\theta_n(t)\|^2 \, dt \leqslant C, \quad \forall t \in [0,T].$$

(Here and throughout the sequel we shall denote by C several positive constants independent of v, θ and n.)

We note that by virtue of (2.8)–(2.10) we have

$$\|Bv\|_* \leqslant C|v| \, \|v\| \quad \forall v \in V, \tag{2.17}$$

$$|Bv| \leqslant C|v|^{1/2} \|v\| \, |Av|^{1/2} \quad \forall v \in D(A). \tag{2.18}$$

Since $(Bv, v) = 0 \ \forall v \in V$ we have get by (2.13) that

$$2^{-1} \frac{d}{dt} |v_n(t)|^2 + v\|v_n(t)\|^2 = (mu_n(t) + \sigma\theta_n(t), v_n(t)), \quad \text{a.e. } t \in (0,T).$$

This yields

$$|v_n(t)|^2 + \int_0^T \|v_n(t)\|^2 \, dt \leqslant C \quad \forall t \in [0,T]. \tag{2.19}$$

Next we multiply Eq. (2.13) by tAv_n and integrate on $(0,t)$ to get after some calculation involving (2.18) and (2.19) that

$$t\|v_n(t)\|^2 + \int_0^t s|Av_n(s)|^2 \, ds \leqslant C \left(1 + \int_0^t s\|v_n(s)\|^4 \, ds\right) \quad \forall t \in [0,T]$$

and so by Gronwall's lemma and (2.19) we have

$$t\|v_n(t)\|^2 \leqslant C \quad \forall t \in [0,T].$$

Since $v_n(0) = v_n(T)$ we infer that $\|v_n(0)\| \leqslant C$. Then multiplying (2.13) by Av_n and integrating on $(0,t)$ we get as above

$$\|v_n(t)\|^2 + \int_0^t |Av_n(s)|^2 \, ds \leqslant C \left(\|v_n(0)\|^2 + \int_0^t \|v_n(s)\|^4 \, ds\right)$$

and therefore

$$\|v_n(t)\|^2 + \int_0^t |Av_n(s)|^2 \, ds \leqslant C \quad \forall t \in [0,T].$$

This yields

$$\|v_n'\|_{L^2(0,T;H)} + \|Bv_n\|_{L^2(0,T;H)} \leqslant C.$$

In the same manner, multiplying (2.14) by $tA_1\theta_n$ and integrating on $(0,t)$ we get

$$t\|\theta_n(t)\|^2 \leqslant C \quad \forall t \in [0,T],$$

which, by periodicity condition $\theta_n(0) = \theta_n(T)$, yields $\|\theta_n(0)\| \leqslant C$. Multiplying (2.14) by $A_1\theta_n$ and integrating on $(0,t)$ we infer

$$\|\theta_n(t)\|^2 + \int_0^t |A_1\theta_n(s)|^2 \, ds \leqslant C \left(\|\theta_n(0)\|^2 + \int_0^t \|\theta_n(s)\|^4 \, ds\right)$$

and therefore

$$\|\theta_n(t)\|^2 + \int_0^t |A_1\theta_n(s)|^2\,\mathrm{d}s \leqslant C \quad \forall t \in [0,T].$$

Finally,

$$\|\theta_n'\|_{L^2(0,T;\,L^2(\Omega))} \leqslant C.$$

Since the injections of V into H and $H_0^1(\Omega)$ into $L^2(\Omega)$ are compact we infer that $\{v_n\},\{\theta_n\}$ are compact in $C([0,T];\,H) \cap L^2(0,T;V)$ and $C([0,T];\,L^2(\Omega)) \cap L^2(0,T;\,H_0^1(\Omega))$, respectively. Thus selecting further subsequences, if necessary, we have

$$v_n \to v^* \quad \text{strongly in } L^2(0,T;V) \cap C([0,T];\,H)$$

$$\operatorname{rot} v_n \to \operatorname{rot} v \quad \text{weakly in } L^2(0,T;\,L^2(\Omega)),$$

$$\theta_n \to \theta^* \quad \text{strongly in } L^2(0,T;\,H_0^1(\Omega)) \cap C([0,T];\,L^2(\Omega)),$$

$$Av_n \to Av^* \quad \text{weakly in } L^2(0,T;\,H),$$

$$A_1\theta_n \to A_1\theta^* \quad \text{weakly in } L^2(0,T;\,L^2(\Omega)),$$

$$v_n' \to (v^*)' \quad \text{weakly in } L^2(0,T;\,H),$$

$$\theta_n' \to (\theta^*)' \quad \text{weakly in } L^2(0,T;\,L^2(\Omega)).$$

Now recalling that by virtue of (2.8)–(2.10) we have

$$\begin{aligned}|(Bv_n - Bv^*, w)| &\leqslant |b(v_n - v^*, v_n, w)| + |b(v^*, v_n - v^*, w)| \\ &\leqslant C((|v_n - v^*|\,\|v_n - v^*\|\,\|v_n\|\,|Av^*|)^{1/2} \\ &\quad + (|v^*|\,\|v^*\|\,\|v_n - v^*\|\,|A(v_n - v^*|)^{1/2})|w| \ \ \forall w \in H\end{aligned}$$

we infer that

$$Bv_n \to Bv^* \quad \text{strongly in } L^2(0,T;\,H).$$

Then letting n go to ∞ in (2.12)–(2.15) we see that (v^*,θ^*,u^*,u_1^*) satisfies system (2.4)–(2.6) and $J(v^*,\theta^*,u^*,u_1^*) = \inf(\mathrm{P})$. This completes the proof. □

3. Necessary conditions for optimality

Here we shall establish a maximum principle result type for problem (P). Instead of (v) we shall use the following stronger hypothesis.

(vi) The function $h : H \to \bar{\mathbb{R}}$ is convex, continuous and there are $\alpha_i, i = 1,2$ such that

$$h(u) \leqslant \alpha_1|u|^2 + \alpha_2 \quad \forall u \in H. \tag{3.1}$$

Theorem 3.1. *Under hypotheses* (i)–(iv), (vi) *if* $(v^*, \theta^*, u^*, u_1^*)$ *are optimal in* (P) *then there are adjoint states* $q \in W^{1,2}([0,T]; H) \cap L^2(0,T; D(A))$ *and* $q_1 \in W^{1,2}([0,T]; L^2(\Omega)) \cap L^2(0,T; D(A_1))$ *such that*

$$q'(t) - vAq(t) - (B'(v^*(t)))^* q(t) - q_1 \cdot \nabla\theta^*$$
$$= v^* + 2\left(-\frac{\partial}{\partial x_2}\,\mathrm{rot}\, v^*, \frac{\partial}{\partial x_1}\,\mathrm{rot}\, v^*\right), \tag{3.2}$$

$$q_1'(t) - \chi A_1 q_1(t) + \mathrm{div}(q_1 v^*) + \sigma \cdot q = \theta^* - \theta^0, \tag{3.3}$$

$$q(0) = q(T), \ \ q_1(0) = q_1(T),$$

$$u^* \in \partial h^*(mq), \ \ u_1^* = q_1 + u_1^0 \quad \text{a.e. } t \in (0,T). \tag{3.4}$$

Here $\partial h : H \to H$ is the subdifferential of h and $(B'(v^*(t)))^* \in L(V, V^*) \cap L(D(A), H)$ is defined by

$$((B'(v^*(t)))^* q, w) = b(w, v^*(t), q) + b(v^*(t), w, q) \quad \forall q \in D(A), \ w \in H. \tag{3.5}$$

In the special case of control system (1.2)–(1.6) the adjoint system (3.1) has the form

$$\frac{\partial q}{\partial t} + v\Delta q - q \cdot \nabla v^* - v^* \cdot \nabla q + \nabla p - q_1 \nabla\theta^* = v^* - v^0,$$

$$\mathrm{div}\, q = 0 \ \text{ in } Q; \ \ q = 0 \ \text{ on } \Sigma, \ \ q(x,0) = q(x,T) \quad \text{in } \Omega,$$

$$\frac{\partial q_1}{\partial t} + \chi\Delta q_1 + \mathrm{div}\,(q_1 v^*) + \sigma \cdot q = \theta^* - \theta^0,$$

$$q_1 = 0 \quad \text{on } \Sigma, \ \ q_1(x,0) = q_1(x,T) \quad \text{in } \Omega. \tag{3.6}$$

We shall prove Theorem 3.1 following several steps.

Let $(v^*, \theta^*, u^*, u_1^*)$ be optimal in problem (P). For each $\varepsilon > 0$ consider the approximating optimization problem:

$$\text{Minimize} \int_0^T (h_\varepsilon(u) + |\mathrm{rot}\, v|^2 + 2^{-1}(|v - v^0|^2 + |\theta - \theta^0|^2 + |u_1 - u_1^0|^2$$
$$+|v - v^*|^2 + |\theta - \theta^*|^2 + |u - u^*|^2 + |u_1 - u_1^*|^2 + \varepsilon^{-1}|\phi|^2 + \varepsilon^{-1}|\psi|^2)\,\mathrm{d}t \tag{3.7}$$

over $v \in W^{1,2}([0,T]; H) \cap L^2(0,T; D(A))$, $\theta \in W^{1,2}([0,T]; L^2(\Omega)) \cap L^2(0,T; D(A_1))$, $u \in L^2(0,T; H)$, $u_1 \in L^2(0,T; L^2(\Omega))$ subject to

$$v'(t) + vAv(t) + Bv(t) = \sigma\theta(t) + mu(t) + \phi(t), \quad t \in (0,T), \ v(0) = v(T), \tag{3.8}$$

$$\theta'(t) + \chi A_1\theta(t) + v \cdot \nabla\theta(t) = u_1(t) + \psi(t), \quad t \in (0,T), \ \theta(0) = \theta(T), \tag{3.9}$$

where $h_\varepsilon \in C^1(\mathbb{R})$ is the Yosida-convex regularization of h, i.e.,

$$h_\varepsilon(r) = \inf\left\{\frac{|r-s|^2}{2\varepsilon} + h(s);\ s\in\mathbb{R}\right\} \quad \forall r\in\mathbb{R}.$$

By Theorem 2.1 for each $\varepsilon > 0$ problem (3.7) has at least one solution $(v_\varepsilon, \theta_\varepsilon, u_\varepsilon, u_{1\varepsilon}, \phi_\varepsilon, \psi_\varepsilon)$.

Lemma 3.1. *For $\varepsilon \to 0$, we have*

$$\begin{aligned}
&u_\varepsilon \to u^* \quad \text{strongly in } L^2(0,T;H),\ u_{1\varepsilon} \to u_1^* \quad \text{strongly in } L^2(0,T;L^2(\Omega)),\\
&v_\varepsilon \to v^* \quad \text{strongly in } L^2(0,T;V)\cap C([0,T];H),\\
&Av_\varepsilon \to Av^*,\ v_\varepsilon' \to (v^*)' \quad \text{weakly in } L^2(0,T;H),\\
&v_\varepsilon(t) \to v^*(t) \quad \text{weakly in } V \quad \forall t\in[0,T],\\
&\operatorname{rot} v_\varepsilon \to \operatorname{rot} v^*,\ \nabla \operatorname{rot} v_\varepsilon \to \nabla\operatorname{rot} v^* \quad \text{weakly in } L^2(0,T;L^2(\Omega)),\\
&\theta_\varepsilon \to \theta^* \quad \text{strongly in } L^2(0,T;H_0^1(\Omega))\cap C([0,T];L^2(\Omega)),\\
&A_1\theta_\varepsilon \to A_1\theta^*,\ \theta_\varepsilon' \to (\theta^*)' \quad \text{weakly in } L^2(0,T;L^2(\Omega)).
\end{aligned} \tag{3.10}$$

Proof. By (3.7) we see that

$$\int_0^T (|u_\varepsilon(t)|^2 + |u_{1\varepsilon}(t)|^2 + |v_\varepsilon(t)|^2 + |\operatorname{rot} v_\varepsilon(t)|^2 + |\theta_\varepsilon(t)|^2 + \varepsilon^{-1}|\phi_\varepsilon|^2$$
$$+\varepsilon^{-1}|\psi_\varepsilon|^2)\,dt \leqslant C$$

while if we multiply Eq. (3.8) by tAv_ε and integrate on $(0,t)$ we get as above that

$$t\|v_\varepsilon(t)\|^2 + \int_0^t s|Av_\varepsilon(s)|^2\,ds \leqslant C\left(1 + \int_0^t s\|v_\varepsilon(s)\|^4\,ds\right).$$

This yields

$$t\|v_\varepsilon(t)\|^2 \leqslant C \quad \forall t\in(0,T]$$

and therefore

$$\|v_\varepsilon(0)\| = \|v_\varepsilon(T)\| \leqslant C \quad \forall\varepsilon > 0.$$

Now if we multiply (3.8) by Av_ε, integrate on (0,t) and use the above estimates, we get

$$\|v_\varepsilon(t)\|^2 + \int_0^T |Av_\varepsilon(s)|\,ds \leqslant C \quad \forall t\in[0,T],\ \varepsilon > 0. \tag{3.11}$$

By (2.18) and (3.8) we have that

$$\int_0^T (|v_\varepsilon'|^2 + |Bv_\varepsilon|^2)\,dt \leqslant C \quad \forall\varepsilon > 0.$$

Finally, if we apply the rotational in Eq. (3.8), multiply by $\operatorname{rot} v_\varepsilon$ and integrate on $(0,T)$, we obtain that

$$\int_0^T |\nabla\operatorname{rot} v_\varepsilon|^2\,dt \leqslant C \quad \forall\varepsilon > 0.$$

Hence on a subsequence convergent to zero, again denoted ε, we have

$$v_\varepsilon \to v_0 \quad \text{strongly in } L^2(0,T;V)\cap C([0,T];H),$$

$$Av_\varepsilon \to Av_0,\ v_\varepsilon' \to v_0' \quad \text{weakly in } L^2(0,T;H),$$

$$\operatorname{rot} v_\varepsilon \to \operatorname{rot} v_0,\ \nabla\operatorname{rot} v_\varepsilon \to \nabla\operatorname{rot} v_0 \quad \text{weakly in } L^2(0,T;L^2(\Omega)),$$

$$u_\varepsilon \to u \quad \text{weakly in } L^2(0,T;H).$$

On the other hand, by (2.8)–(2.10) we see that

$$Bv_\varepsilon \to Bv_0 \quad \text{strongly in } L^2(0,T;H).$$

In the same way, multiplying (3.9) by $tA_1\theta_\varepsilon$ and integrate on $(0,t)$, by the above estimates we get

$$t\|\theta_\varepsilon(t)\|^2 + \int_0^t s|A_1\theta_\varepsilon(s)|^2\,\mathrm{d}s \leqslant C \quad \forall t\in(0,T]$$

and therefore

$$\|\theta_\varepsilon(0)\| = \|\theta_\varepsilon(T)\| \leqslant C \quad \forall\varepsilon>0.$$

Now if we multiply (3.9) by $A_1\theta_\varepsilon$ and integrate on $(0,T)$, by the above estimates we get that

$$\|\theta_\varepsilon(t)\|^2 + \int_0^T |\theta_\varepsilon'(t)|^2\,\mathrm{d}t + \int_0^T |A_1\theta_\varepsilon(t)|^2\,\mathrm{d}t \leqslant C \quad \forall t\in[0,T],\ \varepsilon>0.$$

Therefore, on a subsequence convergent to zero, we have

$$\theta_\varepsilon \to \theta_0 \quad \text{strongly in } L^2(0,T;H_0^1(\Omega))\cap C([0,T];L^2(\Omega)),$$

$$A_1\theta_\varepsilon \to A_1\theta_0,\ \theta_\varepsilon' \to \theta_0' \quad \text{weakly in } L^2(0,T;L^2(\Omega)),$$

$$u_{1\varepsilon} \to u_{10} \quad \text{weakly in } L^2(0,T;L^2(\Omega)).$$

Clearly $(v_0,\theta_0,u_0,u_{10})$ is a solution to state system (2.4)–(2.6). Since by (3.7)

$$\begin{aligned}&\int_0^T (h_\varepsilon(u_\varepsilon) + |\operatorname{rot} v_\varepsilon|^2 + 2^{-1}(|v_\varepsilon - v^0|^2 + |\theta_\varepsilon-\theta^0|^2 + |u_{1\varepsilon}-u_1^0|^2 + \varepsilon^{-1}|\phi_\varepsilon|^2\\ &\quad +\varepsilon^{-1}|\psi_\varepsilon|^2))\,\mathrm{d}t\\ &\quad + 2^{-1}\int_0^T (|v_\varepsilon - v^*|^2 + |\theta_\varepsilon - \theta^*|^2 + |u_\varepsilon - u^*|^2 + |u_{1\varepsilon}-u_1^*|^2)\,\mathrm{d}t\\ &\qquad \leqslant J(v^*,\theta^*,u^*,u_1^*) = \inf(P)\end{aligned}$$

we infer that $v_0=v^*$, $\theta_0=\theta^*$, $u_0=u^*$, $u_{10}=u_1^*$ and so the conclusion of Lemma 3.1 follows. □

Define the operator $B'(v_\varepsilon)\in L(V,V^*)\cap L(D(A),H)$

$$(B'(v_\varepsilon)z,w)=b(z,v_\varepsilon,w)+b(v_\varepsilon,z,w) \quad \forall w\in H,\ z\in D(A).$$

By (2.9) it follows that

$$|B'(v_\varepsilon)z|\leqslant C((|z|\,\|z\|\,\|v_\varepsilon\|\,|Av_\varepsilon|)^{1/2}+(|v_\varepsilon|\,\|v_\varepsilon\|\,\|z\|\,|Az|)^{1/2}) \quad \forall z\in D(A). \tag{3.12}$$

It is readily seen that

$$\lambda^{-1}(B(v_\varepsilon+\lambda z)-B(v_\varepsilon))\to B'(v_\varepsilon)z \quad \text{strongly in } L^2(0,T;\,H) \tag{3.13}$$

for all $z\in L^2(0,T;\,D(A))\cap C([0,T];\,V)$.

The adjoint operator $(B'(v_\varepsilon))^*\in L(V,V^*)\cap L(D(A),H)$ is given as in (3.5), i.e.,

$$((B'(v_\varepsilon))^*q,w)=b(w,v_\varepsilon,q)+b(v_\varepsilon,w,q) \quad \forall q\in D(A),\ \ w\in H \tag{3.14}$$

and satisfies estimate (3.12).

In the space $L^2(0,T;\,H)$ we define the operators

$$\mathscr{A}_\varepsilon\varphi=\varphi'+vA\varphi+B'(v_\varepsilon)\varphi \quad \forall\varphi\in D(\mathscr{A}_\varepsilon)=X, \tag{3.15}$$

$$\mathscr{A}_\varepsilon^*\varphi=-\varphi'+vA\varphi+(B'(v_\varepsilon))^*\varphi \quad \forall\varphi\in X'. \tag{3.15$'$}$$

Here

$$X=\{\varphi\in W^{1,2}([0,T];\,H)\cap L^2(0,T;\,D(A)),\ \operatorname{div}\varphi=0,\ \ \varphi(0)=\varphi(T)\}. \tag{3.16}$$

It is readily seen that

$$\int_0^T(\mathscr{A}_\varepsilon^*q,\varphi)\,\mathrm{d}t=\int_0^T(\mathscr{A}_\varepsilon\varphi,q)\,\mathrm{d}t \quad \forall\varphi,q\in D(\mathscr{A}_\varepsilon)=D(\mathscr{A}_\varepsilon^*)=X. \tag{3.17}$$

The operators $\mathscr{A}$ and $\mathscr{A}^*$ are defined by the same formulae (3.15) and (3.15′) where $v_\varepsilon=v^*$.

Lemma 3.2. *The operators $\mathscr{A}_\varepsilon$, $\mathscr{A}_\varepsilon^*$, $\mathscr{A}$, $\mathscr{A}^*$ are closed, densely defined and have closed ranges in $L^2(0,T;\,H)$. Moreover,* $\dim N(\mathscr{A}_\varepsilon)$, $\dim N(\mathscr{A}_\varepsilon^*)\leqslant n_0$, *independent of* ε, *$\mathscr{A}_\varepsilon^*$ is the adjoint of $\mathscr{A}_\varepsilon$ and the following estimates hold*

$$\|\mathscr{A}_\varepsilon^{-1}g\|_{L^2(0,T;\,D(A))\cap W^{1,2}([0,T];\,H)}\leqslant C\|g\|_{L^2(0,T;\,H)} \quad \forall g\in R(\mathscr{A}_\varepsilon), \tag{3.18}$$

$$\|(\mathscr{A}_\varepsilon^*)^{-1}g\|_{L^2(0,T;\,D(A))\cap W^{1,2}(0,T;\,H)}\leqslant C\|g\|_{L^2(0,T;\,H)} \quad \forall g\in R(\mathscr{A}_\varepsilon^*). \tag{3.18$'$}$$

Similarly, the operators $\mathscr{A}^*,\mathscr{A}$ are mutually adjoint and estimates (3.18) and (3.18′) remain true for $\mathscr{A}$ and $\mathscr{A}^*$.

Here we have used the symbols N and R to denote the null space and the range of the corresponding operators.

Proof. The proof is as in Ref. [4], but it will be sketched for reader's convenience. We shall prove first that $\mathscr{A}_\varepsilon$ has closed range and $N(\mathscr{A}_\varepsilon)$ is finite dimensional. Let

$(g,x) \in L^2(0,T;\, H) \times H$ be arbitrary but fixed. By (2.9), (2.10) and (3.14) we see that

$$\|B'(v_\varepsilon)\varphi\|_* \leqslant C((|\varphi|\,\|\varphi\|)^{1/2}\|v_\varepsilon\| + (|v_\varepsilon|\,\|v_\varepsilon\|)^{1/2}\|\varphi\|) \leqslant C_1\|\varphi\| \quad \forall \varphi \in V, \tag{3.19}$$

$$|(B'(v_\varepsilon)\varphi, \varphi)| \leqslant |\varphi|\,\|\varphi\|\,\|v_\varepsilon\| \leqslant C_1|\varphi|\,\|\varphi\| \quad \forall \varphi \in V \tag{3.20}$$

because by virtue of Lemma 1, $\{v_\varepsilon\}$ is bounded in $C([0,T];V)$. Then by a standard existence result for linear evolution equations we know that the Cauchy problem

$$\varphi' + \nu A\varphi + B'(v_\varepsilon)\varphi = g, \quad \text{a.e. } t \in (0,T);\ \ \varphi(0) = x \tag{3.21}$$

has a unique solution

$$\varphi = \varphi_\varepsilon(t,x,g) \in L^2(0,T;V) \cap W^{1,2}([0,T];V^*) \subset C([0,T];\, H).$$

Moreover, the following estimate holds

$$|\varphi(t)|^2 + \int_0^t \|\varphi(t)\|^2 \leqslant C\left(|x|^2 + \int_0^T |g(t)|^2\, dt\right). \tag{3.22}$$

(Here C is a positive constant independent of ε, x, g.) If $x \in V$ then it follows that

$$\varphi \in W^{1,2}([0,T];\, H) \cap L^2(0,T;\, D(A)) \subset C([0,T];V).$$

In the latter case if we multiply (3.21) by $tA\varphi(t)$ and integrate on $(0,t)$, by (3.12) and (3.22) we get

$$\begin{aligned} &t\|\varphi(t)\|^2 + \int_0^t s|A\varphi(s)|^2\, ds \\ &\quad \leqslant C\left(|x|^2 + \int_0^t |g(s)|^2\, ds + \left(\int_0^t s|A\varphi(s)|^2\, ds\right)^{3/4}\left(\int_0^t \|\varphi(s)\|^2\right)^{1/4}\right. \\ &\qquad \left. + \left(\int_0^t s|A\varphi(s)|^2\, ds\right)^{1/2}\left(\int_0^t \|\varphi(s)\|^2\, ds\right)^{1/4}\left(\int_0^t |Av_\varepsilon(s)|^2\, ds\right)^{1/4}\right). \end{aligned}$$

Since $\{Av_\varepsilon\}$ is bounded in $L^2(0,T;\, H)$ we conclude by (3.22) that

$$\|\varphi(t)\| \leqslant \rho(|x|, \|g\|_{L^2(0,T;\,H)})t^{-1} \quad \forall t > 0.$$

This estimate extends to all solutions φ to (3.21) where $x \in H$ and we have therefore

$$\varphi_\varepsilon(T,x,g) \in V;\, \|\varphi_\varepsilon(T,x,g)\| \leqslant \rho(|x|, \|g\|_{L^2(0,T;\,H)}) \quad \forall \varepsilon > 0. \tag{3.23}$$

We set

$$\varphi_\varepsilon(T,x,g) = \Gamma_\varepsilon x + E_\varepsilon g \tag{3.24}$$

where $\Gamma_\varepsilon x = \varphi_\varepsilon(T,x,0)$, $E_\varepsilon g = \varphi_\varepsilon(T,0,g)$. Clearly, $\Gamma_\varepsilon \in L(H,V)$, $E_\varepsilon \in L(L^2(0,T;\, H),V)$ and estimate (3.23) yields

$$\|E_\varepsilon\|_{L(L^2(0,T;\,H),V)} + \|\Gamma_\varepsilon\|_{L(H,V)} \leqslant C \quad \forall \varepsilon > 0. \tag{3.25}$$

Since the injection of V into H is compact we infer that Γ_ε is completely continuous in H. Now let $(v,g)\in\mathscr{A}_\varepsilon$, i.e., $\mathscr{A}_\varepsilon v=g$. We have, therefore, $v(t)=\varphi_\varepsilon(t,x,g)$ where $(I-\Gamma_\varepsilon)x=E_\varepsilon g$. By Fredholm–Riesz theory we know that $R(I-\Gamma_\varepsilon)$ is closed and $\dim N(I-\Gamma_\varepsilon)<\infty$. Hence $R(\mathscr{A}_\varepsilon)$ is closed in $L^2(0,T;H)$ and $N(\mathscr{A}_\varepsilon)$ is finite dimensional.

Let $(\varphi_n,g_n)\in\mathscr{A}_\varepsilon$ and

$$\varphi_n\to\varphi,\quad g_n\to g\quad \text{strongly in } L^2(0,T;H).$$

(These results depend heavily on the periodicity.)

Then by estimate (3.23) it follows that $\{\varphi_n(0)\}$ is bounded in V and as seen earlier this implies that

$$\{\varphi_n\} \text{ is bounded in } L^2(0,T;D(A))\cap W^{1,2}([0,T];H)$$

$$B'(v_\varepsilon)\varphi_n\to B'(v_\varepsilon)\varphi\quad \text{weakly in } L^2(0,T;H).$$

Hence $(\varphi,g)\in\mathscr{A}_\varepsilon$, i.e., $\mathscr{A}_\varepsilon$ is closed.

Now let $\Gamma\in L(H,H)$ be defined by $\Gamma x=\varphi(T,x,0)$ where φ is the solution to

$$\varphi'+vA\varphi+B'(v^*)\varphi=0,\quad \text{a.e. } t\in(0,T),\ \varphi(0)=x. \tag{3.26}$$

As seen earlier $\Gamma\in L(H,V)$ and so Γ is completely continuous from H to itself. Moreover, by Lemma 3.1 and estimate (3.25) it follows that

$$\Gamma_\varepsilon\to\Gamma\quad \text{in } L(H,H)$$

as $\varepsilon\to 0$. Hence $\dim N(\mathscr{A}_\varepsilon)\leqslant n_0$, $\forall\varepsilon>0$ as claimed. We also have the estimate

$$|(I-\Gamma_\varepsilon)^{-1}g_0|\leqslant C|g_0|\quad \forall g_0\in R(I-\Gamma_\varepsilon). \tag{3.27}$$

Indeed, otherwise there are $x_\varepsilon\in R((I-\Gamma_\varepsilon)^*)$, $f_\varepsilon\in R(I-\Gamma_\varepsilon)$ such that

$$|f_\varepsilon|=1,\ (I-\Gamma_\varepsilon)x_\varepsilon=f_\varepsilon,\ |x_\varepsilon|\to 0$$

as $\varepsilon\to 0$. Then on a subsequence we have

$$x_\varepsilon|x_\varepsilon|^{-1}\to x_0$$

where $|x_0|=1$, $x_0\in R((I-\Gamma)^*)$, $(I-\Gamma)x_0=0$ which leads to a contradiction because $R((I-\Gamma)^*)\oplus N(I-\Gamma)=H$.

Now we recall that $\varphi=\varphi_\varepsilon(t,x,g)$ where $(I-\Gamma_\varepsilon)x=E_\varepsilon g$ is a solution to equation $\mathscr{A}_\varepsilon\varphi=g$ while by (3.25) and (3.27) we have

$$|\varphi_\varepsilon(0)|\leqslant C\|g\|_{L^2(0,T;H)}\quad \forall g\in R(\mathscr{A}_\varepsilon).$$

Then as seen above we have

$$\|\varphi_\varepsilon\|_{W^{1,2}(0,T;H)}+\|\varphi_\varepsilon\|_{L^2(0,T;D(A))}\leqslant C\|g\|_{L^2(0,T;H)}\quad \forall g\in R(\mathscr{A}_\varepsilon)$$

which implies (3.18).

The corresponding properties of the operator $\mathscr{A}_\varepsilon^*$ follows mutatis-mutandis from the previous arguments because in this case Eq. (3.21) is replaced by

$$\varphi'+vA\varphi+(B'(v_\varepsilon))^*\varphi=g;\ \ \varphi(0)=x$$

and so the previous estimates remain valid. In particular, it follows that the operator $\mathcal{A}_\varepsilon^*$ is closed and by (3.17) we conclude that its adjoint is precisely $\mathcal{A}_\varepsilon$. We note also that by Lemma 3.1 and the above estimates we have

$$\mathcal{A}_\varepsilon v \to \mathcal{A} v \quad \text{weakly in } L^2(0,T;H)$$

as $\varepsilon \to 0$ for each $v \in X$. $\square$

In a similar way to (3.15) we define the operators

$$\mathcal{A}_{1\varepsilon}\psi = \psi' + \chi A_1\psi + v_\varepsilon\cdot\nabla\psi \quad \forall\psi\in D(\mathcal{A}_{1\varepsilon}) = Y, \tag{3.28}$$

$$\mathcal{A}_{1\varepsilon}^*\psi = -\psi' + \chi A_1\psi - \operatorname{div}(\psi v_\varepsilon) \quad \forall\psi\in Y', \tag{3.28'}$$

where

$$Y = \{\psi\in W^{1,2}([0,T];L^2(\Omega))\cap L^2(0,T;D(A_1));\ \psi(0)=\psi(T)\}.$$

The operators $\mathcal{A}_1, \mathcal{A}_1^*$ are defined by the same formulae (3.28), (3.28′), where $v_\varepsilon = v^*$.

Lemma 3.3. *The operators $\mathcal{A}_{1\varepsilon}$, $\mathcal{A}_{1\varepsilon}^*$, $\mathcal{A}_1$, $\mathcal{A}_1^*$ are closed, densely defined and have closed ranges in $L^2(0,T;H)$. Moreover,* $\dim N(\mathcal{A}_{1\varepsilon})$, $\dim N(\mathcal{A}_{1\varepsilon}^*) \leqslant n$, *independent of ε and the following estimates hold*

$$\|\mathcal{A}_{1\varepsilon}^{-1}g\|_{L^2(0,T;D(A_1))\cap W^{1,2}([0,T];L^2(\Omega))} \leqslant C\|g\|_{L^2(0,T;L^2(\Omega))} \quad \forall g\in R(\mathcal{A}_{1\varepsilon}), \tag{3.29}$$

$$\|(\mathcal{A}_{1\varepsilon}^*)^{-1}g\|_{L^2(0,T;D(A_1))\cap W^{1,2}([0,T];L^2(\Omega))} \leqslant C\|g\|_{L^2(0,T;L^2(\Omega))} \quad \forall g\in R(\mathcal{A}_{1\varepsilon}^*)'. \tag{3.29'}$$

This result is a consequence of Riesz–Fredholm theory (see [2,3]), and it is based on the compactness of semigroup $e^{-\mathbf{A}t}$ generated by $-\mathbf{A}$ on $L^2(\Omega)$, where $\mathbf{A}\theta = -\Delta\theta + b\cdot\nabla\theta + c\theta, D(\mathbf{A}) = H_0^1(\Omega)\cap H^2(\Omega)$, $b\in W^{1,\infty}(\Omega,\mathbb{R}^2)$, $c\in L^\infty(\Omega)$.

Proof of Theorem 3.1 (continued). For $\lambda\in\mathbb{R}, (v,\theta,u,u_1)\in X\times Y\times L^2(0,T;H)\times L^2(0,T;L^2(\Omega))$ we set

$$\phi^\lambda = (v_\varepsilon+\lambda v)' + vA(v_\varepsilon+\lambda v) + B(v_\varepsilon+\lambda v) - \sigma(\theta_\varepsilon+\lambda\theta) - m(u_\varepsilon+\lambda u)$$

$$\psi^\lambda = (\theta_\varepsilon+\lambda\theta)' + \chi A_1(\theta_\varepsilon+\lambda\theta) + (v_\varepsilon+\lambda v)\cdot\nabla(\theta_\varepsilon+\lambda\theta) - (u_{1\varepsilon}+\lambda u_1).$$

We may write $(\phi_\lambda,\psi_\lambda)$ as

$$\phi^\lambda = \phi_\varepsilon + \lambda\left(v' + vAv + B'(v_\varepsilon)v + \lambda B(v) - \sigma\theta - mu\right)$$

$$\psi^\lambda = \psi_\varepsilon + \lambda\left(\theta' + \chi A_1\theta + v_\varepsilon\cdot\nabla\theta + v\cdot\nabla\theta_\varepsilon + \lambda v\cdot\nabla\theta - u_1\right).$$

So by the optimality of $(v_\varepsilon,\theta_\varepsilon,u_\varepsilon,u_{1\varepsilon},\phi_\varepsilon,\psi_\varepsilon)$ in (3.7) we have that

$$\int_0^T ((2v_\varepsilon - v^0 - v^*, v) + (2\theta_\varepsilon - \theta^0 - \theta^*, \theta) + (u_\varepsilon - u^*, u)$$

$$+ 2(\operatorname{rot} v_\varepsilon, \operatorname{rot} v) + (2u_{1\varepsilon} - u_1^0 - u_1^*, u_1) + h_\varepsilon'(u_\varepsilon, u) \tag{3.30}$$

$$+\frac{1}{\varepsilon}(\phi_\varepsilon, v' + vAv + B'(v_\varepsilon)v - \sigma\theta - mu)$$

$$+\frac{1}{\varepsilon}(\psi_\varepsilon, \theta' + \chi A_1\theta + v_\varepsilon\cdot\nabla\theta + v\cdot\nabla\theta_\varepsilon - u_1))\,\mathrm{d}t \geqslant 0,$$

for all $(v,\theta,u,u_1)\in X\times Y\times L^2(0,T;H)\times L^2(0,T;L^2(\Omega))$.

We set $(q_\varepsilon, q_{1\varepsilon}) = \varepsilon^{-1}(\phi_\varepsilon,\psi_\varepsilon)$. In (3.30) we take $(u,u_1)=0$ we get

$$\int_0^T \big((2v_\varepsilon - v^0 - v^* - 2\Delta v_\varepsilon + q_{1\varepsilon}\nabla\theta_\varepsilon, v) + (\mathscr{A}_\varepsilon v, q_\varepsilon)$$

$$+(2\theta_\varepsilon - \theta^0 - \theta^* - \sigma\cdot q_\varepsilon, \theta) + (\mathscr{A}_{1\varepsilon}\theta, q_{1\varepsilon})\big)\,\mathrm{d}t = 0. \tag{3.31}$$

Hence $q_\varepsilon\in D(\mathscr{A}_\varepsilon^*), q_{1\varepsilon}\in D(\mathscr{A}_{1\varepsilon}^*)$ and

$$\begin{aligned}&\mathscr{A}_\varepsilon^* q_\varepsilon + q_{1\varepsilon}\nabla\theta_\varepsilon = v^0 + v^* - 2v_\varepsilon + 2\Delta v_\varepsilon\\ &\mathscr{A}_{1\varepsilon}^* q_{1\varepsilon} - \sigma\cdot q_\varepsilon = \theta^0 + \theta^* - 2\theta_\varepsilon.\end{aligned} \tag{3.32}$$

Then using once again (3.30) we see that

$$\int_0^T \big((u_\varepsilon - u^* - mq_\varepsilon, u) + (2u_{1\varepsilon} - u_1^0 - u_1^*, u_1) + h_\varepsilon'(u_\varepsilon, u)\big)\,\mathrm{d}t \geqslant 0.$$

which yields

$$\begin{aligned}&mq_\varepsilon = h_\varepsilon'(u_\varepsilon) + u_\varepsilon - u^*,\\ &q_{1\varepsilon} = 2u_{1\varepsilon} - u_1^0 - u_1^*, \quad \text{a.e. } t\in(0,T).\end{aligned} \tag{3.33}$$

Due to assumption (vi) and Lemma 3.1 it follows that

$$\|mq_\varepsilon\|_{L^2(0,T;H)} + \|q_{1\varepsilon}\|_{L^2(0,T;L^2(\Omega))} \leqslant C \quad \forall\varepsilon>0. \tag{3.34}$$

Now, we may write $q_{1\varepsilon} = q_{1\varepsilon}^1 + q_{1\varepsilon}^2$ where $q_{1\varepsilon}^1\in R(\mathscr{A}_{1\varepsilon})$, $q_{1\varepsilon}^2\in N(\mathscr{A}_{1\varepsilon}^*)$. By Lemma 3.3 we have

$$\|q_{1\varepsilon}^1\|_{L^2(0,T;D(A_1))\cap W^{1,2}([0,T];L^2(\Omega))} \leqslant C \quad \forall\varepsilon>0.$$

Since by (3.34) we see that $\{q_{1\varepsilon}^2\}$ is bounded then on a subsequence again denoted $\{\varepsilon\}$ we have

$$\begin{aligned}&q_{1\varepsilon}^1 \to q_1^1 \quad \text{weakly in } L^2(0,T;D(A_1))\cap W^{1,2}([0,T];L^2(\Omega)),\\ &q_{1\varepsilon}^2 \to q_1^2 \quad \text{strongly in } L^2(0,T;L^2(\Omega)).\end{aligned}$$

On the other hand, we may write $q_\varepsilon = q_\varepsilon^1 + q_\varepsilon^2$ where $q_\varepsilon^1\in R(\mathscr{A}_\varepsilon)$, $q_\varepsilon^2\in N(\mathscr{A}_\varepsilon^*)$. By Lemma 3.2, part (3.18), we know that

$$\|q_\varepsilon^1\|_{L^2(0,T;D(A))\cap W^{1,2}([0,T];H)} \leqslant C \quad \forall\varepsilon>0.$$

Then by (3.34) we see that $\{mq_\varepsilon^2\}$ is bounded in $L^2(0,T;H)$. In order to prove that $\{q_\varepsilon^2\}$ is bounded in $L^2(0,T;H)$, we will write that $\mathscr{A}_\varepsilon^* q_\varepsilon^2 = 0$ in the following form

$$\begin{aligned}&\partial_t(\Delta p(x,t)) + \Delta^2 p + \partial_{x_1}(\Delta\psi\,\partial_{x_2}p) - \partial_{x_2}(\Delta\psi\,\partial_{x_1}p)\\ &\qquad + \Delta(\partial_{x_1}p\partial_{x_2}\psi - \partial_{x_2}p\partial_{x_1}\psi) = 0,\\ &p|_{\partial\Omega} = 0, \;\; p(T) = p(0).\end{aligned} \tag{3.35}$$

where $q_\varepsilon^2 = (\partial_{x_2} p, -\partial_{x_1} p)$, $v_\varepsilon = (\partial_{x_2}\psi, -\partial_{x_1}\psi)$, $\operatorname{rot} v_\varepsilon = -\Delta\psi$, $\operatorname{rot} q_\varepsilon^2 = -\Delta p$. Moreover, let us define functions $\beta \in C^2(\bar{\Omega})$,

$$\beta(x) > 0 \quad \forall x \in \Omega, \ \beta|_{\partial\Omega} = 0, \ |\nabla\beta(x)| > 0 \quad \forall x \in \Omega \setminus \omega_0, \ \omega_0 \subset\subset \omega \tag{3.36}$$

and $\alpha(t,x) = (\mathrm{e}^{\lambda\beta} - \mathrm{e}^{\lambda^2\|\beta\|_{C(\bar{\Omega})}})/(t(T-t))^3$, with $\lambda > 1$ satisfying a certain inequality. From [6] we have the following Carleman estimate

Proposition 3.1. *There exists $s > 0$ such that the solution of (3.35) satisfy*

$$\int_Q \left(\frac{1}{((T-t)t)^{12}}|\nabla p|^2\right) \mathrm{e}^{2s\alpha}\,\mathrm{d}x\,\mathrm{d}t$$

$$\leqslant C(s)\int_{Q_{\omega_1}} \left(\frac{1}{((T-t)t)^{21}}p^2\right) \mathrm{e}^{2s\alpha}\,\mathrm{d}x\,\mathrm{d}t. \tag{3.37}$$

This imply that $\|q_\varepsilon^2\|_{L^2(0,T;L^2(\Omega))} \leqslant C(s,\omega)\|q_\varepsilon^2\|_{L^2(0,T;L^2(\omega))}$. Therefore $\{q_\varepsilon^2\}$ is bounded in $L^2(0,T;H)$ and on a subsequence, again denoted $\{\varepsilon\}$, we have

$$q_\varepsilon^1 \to q^1 \quad \text{weakly in } L^2(0,T;D(A)) \cap W^{1,2}([0,T];H)$$

$$q_\varepsilon^2 \to q^2 \quad \text{strongly in } L^2(0,T;H)$$

because $\{q_\varepsilon^2\} \subset N(\mathscr{A}_\varepsilon^*)$ and $\dim N(\mathscr{A}_\varepsilon^*) \leqslant n_0$.

Now letting ε tend to 0 into (3.32) and (3.33) it follows by Lemmas 3.1 and 3.2 that

$$\mathscr{A}^* q + q_1 \nabla\theta^* = v^0 - v^* + 2\left(-\frac{\partial}{\partial x_2}\operatorname{rot} v_*, \frac{\partial}{\partial x_1}\operatorname{rot} v_*\right),$$

$$\mathscr{A}_1^* q_1 - \sigma \cdot q = \theta^0 - \theta^*$$

and the optimality condition

$$u^* \in \partial h^*(mq), \ u_1^* = q_1 + u_1^0 \quad \text{a.e. } t \in (0,T).$$

Hence q and q_1 satisfies Eqs. (3.2) and (3.4). This completes the proof of Theorem 3.1. □

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