# BOUNDS ON ENERGY, MAGNETIC HELICITY AND CROSS HELICITY DISSIPATION RATES OF APPROXIMATE DECONVOLUTION MODELS OF TURBULENCE FOR MHD FLOWS

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**Abstract.** We consider a family of high accuracy, approximate deconvolution models of turbulent magnetohydrodynamic flows. For body force driven turbulence, we prove directly from the model's equations of motion the following bounds on the model's time averaged energy dissipation rate $\langle \varepsilon_{\mathtt{ADM}} \rangle$, time averaged cross helicity dissipation rate $\langle \gamma_{\times,\mathtt{ADM}} \rangle$ and magnetic helicity dissipation rate $\langle \gamma_{\mathtt{mag,ADM}} \rangle$, where $\mathscr{U}, \mathscr{B}, L$ are the global velocity scale, global magnetic field scale and length scale, $R$ is a dimensionless constant related to fluid and magnetic Reynolds numbers, given precisely in Section 2.2, $S$ is the reciprocal of the product of fluid density times free-space permeability and $\delta$ is the LES filter radius.

$$\langle \varepsilon_{\mathtt{ADM}} \rangle \sim \langle \gamma_{\times,\mathtt{ADM}} \rangle \sim \frac{\sqrt{S}}{L} \langle \gamma_{\mathtt{mag,ADM}} \rangle \sim \frac{(\mathscr{U}^2 + S\mathscr{B}^2)^{3/2}}{L} \left(1 + \frac{1}{R^2}(1 + \frac{\delta^2}{L^2})\right)^{1/2}$$

**Key words.** energy dissipation rate, helicity, helicity dissipation rate, large eddy simulation, turbulence, deconvolution, MHD

**1. Introduction.** The turbulent flow of an electrically and magnetically conducting fluid (MHD turbulence) is significantly more complex with many more parameter regimes and more uncertainties about basic physical mechanisms than turbulent flow of a non conducting fluid. The added complexities, length scales, dynamics and the extra variables needed for MHD flows make accurate simulation of MHD turbulence still more challenging. Because of these (and other) factors, the development of accurate and reliable models for MHD turbulence is correspondingly more important than for ordinary turbulent flows and correspondingly less well developed.

We consider herein one recent family of high-accuracy, parameter free models, Approximate Deconvolution Models (ADMs) of magnetohydrodynamic (MHD) turbulence. Considerable experience with turbulence models suggests that one critical factor for a reliable model is that it contains sufficient model dissipation for the required scale truncation without over dissipation of critical structures and important dynamics. This report studies the rates of model dissipation of the three critical integral invariants of MHD flows for approximate deconvolution models.

We derive rigorous bounds, consistent with dimensional analysis and turbulence phenomenology, on the ADM's time averaged energy, magnetic and cross helicity dissipation rates. To begin, consider the magnetohydrodynamic equations in a periodic box in $\mathbb{R}^3, \Omega = (0, L_\Omega)^3, t > 0$,

$$\begin{aligned} &u_t + \nabla \cdot (uu^T) - \nu \Delta u + \frac{S}{2}\nabla(B^2) - S\nabla \cdot (BB^T) + \nabla p = f(x), \\ &B_t + \nu_{\mathtt{m}} \nabla \times (\nabla \times B) + \nabla \times (B \times u) = \nabla \times g(x), \\ &\nabla \cdot u = 0, \end{aligned} \tag{1.1}$$

subject to periodic (with zero mean) conditions

$$\phi(t, x + L_\Omega \mathrm{e}_i) = \phi(t, x), i = 1, 2, 3, \quad \int_\Omega \phi(t, x) dx = 0, \tag{1.2}$$

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‡Partially supported by Air Force grant FA 9550-09-1-0058

for $\phi = u, p, B, f, g$. The physical constants $\nu$, $\nu_{\mathtt{m}}$, and $S$ represent kinematic viscosity of the fluid, magnetic diffusivity, and permeability times density, respectively, and the variables $u$, $B$ represent fluid velocity and magnetic field, respectively.

We suppose throughout that the data $u_0(x), B_0(x), f(x), \nabla \times g(x)$ are smooth and satisfy (1.2) as well as

$$\nabla \cdot u_0 = 0, \quad \nabla \cdot B_0 = 0, \quad \nabla \cdot g = 0, \text{ and } \nabla \cdot f = 0.$$

Note that $\nabla \cdot B_0 = 0$ immediately implies $\nabla \cdot B = 0$ for $t > 0$. Many averaging operators are used in LES, see, e.g., Sagaut [46], John [26], and [5]. Herein we consider a differential filter due to Germano [21] that is associated with length scale $\delta > 0$ and is related to the Yoshida regularization (and sometimes called a Helmholz-filter in the alpha-model literature, e.g., Cheskidov, Holm, Olson and Titi [10]) defined as follows. Given $\phi(x), \overline{\phi(x)}$ is the unique $L_\Omega$-periodic solution of

$$\phi = A\overline{\phi} = -\delta^2 \Delta \overline{\phi} + \overline{\phi}, \quad \text{in } \Omega.$$

As is commonly done in the LES literature, we denote the filtering operator by $G$, so $\overline{\phi} = G\phi$. Averaging the MHD (i.e., applying $G = A^{-1}$ to (1.1)) gives the exact space filtered MHD for $\overline{u}, \overline{B}$

$$\begin{aligned}
&\overline{u}_t + \overline{\nabla \cdot (uu^T)} - \nu \Delta \overline{u} - S \nabla \cdot (\overline{BB^T}) + \nabla \left( \frac{S}{2} \overline{B^2} + \overline{p} \right) = \overline{f}, \\
&\overline{B}_t + \nu_{\mathtt{m}} \nabla \times (\nabla \times \overline{B}) - \nabla \cdot (\overline{Bu^T}) + \nabla \cdot (\overline{uB^T}) = \nabla \times \overline{g}, \\
&\nabla \cdot \overline{u} = 0, \nabla \cdot \overline{B} = 0.
\end{aligned}$$

This is not closed since

$$\overline{uu} \neq \overline{u}\,\overline{u}, \quad \overline{BB} \neq \overline{B}\,\overline{B}, \quad \overline{uB} \neq \overline{u}\,\overline{B}, \quad \overline{Bu} \neq \overline{B}\,\overline{u}.$$

There are many closure models used in LES, see Sagaut [46], John [26], Lesieur, Metais and Comte [37] and [5, 22] for surveys. Few have been extended to the *full* MHD system including the needed truncation of both fluid and electromagnetic scales. Another approach that has proven valuable is to use simpler regularizations of the fluid equations as a basis for numerical simulation. For MHD turbulence, for example, Linschiz and Titi [38] have studied the NS-$\alpha$ regularization of the momentum equation with no further averaging of the other MHD system's couple equations. The Lagrangian-averaged magnetohydrodynamics (LAMHD) $\alpha$ model is studied in [23].

Approximate deconvolution models, studied herein, are used, with success, in many simulations of turbulent flows, e.g., the works of Adams, Kleiser and Stolz [1, 2, 49, 51, 50, 52]. They are among the most accurate of turbulence models, and one of the few turbulence models for which a mathematical confirmation of their effectiveness is known [34] and Dunca and Epshteyn [15]. Briefly, an approximate deconvolution operator (constructed in Section 2.1) denoted by $D_N$ is an operator satisfying

$$\phi = D_N \overline{\phi} + O(\delta^{2N+2}) \qquad \text{for smooth } \phi.$$

Since $D_N u$ approximates $u$ and $D_N B$ approximates $B$ to accuracy $O(\delta^{2N+2})$ in the smooth flow regions it is justified to consider the closure approximations:

$$\begin{aligned}
\overline{uu} &\simeq \overline{D_N \overline{u} D_N \overline{u}} + O(\delta^{2N+2}), \qquad & \overline{BB} &\simeq \overline{D_N \overline{B} D_N \overline{B}} + O(\delta^{2N+2}), \\
\overline{uB} &\simeq \overline{D_N \overline{u} D_N \overline{B}} + O(\delta^{2N+2}), \qquad & \overline{Bu} &\simeq \overline{D_N \overline{B} D_N \overline{u}} + O(\delta^{2N+2}).
\end{aligned}$$

Using this closure approximation, the resulting family of ADMs is given by *

$$
\begin{aligned}
&w_t + \nabla\cdot(\overline{D_N w D_N w^T}) - \nu\Delta w - S\nabla\cdot(\overline{D_N W\, D_N W^T}) + \nabla q = \overline{f},\\
&W_t + \nu_{\mathtt{m}}\nabla\times(\nabla\times W) - \nabla\cdot(\overline{D_N W D_N w^T}) + \nabla\cdot(\overline{D_N w D_N W^T}) = \nabla\times\overline{g},\\
&\nabla\cdot w = 0, \quad \nabla\cdot W = 0, \qquad N = 0,1,2,\ldots.
\end{aligned}
$$

As a special case, $D_0\overline{u} = \overline{u} + O(\delta^2), D_0\overline{B} = \overline{B} + O(\delta^2)$ gives the zeroth order ADM:

$$
\begin{aligned}
&w_t + \nabla\cdot(\overline{ww^T}) - \nu\Delta w - S\nabla\cdot(\overline{W\,W^T}) + \nabla q = \overline{f},\\
&W_T + \nu_{\mathtt{m}}\nabla\times(\nabla\times W) + \nabla\cdot(\overline{Ww^T}) - \nabla\cdot(\overline{wW^T}) = \nabla\times\overline{g},\\
&\nabla\cdot w = 0, \quad \nabla\cdot W = 0.
\end{aligned}
\tag{1.3}
$$

We consider three important flow statistics: the time averaged energy, magnetic and cross helicity dissipation rates. The energy dissipation rate is a fundamental statistic in experimental and theoretical studies of turbulence, e.g., Sreenivasan [47, 48], Bourne and Orszag [8], Pope [44], Frisch [19], Lesieur [36]. In the early 1990's Constantin and Doering [12] (see also Doering and Gibbon [14]) established a direct link between the phenomenology of energy dissipation and energy dissipation predicted for shear flows directly from the NSE. That work builds on earlier work of Busse [9], Howard [25] (and others) and has developed in many important directions, including Childress, Kerswell and Gilbert [11], Kerswell [27] and Wang [55] (shear flows) and Foias [17], Doering and Foias [13] (body force driven flows).

A good introduction to the subject of magnetohydrodynamic turbulence can be found in Biskamp [7]. A discussion of invariant quantities can be found in Oughton and Prandl [43], who explain that, in the ideal (i.e. $\nu = \nu_{\mathtt{m}} = 0$) continuum limit, (1.1) conserves an infinite number of invariants and the ideal evolution is constrained by all these quantities. However, when even a small amount of dissipation is present, only the linear and quadratic invariants still play a key role (Kraichnan [28]). This follows because the invariance of higher-order quantities requires the existence of excitation at arbitrarily small scales. However, such scales are dynamically smoothed by the dissipative terms. Similarly, since numerical simulations are inherently restricted to a finite set of length scales, they too can only conserve at most the linear and quadratic invariants completely independent of the presence or absence of numerical dissipation. For $3d$ MHD, the quadratic invariants (termed 'rugged invariants') are the total energy $E = E^v + E^B$, the magnetic helicity $H_{\mathtt{mag}} = \mathbb{A}\cdot B$ (where $\mathbb{A}$ is the vector potential of the magnetic field $B = \nabla\times\mathbb{A}$), and the cross helicity $H_\times = u\times B$ (see e.g. Frisch *et al.* [20]). Although the kinetic helicity $H_{\mathtt{kin}} = u\cdot\nabla\times u$ is a rugged invariant for $3d$ Euler flows, it is not one for MHD systems. It is nonetheless an important quantity, as noted in [43].

Note that magnetic helicity is not conserved when a mean magnetic field is present (Matthaeus and Goldstein [40]; Stribling, *et al.* [53], [54]), a feature that has recently provoked interest (Berger [3, 4]; Matthaeus, *et al.* [41]; Montgomery and Bates [42]). Indeed, the presence of a mean magnetic field introduces nonlinear interactions that fundamentally alter the spectral behavior of the fluid. The present work assumes the absence of a mean magnetic field, so these additional considerations do not arise.

*In practical computations with ADMs an additional time relaxation term, $\chi(w - \overline{w})$, has often been added to (1.4). This term can be used as a numerical regularization in any model and is studied in [35, 16], Adams and Stolz [2], Pruett [45] and Guenaff [24].

Let $\langle\cdot\rangle$ denote long time averaging (defined in Section 2). K41 phenomenology, e.g., Frisch [19], Pope [44], in [35] suggests the scaling of the energy dissipation rate for NSE $\langle\varepsilon\rangle$

$$\langle\varepsilon\rangle \sim \frac{U^3}{L}.$$

In Section 3, we prove directly from the equations of motion (1.3) that the average energy dissipation rate of the ADM MHD model and the average cross helicity dissipation rate are both bounded by the quantity

$$\frac{\sqrt{3}}{L_N}(\mathscr{U}_N^2 + S\mathscr{B}_N^2)^{3/2}\left(2 + (\frac{1}{\mathrm{Re}_N^2} + \frac{1}{\mathrm{Re}_{\mathtt{m}N}^2})(1 + \frac{\delta^2}{L_N^2})\right)^{1/2}.$$

Here $\mathscr{U}_N$, $\mathscr{B}_N$, $L_N$ denote natural velocity, magnetic field and length scales, and $\mathrm{Re}_N$ and $\mathrm{Re}_{\mathtt{m}N}$ are dimensionless values similar in concept to the Reynolds number and magnetic Reynolds number associated with the largest scales of the model (1.3), and defined precisely in Section 2.2. The average magnetic helicity dissipation rate is bounded by the quantity

$$\frac{\sqrt{3}\,L_\Omega}{\pi L_N\sqrt{S}}\mathscr{B}_N(\mathscr{U}_N^2 + S\mathscr{B}_N^2)\left(2 + (\frac{1}{\mathrm{Re}_N^2} + \frac{1}{\mathrm{Re}_{\mathtt{m}N}^2})(1 + \frac{\delta^2}{L_N^2})\right)^{1/2}.$$

Aside from the family of ADMs, we also believe that the techniques used herein (beginning with their similar kinetic energy balances) can be used to prove parallel estimates of energy and cross and magnetic helicity dissipation rates for the alpha-model [18].

**2. Notation and preliminaries.** The time average of a function $\phi(t)$ is defined by

$$\langle\phi\rangle = \limsup_{T\to\infty}\frac{1}{T}\int_0^T \phi(t)dt.$$

**2.1. Approximate Deconvolution Operators.** Typical filtering or convolution operators $G : \phi \to \overline{\phi}$ are bounded (and even smoothing, thus compact) maps: $L^2(\Omega) \to L^2(\Omega)$. If $G$ is smoothing, hence compact, the inverse of $G$ cannot be bounded. An *approximate deconvolution operator* $D_N$ is an approximate inverse $\overline{\phi} \to D_N(\overline{\phi}) \approx \phi$ which

- is a bounded operator on $L^2(\Omega)$,
- approximates $\phi$ in some useful (typically asymptotic) sense, and
- satisfies other conditions necessary for the application at hand.

The deconvolution operator we consider was studied by van Cittert in 1931, e.g., Bertero and Boccacci [6], and its use in LES pioneered by Adams, Kleiser and Stolz [1, 49, 2, 51, 50, 52]. The $N^{th}$ van Cittert approximate deconvolution operator $D_N$ is given explicitly by

$$D_N\phi = \sum_{n=0}^{N}(I - G)^n\phi.$$

Its action is calculated by $N$ steps of the following algorithm.

Algorithm 2.1 (van Cittert Approximate Deconvolution Operator). $\phi_0 = \overline{\phi}$, *for* $n = 1, 2, \ldots, N-1$, *perform*

$$\phi_{n+1} = \phi_n + \{\overline{\phi} - G\phi_n\}$$

*Define* $D_N\overline{\phi} = \phi_N$.

DEFINITION 2.2. *The deconvolution weighted inner product and norm,* $(\cdot,\cdot)_N$ *and* $\|\cdot\|_N$ *are*

$$(\phi,\psi)_N = (\phi, D_N\psi), \qquad \|\phi\|_N = (\phi,\phi)_N^{\frac{1}{2}}. \tag{2.1}$$

LEMMA 2.3. *Consider the approximate deconvolution operator*

$$D_N : L^2(\Omega) \to L^2(\Omega)$$

$D_N$ *is a bounded, self-adjoint, positive-definite operator and satisfies*

$$\|\phi\|^2 \le \|\phi\|_N^2 \le (N+1)\|\phi\|^2, \qquad \forall\phi \in L^2(\Omega). \tag{2.2}$$

*Proof.* See [31]. ☐

The deconvolution weighted scales of the body force and large scale velocity and magnetic field are defined by

$$\mathscr{F}_N = (\frac{1}{|\Omega|}\|f\|_N^2)^{\frac{1}{2}}, \qquad \mathscr{G}_N = (\frac{1}{|\Omega|}\|\nabla\times g\|_N^2)^{\frac{1}{2}},$$
$$\mathscr{U}_N = \langle\frac{1}{|\Omega|}\|w\|_N^2\rangle^{\frac{1}{2}}, \qquad \mathscr{B}_N = \langle\frac{1}{|\Omega|}\|W\|_N^2\rangle^{\frac{1}{2}}.$$

For $N = 0$, these quantities are generally written without subscripts. Note that, by (2.2) these quantities are related by

$$\mathscr{F} \le \mathscr{F}_N \le (N+1)^{\frac{1}{2}}\mathscr{F}, \qquad \mathscr{G} \le \mathscr{G}_N \le (N+1)^{\frac{1}{2}}\mathscr{G},$$
$$\mathscr{U} \le \mathscr{U}_N \le (N+1)^{\frac{1}{2}}\mathscr{U}, \qquad \mathscr{B} \le \mathscr{B}_N \le (N+1)^{\frac{1}{2}}\mathscr{B},$$

The deconvolution-weighted global length scale associated with the power input to the large scales, i.e., with $f(x), \nabla\times g(x)$, is defined to be

$$L_N = \min\Bigg\{L_\Omega, \frac{\mathscr{F}_N}{\|D_N\nabla f\|_{L^\infty(\Omega)}}, \sqrt{\frac{\mathscr{F}_N}{\|\Delta f\|_N}}, \sqrt[4]{\frac{\mathscr{F}_N}{\|\Delta^2 f\|_N}},$$
$$\frac{\mathscr{G}_N}{\|D_N\nabla\nabla\times g\|_{L^\infty(\Omega)}}, \sqrt{\frac{\mathscr{G}_N}{\|\Delta\nabla\times g\|_N}}, \sqrt[4]{\frac{\mathscr{G}_N}{\|\Delta^2\nabla\times g\|_N}}\Bigg\} \tag{2.3}$$

It is easy to check that $L_N$ has units of length and satisfies the following inequalities that will be used in the proof of lemma 4.2.

$$\|D_N\nabla f\|_\infty \le \frac{\mathscr{F}_N}{L_N} \qquad \|\Delta f\|_N \le \frac{\mathscr{F}_N}{L_N^2} \qquad \|\Delta^2 f\|_N \le \frac{\mathscr{F}_N}{L_N^4}$$
$$\|D_N\nabla(\nabla\times g)\|_\infty \le \frac{\mathscr{G}_N}{L_N} \qquad \|\Delta(\nabla\times g)))\|_N \le \frac{\mathscr{G}_N}{L_N^2} \qquad \|\Delta^2(\nabla\times g)\|_N \le \frac{\mathscr{G}_N}{L_N^4}$$

**2.2. The Reynolds and Magnetic Reynolds numbers.** With the deconvolution weighted length scales given above, we define the deconvolution weighted Reynolds number as

$$\mathrm{Re}_N := \mathscr{U}_N L_N/\nu.$$

The analysis (following) suggests the importance of an apparently new non dimensional parameter playing the role of a magnetic Reynolds number and given by

$$\mathrm{Re}_{\mathtt{m}N} := \sqrt{S}\mathscr{B}_N L_N/\nu_{\mathtt{m}}.$$

This quantity differs from the usual definition of magnetic Reynolds number as $UL/\nu_{\mathtt{m}}$ but can be seen to appear in a manner similar to the Reynolds number in several expressions (for example in Proposition 4.1 below).

From these two we form an overall parameter "R", referred to in the abstract, as the harmonic mean

$$R := \left[\frac{1}{2}\frac{1}{\mathrm{Re}_N} + \frac{1}{2}\frac{1}{\mathrm{Re}_{\mathtt{m}N}}\right]^{-1}.$$

This quantity arises naturally in the proof of Lemma 3.4 where it helps provide an upper bound for $1/\nu$ and $1/\nu_{\mathtt{m}}$ and also a lower bound for $\nu$ and $\nu_{\mathtt{m}}$.

In the following section, kinetic energy, cross helicity and magnetic helicity for the ADM are defined along with their respective dissipation rates, and balance equations are derived.

**3. Kinetic energy, cross helicity, and magnetic helicity balance.** To see the mathematical key to the estimates of energy and helicity dissipation rates we first recall from [32], [15] (see also [33], and [39] for the difficult case of no-slip boundary conditions) the energy balance for the ADM (1.3). The preservation of the Alfvèn waves by the model and the existence of unique, smooth strong solutions and the has been proven for the continuum MHD ADM in [29] and [30]. Thus, the derivations in the various balances satisfied by model solutions below proceeds with no hidden subtleties.

PROPOSITION 3.1. *If $w, W$ is a weak or strong solution*[†] *of (1.3), $w, W$ satisfies*

$$\begin{aligned}
&\frac{1}{2}\left(\|w(T)\|_N^2 + \delta^2\|\nabla w(T)\|_N^2 + S\|W(T)\|_N^2 + \delta^2 S\|\nabla W(T)\|_N^2\right) \\
&\qquad + \int_0^T \nu\left(\|\nabla w(t)\|_N^2 + \delta^2\|\Delta w(t)\|_N^2\right) + S\nu_{m}\left(\|\nabla W(t)\|_N^2 + \delta^2\|\Delta W(t)\|_N^2\right) dt \\
&= \frac{1}{2}\left(\|w(0)\|_N^2 + \delta^2\|\nabla w(0)\|_N^2 + S\|W(0)\|_N^2 + \delta^2 S\|\nabla W(0)\|_N^2\right) \\
&\qquad + \int_0^T (f, w(t))_N + S(\nabla\times g, W(t))_N dt.
\end{aligned}$$

*Proof.* Let $w, W$ denote a $L_\Omega$ periodic solution to the ADM (1.3). Multiplying the first equation in (1.3) by $AD_N w$, the second by $SAD_N W$, and integrating over $\Omega$

[†]It is known that weak = strong solution for the MHD ADM, see [29].

gives

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|w\|_N^2+\delta^2\|\nabla w\|_N^2)+(\nabla\cdot\overline{D_N w D_N w^T},AD_N w)+\nu(\|\nabla w\|_N^2+\delta^2\|\Delta w\|_N^2)\\
&\quad+(\nabla q,AD_N w)-S(\nabla\cdot(\overline{D_N W D_N W^T}),AD_N w)=(\overline{f},AD_N w),\\
&\frac{S}{2}\frac{d}{dt}(\|W\|_N^2+\delta^2\|\nabla W\|_N^2)+S\nu_{\mathtt{m}}(\|\nabla W\|_N^2+\delta^2\|\Delta W\|_N^2)\\
&\quad-S(\nabla\cdot\overline{D_N W D_N w^T},AD_N W)+S(\nabla\cdot\overline{D_N w D_N W^T},AD_N W)\\
&\quad=S(\nabla\times\overline{g},AD_N W).
\end{aligned}
\tag{3.1}
$$

The sum of nonlinear terms vanishes exactly because of the divergence free condition

$$
\begin{aligned}
&(\nabla\cdot(\overline{D_N w D_N w^T}),AD_N w)-S(\nabla\cdot(\overline{D_N W D_N W^T}),AD_N w)\\
&\quad-S(\nabla\cdot\overline{D_N W D_N w^T},AD_N W)+S(\nabla\cdot\overline{D_N w D_N W^T},AD_N W)\\
&=(\nabla\cdot(D_N w D_N w^T),D_N w)-S(\nabla\cdot(D_N W D_N W^T),D_N w)\\
&\quad-S(\nabla\cdot(D_N W D_N w^T),D_N W)+S(\nabla\cdot(D_N w D_N W^T),D_N W)=0.
\end{aligned}
$$

The result follows by summing and integrating from 0 to $T$. □

From Proposition 3.1, the ADMs kinetic energy, energy dissipation rate and power input are clearly identified.
ADM energy:

$$
\begin{aligned}
&E_{ADM-N}(w,W)(t)\\
&\quad=\frac{1}{2|\Omega|}\left(\|w(t)\|_N^2+\delta^2\|\nabla w(t)\|_N^2+S\|W(t)\|_N^2+\delta^2 S\|\nabla W(t)\|_N^2\right),
\end{aligned}
\tag{3.2}
$$

ADM dissipation rate:

$$
\begin{aligned}
&\varepsilon_{ADM-N}(w,W)(t)\\
&=\frac{\nu}{|\Omega|}\left(\|\nabla w(t)\|_N^2+\delta^2\|\Delta w(t)\|_N^2\right)+\frac{S\nu_{\mathtt{m}}}{|\Omega|}\left(\|\nabla W(t)\|_N^2+\delta^2\|\Delta W(t)\|_N^2\right),
\end{aligned}
\tag{3.3}
$$

Time averaged dissipation rate:

$$
\langle\varepsilon_{ADM-N}\rangle=\langle\varepsilon_{ADM-N}(w,W)(t)\rangle,
\tag{3.4}
$$

ADM power input:

$$
P_{ADM-N}(w,W)(t)=\frac{1}{|\Omega|}\left((f,w(t))_N+S(\nabla\times g,W(t))_N\right).
\tag{3.5}
$$

A similar approach results in balance equations for cross and magnetic helicity.

PROPOSITION 3.2. *The ADM cross helicity:*

$$
H_{\times,\mathtt{ADM}}(w,W)(t)=\frac{1}{|\Omega|}\left((w,W)_N+\delta^2(\nabla w,\nabla W)_N\right).
\tag{3.6}
$$

*satisfies the balance equation*

$$\begin{aligned} &H_{\times,\mathtt{ADM}}(w,W)(T) + \int_0^T \gamma_{\times,\mathtt{ADM}}(w,W)(t)dt \\ &\qquad = H_{\times,\mathtt{ADM}}(w_0,W_0) + \int_0^T \frac{1}{|\Omega|}\Big((f,W)_N + (\nabla\times g, w)_N\Big)dt \end{aligned} \tag{3.7}$$

*where* $\gamma_{\times,\mathtt{ADM}}(w,W)$ *is the cross helicity dissipation rate given by*

$$\gamma_{\times,\mathtt{ADM}}(w,W) = (\nu+\nu_m)\frac{1}{|\Omega|}\left((\nabla\times w,\nabla\times W)_N + \delta^2(\Delta w,\Delta W)_N\right). \tag{3.8}$$

*Proof.* Multiplying the first and second equation in (1.3) by $AD_NW$ and $AD_Nw$, respectively, integrating over $\Omega$, adding up and using the skew symmetry of the nonlinearity terms we obtain

$$\begin{aligned} &\frac{d}{dt}\left((w,W)_N + \delta^2(\nabla w,\nabla W)_N\right) + \Big(\nu+\nu_{\mathtt{m}}\Big)\left((\nabla\times w,\nabla\times W)_N + \delta^2(\Delta w,\Delta W)_N\right) \\ &\qquad = (f,W)_N + (\nabla\times g, w)_N. \end{aligned}$$

The result follows by integrating over time. ◻

PROPOSITION 3.3. *The ADM magnetic helicity:*

$$H_{\mathtt{mag},\mathtt{ADM}}(W)(t) = \frac{1}{|\Omega|}\left((\mathbb{A},W)_N + \delta^2(W,\nabla\times W)_N\right) \tag{3.9}$$

*(where* $\mathbb{A}$ *is the vector potential,* $W=\nabla\times\mathbb{A}$*) satisfies the balance equation*

$$H_{\mathtt{mag},\mathtt{ADM}}(W)(T) + \int_0^T \gamma_{\mathtt{mag},\mathtt{ADM}}(W)(t)dt = H_{\mathtt{mag},\mathtt{ADM}}(W_0) + \int_0^T \frac{1}{|\Omega|}(\nabla\times g,\mathbb{A})_N dt \tag{3.10}$$

*where* $\gamma_{\mathtt{mag},\mathtt{ADM}}(W)$ *is the ADM magnetic helicity dissipation rate given by*

$$\gamma_{\mathtt{mag},\mathtt{ADM}}(W) = \frac{\nu_m}{|\Omega|}\left((\nabla\times W,W)_N + (\nabla\times(\nabla\times W),\nabla\times W)_N\right). \tag{3.11}$$

*Proof.* Multiplying the second equation in (1.3) by $AD_N\mathbb{A}$ and integrating over $\Omega$ gives

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(W,AD_N\mathbb{A}) + \nu_{\mathtt{m}}(\nabla\times\nabla\times W, AD_N\mathbb{A}) + (\nabla\cdot(\overline{D_NWD_Nw^T}), AD_N\mathbb{A}) \\ &\qquad -\nabla\cdot(\overline{D_NwD_NW^T}), AD_N\mathbb{A}) = (\nabla\times\overline{g}, AD_N\mathbb{A}). \end{aligned} \tag{3.12}$$

Using the identity

$$((\nabla\times a)\times b, c) = (b\cdot\nabla a, c) - (c\cdot\nabla a, b),$$

the nonlinear terms become

$$\begin{aligned} &(D_Nw\cdot\nabla(D_NW), D_N\mathbb{A}) - (D_NW\cdot\nabla(D_Nw), D_N\mathbb{A}) \\ &= -(\underbrace{D_Nw}_{b}\cdot\nabla(\underbrace{D_N\mathbb{A}}_{a}), \underbrace{\nabla\times D_N\mathbb{A}}_{c}) - (D_NW\cdot\nabla(D_Nw), D_N\mathbb{A}) \\ &= -((\nabla\times D_N\mathbb{A})\times D_Nw, \nabla\times D_N\mathbb{A}) - ((\nabla\times D_N\mathbb{A})\cdot\nabla(D_N\mathbb{A}), D_Nw) \\ &\qquad - (D_NW\cdot\nabla(D_Nw), D_N\mathbb{A}) = 0, \end{aligned}$$

since the cross product of two vectors is orthogonal to each of them, and $W = \nabla \times \mathbb{A}$. Substituting this in (3.12) and integrating in time we conclude (3.10). □

The following lemma shows that $w$ and $W$ are bounded in time and justifies the definitions of $\mathscr{U}_N$ and $\mathscr{B}_N$. The lemma is a consequence of the estimates in Proposition 3.1.

LEMMA 3.4. *Let $f = f(x), \nabla \times g(x) \in L^2(\Omega)$ and $w, W$ be a solution of the ADM turbulence model (1.3) then*

$$\sup_{t\in(0,\infty)} E_{\mathtt{ADM}-N}(w,W)(t) \leq E_{\mathtt{ADM}-N}(0) + C_p^4(1/\nu + 1/\nu_{\mathtt{m}})^2(\mathscr{F}_N^2 + S\mathscr{G}_N^2) < \infty, \quad (3.13)$$

$$\begin{aligned}\frac{1}{T}\int_0^T \varepsilon_{\mathtt{ADM}-N}(w,W)(t)dt \leq& \frac{1}{T}E_{\mathtt{ADM}-N}(0) + (2E_{\mathtt{ADM}-N}(0))^{\frac{1}{2}}(\mathscr{F}_N^2 + S\mathscr{B}_N^2)^{\frac{1}{2}} + \\ &+ C_p^2(1/\nu + 1/\nu_{\mathtt{m}})(\mathscr{F}_N^2 + S\mathscr{G}_N^2) < \infty, \quad (3.14)\end{aligned}$$

*where $C_P = L_\Omega/\pi$ is the Poincaré constant.*

*Proof.* We begin with (3.1) from the proof of the Proposition 3.1. Using the Poincaré inequality we have from (3.1)

$$\frac{d}{dt}E_{\mathtt{ADM}-N}(w,W)(t) + CE_{\mathtt{ADM}-N}(w,W)(t) \leq \frac{1}{|\Omega|}\left((f,w)_N + S(\nabla\times g, W)_N\right),$$

where $C = \frac{2}{C_p^2(1/\nu+1/\nu_{\mathtt{m}})}$. Applying the Cauchy-Schwarz and Young inequalities yields a differential inequality that admits an integrating factor to confirm (3.13). For (3.14), divide the energy estimate of the ADM turbulence model energy equality from Proposition 3.1 by $T|\Omega|$:

$$\begin{aligned}&\frac{1}{T}E_{\mathtt{ADM}-N}(w,W)(T) + \frac{1}{T}\int_0^T \varepsilon_{\mathtt{ADM}-N}(w,W)(t)dt \\ &\leq \frac{1}{T}E_{\mathtt{ADM}-N}(w,W)(0) + \left(\mathscr{F}_N^2 + S\mathscr{G}_N^2\right)^{\frac{1}{2}}\left(\frac{1}{T|\Omega|}\int_0^T \|w(t)\|_N^2 + S\|W(t)\|_N^2 dt\right)^{\frac{1}{2}} \quad (3.15)\end{aligned}$$

and (3.13) can be used to estimate the integral in the second term of (3.15). □

**4. Bounds on dissipation rates.** We prove the following estimate on the model's time averaged energy, cross helicity, and magnetic helicity dissipation rates.

PROPOSITION 4.1. *With $Re_N$ and $Re_{\mathtt{m}N}$*

$$\langle \varepsilon_{\mathtt{ADM}-N}\rangle \leq \frac{\sqrt{3}}{L_N}(\mathscr{U}_N^2 + S\mathscr{B}_N^2)^{3/2}\left(2(N+1)^2 + (\frac{1}{Re_N^2} + \frac{1}{Re_{\mathtt{m}N}^2})(1 + \frac{\delta^2}{L_N^2})\right)^{1/2} \quad (4.1)$$

$$\langle \gamma_{\times,\mathtt{ADM}-N}\rangle \leq \frac{\sqrt{3}}{L_N}(\mathscr{U}_N^2 + S\mathscr{B}_N^2)^{3/2}\left(2(N+1)^2 + (\frac{1}{Re_N^2} + \frac{1}{Re_{\mathtt{m}N}^2})(1 + \frac{\delta^2}{L_N^2})\right)^{1/2} \quad (4.2)$$

$$\langle \gamma_{\mathtt{mag},\mathtt{ADM}-N}\rangle \leq \frac{\sqrt{3}\,L_\Omega}{\pi L_N\sqrt{S}}\mathscr{B}_N(\mathscr{U}_N^2 + S\mathscr{B}_N^2)\left(2 + (\frac{1}{Re_N^2} + \frac{1}{Re_{\mathtt{m}N}^2})(1 + \frac{\delta^2}{L_N^2})\right)^{1/2} \quad (4.3)$$

This proposition is a consequence of the following lemma.

Lemma 4.2.

$$\mathscr{F}_N^2 + S\mathscr{G}_N^2 \leq \frac{3}{L_N^2}(\mathscr{U}_N^2 + S\mathscr{B}_N^2)^2\left(2(N+1)^2 + (\frac{1}{Re_N^2} + \frac{1}{Re_{\mathtt{m}N}^2})(1+\frac{\delta^2}{L_N^2})^2\right).$$

*Proof of Lemma 4.2*. Assume $w$ and $W$ are solutions to (1.3) so that their long time averages satisfy the following equations.

$$\begin{aligned}
&\langle w_t\rangle + \nabla\cdot\langle\overline{D_N w D_N w^T}\rangle - \nu\Delta\langle w\rangle - S\nabla\cdot\langle\overline{D_n W\, D_N W^T}\rangle + \nabla\langle q\rangle = \langle\overline{f}\rangle,\\
&\langle W_t\rangle + \nu_{\mathtt{m}}\nabla\times(\nabla\times\langle W\rangle) + \nabla\cdot\langle\overline{D_N W D_N w^T}\rangle - \nabla\cdot\langle\overline{D_N w D_N W^T}\rangle = \nabla\times\langle\overline{g}\rangle.
\end{aligned}\tag{4.4}$$

Take the inner product of the time averaged model (4.4) with $D_N Af$ and $D_N A\nabla\times g$. Note that $\phi = \phi(x)$, $(\overline{\phi}, A\phi) = (A^{-1}\phi, A\phi) = \|\phi\|^2$, analogously for the nonlinear term, and that $\nabla\cdot f = 0$ so the pressure term vanishes. This gives

$$\begin{aligned}
\frac{1}{|\Omega|}\|f\|^2 &= \frac{1}{|\Omega|}(Af, \langle w_t\rangle)_N\\
&\quad - \frac{1}{|\Omega|}(\nabla f, \langle D_N w\, D_N w^T\rangle)_N - \frac{\nu}{|\Omega|}(Af, \Delta\langle w\rangle)_N + \frac{S}{|\Omega|}(\nabla f, \langle D_N W\, D_N W^T\rangle)_N,\\
\frac{1}{|\Omega|}\|\nabla\times g\|^2 &= \frac{1}{|\Omega|}(A\nabla\times g, \langle W_t\rangle)_N + \frac{\nu_{\mathtt{m}}}{|\Omega|}(A\nabla\times(\nabla\times g), \nabla\times\langle W\rangle)_N\\
&\quad - \frac{1}{|\Omega|}(\nabla\nabla\times g, \langle D_N W\, D_N w^T\rangle)_N + \frac{1}{|\Omega|}(\nabla\nabla\times g, \langle D_N\, w D_N W^T\rangle)_N.
\end{aligned}$$

The time derivative terms vanish in the limit as $T\to\infty$ by the Cauchy-Schwarz inequality and Corollary 3.4. The remaining terms on the RHS are integrated by parts (as in the derivation of the energy equality):

$$\begin{aligned}
\frac{1}{|\Omega|}\|f\|^2 &= -\frac{1}{|\Omega|}(\nabla f, \langle D_N w\, D_N w^T\rangle)_N\\
&\quad + \frac{\nu}{|\Omega|}\left[(\nabla f, \nabla\langle w\rangle)_N + \delta^2(\Delta f, \Delta\langle w\rangle)_N\right] + \frac{S}{|\Omega|}(\nabla f, \langle D_N W\, D_N W^T\rangle)_N,\\
\frac{1}{|\Omega|}\|\nabla\times g\|^2 &= \frac{\nu_{\mathtt{m}}}{|\Omega|}\left[(\nabla\times(\nabla\times g), \nabla\times\langle W\rangle)_N + \delta^2(\Delta(\nabla\times g), \Delta\langle W\rangle)_N\right]\\
&\quad - \frac{1}{|\Omega|}(\nabla(\nabla\times g), \langle D_N W\, D_N w^T\rangle)_N + \frac{1}{|\Omega|}(\nabla(\nabla\times g), \langle D_N\, w D_N W^T\rangle)_N.
\end{aligned}$$

Adding the first and $S$ times the second yields

$$\begin{aligned}
\frac{1}{|\Omega|}\|f\|_N^2 + \frac{S}{|\Omega|}\|\nabla\times g\|_N^2 &= -\frac{1}{|\Omega|}(\nabla f, \langle D_N w\, D_N w^T\rangle)_N + \frac{S}{|\Omega|}(\nabla f, \langle D_N W\, D_N W^T\rangle)_N\\
&\quad - \frac{S}{|\Omega|}(\nabla(\nabla\times g), \langle D_N w\, D_N W^T\rangle)_N + \frac{S}{|\Omega|}(\nabla(\nabla\times g), \langle D_N W\, D_N w^T\rangle)_N\\
&\quad + \frac{\nu}{|\Omega|}(\nabla f, \nabla\langle w\rangle)_N + \frac{\nu}{|\Omega|}\delta^2(\Delta f, \Delta\langle w\rangle)_N\\
&\quad + \frac{\nu_{\mathtt{m}}S}{|\Omega|}\langle\nabla\times(\nabla\times g), \nabla\times\langle W\rangle)_N + \frac{\nu_{\mathtt{m}}S}{|\Omega|}\delta^2(\Delta\nabla\times g, \Delta\langle W\rangle)_N.
\end{aligned}$$

Integrating by parts so that no derivatives of $w$ or $W$ appear yields

$$\frac{1}{|\Omega|}\|f\|_N^2+\frac{S}{|\Omega|}\|\nabla\times g\|_N^2=-\frac{1}{|\Omega|}(\nabla f,\langle D_N w\, D_N w^T\rangle)_N+\frac{S}{|\Omega|}(\nabla f,\langle D_N W\, D_N W^T\rangle)_N$$
$$-\frac{S}{|\Omega|}(\nabla(\nabla\times g),\langle D_N w\, D_N W^T\rangle)_N+\frac{S}{|\Omega|}(\nabla(\nabla\times g),\langle D_N W\, D_N w^T\rangle)_N$$
$$-\frac{\nu}{|\Omega|}(\Delta f,\langle w\rangle)_N+\frac{\nu}{|\Omega|}\delta^2(\Delta^2 f,\langle w\rangle)_N$$
$$+\frac{\nu_{\mathtt{m}} S}{|\Omega|}(\nabla\times(\nabla\times(\nabla\times g)),\langle W\rangle)_N+\frac{\nu_{\mathtt{m}} S}{|\Omega|}\delta^2(\Delta^2\nabla\times g,\langle W\rangle)_N.$$

Next, consider the nonlinear terms on the above RHS. By the definitions of $L_N$, $\mathscr{F}_N$, $\mathscr{G}_N$, $\mathscr{U}_N$ and $\mathscr{B}_N$, and the norm equivalence (2.2) we have

$$\begin{aligned}
&\tfrac{1}{|\Omega|}(\nabla f,\langle D_N w\, D_N w^T\rangle)_N\le\|D_N\nabla f\|_{L^\infty}\langle\tfrac{1}{|\Omega|}\|(D_N w)^2\|\rangle\le\tfrac{\mathscr{F}_N}{L_N}\langle\tfrac{1}{|\Omega|}\|D_N w\|^2\rangle\\
&\qquad=\tfrac{\mathscr{F}_N}{L_N}\langle\tfrac{1}{|\Omega|}\|D_N^{\frac12}w\|_N^2\rangle\le\tfrac{(N+1)\mathscr{F}_N}{L_N}\langle\tfrac{1}{|\Omega|}\|D_N^{\frac12}w\|^2\rangle=\tfrac{(N+1)\mathscr{F}_N\mathscr{U}_N^2}{L_N},\\
&\tfrac{S}{|\Omega|}(\nabla f,\langle D_N W\, D_N W^T\rangle)_N\le\|D_N\nabla f\|_{L^\infty}\langle\tfrac{S}{|\Omega|}\|(D_N W)^2\|_N\rangle\le S\tfrac{(N+1)\mathscr{F}_N\mathscr{B}_N^2}{L_N},\\
&\tfrac{S}{|\Omega|}(\nabla(\nabla\times g),\langle D_N w\, D_N W^T\rangle)_N\le S\|D_N\nabla(\nabla\times g)\|_{L^\infty}\langle\tfrac{1}{|\Omega|}\|w\|\,\|W\|\rangle\\
&\qquad\le S\tfrac{(N+1)\mathscr{G}_N\mathscr{U}_N\mathscr{B}_N}{L_N}.
\end{aligned}\tag{4.5}$$

Similarly estimating the linear terms and using (2.3) yields

$$\mathscr{F}_N^2+S\mathscr{G}_N^2\le\frac{1}{L_N}\mathscr{F}_N\mathscr{U}_N^2+\frac{S}{L_N}\mathscr{F}_N\mathscr{B}_N^2+\frac{2S}{L_N}\mathscr{G}_N\mathscr{U}_N\mathscr{B}_N$$
$$+\frac{\nu}{L_N^2}\mathscr{F}_N\mathscr{U}_N+\frac{\nu\delta^2}{L_N^4}\mathscr{F}_N\mathscr{U}_N+\frac{\nu_{\mathtt{m}}S}{L_N^2}\mathscr{G}_N\mathscr{B}_N+\frac{\nu_{\mathtt{m}}S\delta^2}{L_N^4}\mathscr{G}_N\mathscr{B}_N.\tag{4.6}$$

At this point, it is convenient to employ the quantities introduced in Section 2.2. The deconvolution weighted Reynolds number $\mathrm{Re}_N=\mathscr{U}_N L_N/\nu$, and a quantity similar to, but not the same as, the magnetic Reynolds number, $\mathrm{Re}_{\mathtt{m}N}=\sqrt{S}\mathscr{B}_N L_N/\nu_{\mathtt{m}}$. With these quantities, (4.6) becomes

$$\mathscr{F}_N^2+S\mathscr{G}_N^2\le\frac{1}{L_N}\mathscr{F}_N\mathscr{U}_N^2+\frac{S}{L_N}\mathscr{F}_N\mathscr{B}_N^2+\frac{2S}{L_N}\mathscr{G}_N\mathscr{U}_N\mathscr{B}_N$$
$$+\frac{\mathscr{F}_N\mathscr{U}_N^2}{\mathrm{Re}_N L_N}(1+\frac{\delta^2}{L_N^2})+\frac{\sqrt{S}\mathscr{G}_N S\mathscr{B}_N^2}{\mathrm{Re}_{\mathtt{m}N}L_N}(1+\frac{\delta^2}{L_N^2}).$$

Next, applying Young's inequality in the form $ab\le a^2/6+3b^2/2$ to each of the terms yields the following

$$\mathscr{F}_N^2+S\mathscr{G}_N^2\le\frac{1}{6}\mathscr{F}_N^2+\frac{3(N+1)^2}{2L_N^2}\mathscr{U}_N^4+\frac{1}{6}\mathscr{F}_N^2+\frac{3S^2(N+1)^2}{2L_N^2}\mathscr{B}_N^4$$
$$+\frac{S}{6}\mathscr{G}_N^2+\frac{6(N+1)^2S}{L_N^2}\mathscr{U}_N^2\mathscr{B}_N^2$$
$$+\frac{1}{6}\mathscr{F}_N^2+\frac{3}{2\mathrm{Re}_N^2L_N^2}\mathscr{U}_N^4(1+\frac{\delta^2}{L_N^2})^2+\frac{S}{6}\mathscr{G}_N^2+\frac{3S^2}{2\mathrm{Re}_{\mathtt{m}N}^2L_N^2}\mathscr{B}_N^4(1+\frac{\delta^2}{L_N^2})^2.$$

Collecting terms and applying the Cauchy-Schwarz inequality gives

$$\begin{aligned}
&\mathscr{F}_N^2 + S\mathscr{G}_N^2 \\
&\quad \le \frac{3}{L_N^2}\left((N+1)^2(\mathscr{U}_N^4 + S^2\mathscr{B}_N^4 + 4S\mathscr{U}_N^2\mathscr{B}_N^2) + (1+\frac{\delta^2}{L_N^2})^2(\frac{\mathscr{U}_N^4}{\mathrm{Re}_N^2} + \frac{S^2\mathscr{B}_N^4}{\mathrm{Re}_{\mathtt{m}N}^2})\right) \\
&\quad \le \frac{3}{L_N^2}\left((N+1)^2(\mathscr{U}_N^2 + S\mathscr{B}_N^2)^2 + (1+\frac{\delta^2}{L_N^2})^2\sqrt{\frac{1}{\mathrm{Re}_N^4} + \frac{1}{\mathrm{Re}_{\mathtt{m}N}^4}}\sqrt{\mathscr{U}_N^8 + S^4\mathscr{B}_N^8}\right).
\end{aligned}$$

With further simplification, the estimate becomes

$$\mathscr{F}_N^2 + S\mathscr{G}_N^2 \le \frac{3}{L_N^2}(\mathscr{U}_N^2 + S\mathscr{B}_N^2)^2\left(2(N+1)^2 + (\frac{1}{\mathrm{Re}_N^2} + \frac{1}{\mathrm{Re}_{\mathtt{m}N}^2})(1+\frac{\delta^2}{L_N^2})^2\right),$$

and the lemma is proved. □

*Proof of Proposition 4.1.* Consider first the energy balance expression (3.15) in Corollary 3.4. Letting $T \to \infty$ gives

$$\langle \varepsilon_{\mathtt{ADM}-N}\rangle^2 \le (\mathscr{F}_N^2 + S\mathscr{G}_N^2)(\mathscr{U}_N^2 + S\mathscr{B}_N^2).$$

Putting the estimate from Lemma 4.2 proves (4.1).

$$\langle \varepsilon_{\mathtt{ADM}-N}\rangle^2 \le \frac{3}{L_N^2}(\mathscr{U}_N^2 + S\mathscr{B}_N^2)^3\left(2(N+1)^2 + (\frac{1}{\mathrm{Re}_N^2} + \frac{1}{\mathrm{Re}_{\mathtt{m}N}^2})(1+\frac{\delta^2}{L_N^2})\right).$$

The definition of cross helicity (3.6) in Proposition 3.2 shows that $H_{\times,\mathtt{ADM}}(w,W)$ is bounded in time, so that (3.7) shows that

$$\langle \gamma_{\times,\mathtt{ADM}}\rangle^2 \le (\mathscr{F}_N^2 + S\mathscr{G}_N^2)(\mathscr{U}_N^2 + S\mathscr{B}_N^2),$$

the same as $\langle \varepsilon_{\mathtt{ADM}-N}\rangle$, proving (4.2).

Finally, the definition of magnetic helicity (3.9) in Proposition 3.3, along with the Poincaré inequality‡ shows that $H_{\mathtt{mag},\mathtt{ADM}}(W)$ is bounded in time. Thus, (3.10) shows that

$$\langle \gamma_{\mathtt{mag},\mathtt{ADM}}(W)\rangle^2 \le \frac{L_\Omega^2}{\pi^2}\mathscr{G}_N^2\mathscr{B}_N^2 \le \frac{L_\Omega^2}{\pi^2 S}(\mathscr{F}_N^2 + S\mathscr{G}_N^2)\mathscr{B}_N^2,$$

proving (4.3). □

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‡The Poincaré inequality is being used in the form $\|g\|_N \le (L_\Omega/\pi)\|\nabla\times g\|_N$, where $g$ is periodic, has trivial average and is divergence-free.

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