THE VELOCITY AND MAGNETIC FIELD TRACKING PROBLEM FOR MHD FLOWS WITH DISTRIBUTED CONTROLS

M. GUNZBURGER,* J. PETERSON,* AND C. TRENCHEA†

Abstract. We consider the mathematical formulation and the analysis of an optimal control problem associated with the tracking of the velocity and the magnetic field of a viscous, incompressible, electrically conducting fluid in a bounded two-dimensional domain through the adjustment of distributed controls. Existence of optimal solutions is proved and first-order necessary conditions for optimality are used to derive an optimality system of partial differential equations whose solutions provide optimal states and controls. Semidiscrete-in-time and fully discrete space-time approximations are defined and their convergence to the exact optimal solutions is shown. The results of some computational experiments are provided.

Key words. Optimal control, magnetohydrodynamics, Navier-Stokes, Maxwell equations, Gâteaux derivatives.

1. Introduction. Control methods for fluid dynamics have attracted substantial interest in recent years due to their wide range of applications in engineering and science. In the literature, examples are found related to magnetic control of plasma current in tokamak, combustion, chemical reacting flows, design problems, reduction of turbulence, controllability, drag reduction; see, e.g. [3, 19, 21, 10, 8, 32]. A remarkable amount of attention was dedicated to both experimental and theoretical investigations of electromagnetic force on flows; see, e.g. [4, 6, 11, 15, 22, 23, 24, 27, 30, 31]. However there is little work on optimal control and feedback control methods for magnetohydrodynamics except in the works [9, 13, 14, 16, 17, 18, 25, 27]. The majority of these papers treat the stationary case.

In this paper we study an optimal control problem for unsteady viscous, incompressible, electrically conducting fluids. The controls applied are distributed force and current, and the goal is to match the controlled velocity and magnetic fields to some given fields. The controls and the states are constrained to satisfy a coupled system of partial differential equations consisting of the Navier-Stokes system and Maxwell's equations.

The mathematical description of the control problem proceeds as follows. Let Ω be a bounded, connected open domain with a boundary Γ. We assume Ω is of class $C^{1,1}$ or a convex polyhedron. Let u denote the velocity field, p the pressure field, and B the magnetic field, respectively. Denote by φ the velocity control and by $\operatorname{curl}\psi$ the magnetic field control. Also let ν denote the outward unit normal vector on Γ. For given T, the cost functional is defined by:

$$\mathcal{J}(u, B, \varphi, \operatorname{curl}\psi) = \int_0^T \int_\Omega \Big(\frac{\alpha_1}{2}|u-u_d|^2 + \frac{\alpha_2 S}{2}|B-B_d|^2 + \frac{\beta_1}{2}|\varphi|^2 + \frac{\beta_2 S}{2}|\operatorname{curl}\psi|^2 \Big)\, dx\, dt \tag{1.1}$$

where u_d is some desired velocity field, B_d some desired magnetic field.

We wish to minimize (1.1) subject to the constraints which are the MHD equa-

*Department of Scientific Computing, Florida State University, Tallahassee, FL 32306, USA
†Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA

tions:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \frac{1}{\mathrm{Re}}\Delta u - S \,\mathrm{curl}\, B \times B + \nabla p = \varphi \quad \text{in } \Omega \times (0,T), \tag{1.2a}$$

$$\frac{\partial B}{\partial t} + \frac{1}{\mathrm{Re_m}}\mathrm{curl}(\mathrm{curl}\, B) - \mathrm{curl}\,(u \times B) = \mathrm{curl}\,\psi \quad \text{in } \Omega \times (0,T), \tag{1.2b}$$

$$\mathrm{div}\, u = 0, \quad \text{in } \Omega \times (0,T), \tag{1.2c}$$

$$\mathrm{div}\, B = 0, \quad \text{in } \Omega \times (0,T), \tag{1.2d}$$

and the boundary conditions:

$$u = 0 \quad \text{on } \Gamma \times (0,T), \tag{1.3a}$$

$$B \cdot \nu = 0 \quad \text{and } \mathrm{curl}\, B \times \nu = 0 \quad \text{on } \Gamma \times (0,T), \tag{1.3b}$$

and the initial data:

$$u = u_0, \quad B = B_0 \quad \text{on } \Omega. \tag{1.4}$$

Here Re, $\mathrm{Re_m}$, and S are nondimensional constants that characterize the flow: the Reynolds number, the magnetic Reynolds number and the coupling number, respectively. For derivation of (1.2a)-(1.4), physical interpretation and mathematical analysis, see [20, 26].

In Section 2 we will treat the continuous optimal control problem. In Section 3 and Section 4 we will analyze semidiscrete and fully discrete approximations, respectively. In Section 5 a gradient method for the solution of the fully discrete optimal control problem is presented and analyzed. Finally, in Section 6, the results of some computational experiments are presented.

2. The optimal control problem.

2.1. Notation and formulation of the optimal control problem. We shall use the standard notations for the Sobolev spaces $H^m(\Omega)$ with inner products and norms denoted by $(\cdot,\cdot)_m$ and $\|\cdot\|_m$, respectively. $H^m(\Omega)$ denotes the vector-valued ($\mathbb{R}^2$-valued) counterparts whose inner products and norms will also be denoted by $(\cdot,\cdot)_m$ and $\|\cdot\|_m$ respectively. In particular, for $m = 1$, $H^1_0(\Omega)$ denotes the subspace of vector functions in $H^1(\Omega)$ which vanish on Γ, whereas in $H^1_\nu(\Omega)$ only the normal component of the vector field is assumed to vanish along the boundary. For details concerning these spaces, see [2, 7].
We define the subspace

$$L^2_0(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q \, d\Omega = 0 \right\},$$

equipped with the $L^2(\Omega)$ inner product. We introduce the solenoidal spaces

$$H = \{u \in L^2(\Omega);\ \mathrm{div}\ u = 0 \text{ and } u \cdot \nu = 0\},$$
$$V = (H \times H) \cap (H^1_0(\Omega) \times H^1_\nu(\Omega)).$$

The boundary condition $\mathrm{curl}\, B \times \nu = 0$ is enforced weakly through the variational equation and is not imposed on the definition of V.

On $H^1_\nu(\Omega)$ we have that $(\|\nabla \times (\cdot)\|_0^2 + \|\nabla \cdot (\cdot)\|_0^2)^{1/2}$ is a equivalent norm to $\|\cdot\|_1$ (see [7, 11]), and also we recall the Poincaré inequalities

$$\|u\|_0 \le \lambda_1^{-1/2} \|\nabla u\|_0 \quad \text{and} \quad \|B\|_0 \le \lambda_2^{-1/2} \|\mathrm{curl}\, B\|_0. \tag{2.1}$$

In order to define a weak form of the MHD equations we introduce the following linear forms. For $(u_i, B_i) \in H_0^1(\Omega) \times H_\nu^1(\Omega)$, $q \in L_0^2(\Omega)$

$$a((u_1, B_1), (u_2, B_2)) = \frac{1}{\mathrm{Re}} \int_\Omega \nabla u_1 : \nabla u_2 \, dx + \frac{S}{\mathrm{Re_m}} \int_\Omega \mathrm{curl}\, B_1 \mathrm{curl}\, B_2 \, dx, \tag{2.2a}$$

$$b((u_1, B_1), q) = -\int_\Omega q \nabla \cdot u_1 \, dx, \tag{2.2b}$$

$$\begin{aligned} c((u_1, B_1), (u_2, B_2), (u_3, B_3)) &= \int_\Omega (u_1 \cdot \nabla) u_2 \cdot u_3 \, dx \\ &- S \int_\Omega \mathrm{curl}\, B_2 \times B_1 \cdot u_3 \, dx - S \int_\Omega \left[(u_1 \cdot \nabla) B_2 \cdot B_3 - (B_1 \cdot \nabla) u_2 \cdot B_3 \right] dx. \end{aligned} \tag{2.2c}$$

The following properties of the trilinear form c hold (see [26])

$$\begin{aligned} &c((u_1, B_1), (u_2, B_2), (u_3, B_3)) \\ &\quad = -c((u_1, B_1), (u_3, B_3), (u_2, B_2)) \quad \forall (u_1, B_1) \in H, \forall (u_2, B_2), (u_3, B_3) \in V, \end{aligned} \tag{2.3a}$$

$$\begin{aligned} &|c((u_1, B_1), (u_2, B_2), (u_3, B_3))| \\ &\quad \le K_0 \|(u_1, B_1)\|_{1/2} \|(u_2, B_2)\|_1 \|(u_3, B_3)\|_{1/2} \quad \forall (u_i, B_i) \in V. \end{aligned} \tag{2.3b}$$

We introduce the diagonal matrix $\mathcal{S} \in \mathcal{M}_4(\mathbb{R}^2)$, defined by

$$m_{ii} = 1 \text{ if } 1 \le i \le 2, \qquad m_{ii} = S \text{ if } 3 \le i \le 4.$$

Finally we recall the identity

$$\int_\Omega B \times u \cdot \mathrm{curl}\, C \, dx = \int_\Omega \mathrm{curl}\, (B \times u) \cdot C \, dx = \int_\Omega \left[(u \cdot \nabla B) \cdot C - (B \cdot \nabla u) \cdot C \right] dx.$$

A weak formulation of the MHD problem is defined as follows: seek $(u, B, p) \in L^2(0, T; H_0^1(\Omega)) \times L^2(0, T; H_\nu^1(\Omega)) \times L^2(0, T; L_0^2(\Omega))$ satisfying

$$\begin{aligned} &\langle \frac{d}{dt}(u, B), \mathcal{S}(v, C) \rangle + a((u, B), (v, C)) + c((u, B), (u, B), (v, C)) + b((v, C), p) \\ &\qquad = \langle (\varphi, \mathrm{curl}\, \psi), \mathcal{S}(v, C) \rangle \quad \forall (v, C) \in H_0^1(\Omega) \times H_\nu^1(\Omega), \end{aligned} \tag{2.4a}$$

$$b((u, B), q) = 0 \quad \forall q \in L_0^2(\Omega), \tag{2.4b}$$

$$(u, B)(0, \cdot) = (u_0, B_0)(\cdot), \tag{2.4c}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. The homogeneous boundary conditions for the velocity and the zero-mean condition for the pressure are satisfied due to the choice of spaces where solutions are sought.

The set of all possible target velocities and magnetic fields $L^\infty(0, T; L^2(\Omega))$ is denoted $\mathcal{U}_{ad}$. There are no particular requirements on the target velocities u_d and magnetic fields B_d other than the fact that the cost functional must be bounded.The target needs not to be a solution to the MHD system. In particular, non solenoidal fields that satisfy boundary and initial conditions different from those in (1.3a-1.3b) and (1.4) can be used as desired target velocities and magnetic fields.

Given Ω, T, $(u_0, B_0) \in H_0^1(\Omega) \times H_\nu^1(\Omega)$, and (u_d, B_d) the set of all admissible solutions is defined by

$$\begin{aligned} \mathcal{A}_{ad} = \{ (u, B, \varphi, \mathrm{curl}\, \psi) &\in L^2(0, T; H_0^1(\Omega)) \times L^2(0, T; H_\nu^1(\Omega)) \times \left(L^2(0, T; L^2(\Omega)) \right)^2, \\ &\text{such that } \mathcal{J}(u, B, \varphi, \mathrm{curl}\, \psi) < \infty, \text{and} \\ &\exists\, p \in L^2(0, T; L_0^2(\Omega)) \text{ such that (2.4a)-(2.4c) are satisfied} \}. \end{aligned}$$

With this notation, the formulation of the optimal control problem is given by

given $\Omega, T, (u_0, B_0) \in V$, *and* $(u_d, B_d) \in \mathcal{U}_{ad}$,

find $(\hat{u}, \hat{B}, \hat{\varphi}, \operatorname{curl}\hat{\psi}) \in \mathcal{A}_{ad}$ *such that the functional* (1.1) *is minimized.*

2.2. The existence of an optimal solution. We recall that if Γ is Lipschitz continuous, $(\varphi, \operatorname{curl}\psi) \in L^2(0,T;V')$, and $(u_0, B_0) \in H$, then the unique solution to (2.4a)-(2.4c) satisfies $(u, B) \in C([0,T];H) \cap L^2(0,T;V)$ and $(u_t, B_t) \in L^2(0,T;V')$. If Γ is C^2, $(u_0, B_0) \in V$ and $(\varphi, \operatorname{curl}\psi) \in L^2(0,T;H)$, then $(u, B) \in C([0,T],V) \cap L^2(0,T;H^2(\Omega) \cap V)$. See e.g. [26].

THEOREM 2.1. *Given* $T > 0$, $(u_0, B_0) \in V$, *and* $(u_d, B_d) \in \mathcal{U}_{ad}$, *then there exists a solution* $(\hat{\varphi}, \mathit{curl}\hat{\psi}) \in L^2(0,T;H)$ *and* $(\hat{u}, \hat{B}) \in C([0,T];H) \cap L^2(0,T;V)$ *of the optimal control problem.*

Proof. The admissible set $\mathcal{A}_{ad}$ is bounded and nonempty, e.g., $(u, B, 0, 0) \in \mathcal{A}_{ad}$. Let $\{(\varphi_n, \operatorname{curl}\psi_n)\}$ be a minimizing sequence for the optimal control problem and denote $(u_n, B_n, p_n) = (u(\varphi_n, \operatorname{curl}\psi_n),\ B(\varphi_n, \operatorname{curl}\psi_n), p(\varphi_n, \operatorname{curl}\psi_n))$. The sequence $\{(\varphi_n, \operatorname{curl}\psi_n)\}$ is bounded in $L^2(0,T;H)$ and the corresponding solution (u_n, B_n) is bounded in $C([0,T];H) \cap L^2(0,T;V)$. This follows from the existence and uniqueness theorems of the two-dimensional unsteady MHD equations, see [26]. Also we have that the corresponding sequence p_n in bounded in $L^2(0,T;L^2_0(\Omega))$. Therefore on a subsequence, again denoted n, we have

$$\begin{array}{ll} (\varphi_n, \operatorname{curl}\psi_n) \to (\hat{\varphi}, \operatorname{curl}\hat{\psi}) & \text{weakly in } L^2(0,T;H), \\ (u_n, B_n) \to (\hat{u}, \hat{B}) & \text{weakly in } L^2(0,T;V), \\ (u_n, B_n) \to (\hat{u}, \hat{B}) & \text{weak-* in } L^\infty(0,T;H), \\ p_n \to \hat{p} & \text{weakly in } L^2(0,T;L^2_0(\Omega)). \end{array}$$

Now $(\hat{u}, \hat{B}, \hat{p}, \hat{\varphi}, \hat{\psi})$ satisfies the MHD equations (2.4a)-(2.4c) and minimizes the functional (1.1). Indeed, by the lower semicontinuity of the norms we get

$$\mathcal{J}(\hat{u}, \hat{B}, \hat{\varphi}, \hat{\psi}) \leq \liminf_{n\to\infty} \mathcal{J}(u_n, B_n, \varphi_n, \operatorname{curl}\psi_n).$$

A priori estimates in a fractional time-order Sobolev space yields that $\{(u_n, B_n)\}$ converges strongly to $(\hat{u}, \hat{B})$ in $L^2(0,T;H)$, (see [28]). Let $\chi(t)$ be a continuously differentiable function on $(0,T)$ such that $\chi(T) = 0$. We consider the weak MHD equations (2.4a)-(2.4c) with state (u_n, B_n, p_n) and control $(\varphi_n, \operatorname{curl}\psi_n)$. Then multiply it by $\chi(t)$ and integrate by parts with respect to t to obtain

$$\begin{aligned} &-\int_0^T \langle (u_n, B_n), \chi'(t)\mathcal{S}(v, C)\rangle dt + \int_0^T a((u_n, B)_n, \chi(t)(v, C))dt \\ &+ \int_0^T c((u_n, B_n), (u_n, B_n), \chi(t)(v, C))dt + \int_0^T b(\chi(t)(v, C), p_n)dt \\ &= ((u_0, B_0), \chi(0)\mathcal{S}(v, C)) + \int_0^T ((\varphi_n, \operatorname{curl}\psi_n), \chi(t)\mathcal{S}(v, C))dt. \end{aligned}$$

Now we can pass to the limit inside the linear and nonlinear terms. Since (u_n, B_n) converges to $(\hat{u}, \hat{B})$ in $L^2(0,T;V)$ weakly and $L^2(0,T;H)$ strongly, then

$$\lim_{n\to\infty} \int_0^T c((u_n, B_n), (u_n, B_n), (v(t), C(t)))\, dt = \int_0^T c((\hat{u}, \hat{B}), (\hat{u}, \hat{B}), (v(t), C(t)))\, dt$$

for all $(v, C) \in L^2(0,T;V)$, and $(\hat{u}, \hat{B}, \hat{p})$ satisfies the MHD equations in the weak sense. ∎

2.3. First-order necessary condition. We shall show that the optimal solution must satisfy the first-order necessary condition associated with the optimal control problem. By studying the case in which the Gâteaux derivative of the cost functional vanishes, we get a possible candidate solution for the optimal control, (see [29]). First let prove the existence of the Gâteaux derivative.

THEOREM 2.2. *Let* $(u_0, B_0) \in V$. *The mapping* $(\widetilde{\varphi}, \operatorname{curl} \widetilde{\psi}) \mapsto ((u,B))(\widetilde{\varphi}, \operatorname{curl} \widetilde{\psi})$ *from* $L^2(0,T;L^2(\Omega))^2$ *to* $L^2(0,T;V)$ *has a Gâteaux derivative* $\Big(d(u,B)/d(\widetilde{\varphi}, \operatorname{curl} \widetilde{\psi})\Big) \cdot (\varphi, \operatorname{curl} \psi)$ *for every* $(\varphi, \operatorname{curl} \psi) \in L^2(0,T;L^2(\Omega))^2$. *Moreover,* $((\breve{u}, \breve{B}))(\varphi, \operatorname{curl} \psi) = \Big(d(u,B)/d(\widetilde{\varphi}, \operatorname{curl} \widetilde{\psi})\Big) \cdot (\varphi, \operatorname{curl} \psi)$ *is the solution of the linear problem*

$$\begin{aligned} &\langle \frac{d}{dt}(\breve{u}, \breve{B}), \mathcal{S}(v,C)\rangle + a((\breve{u}, \breve{B}), (v,C)) + c((\breve{u}, \breve{B}), (u,B), (v,C)) \\ &\qquad + c((u,B), (\breve{u}, \breve{B}), (v,C)) + b((v,C), \check{r}) = \langle (\varphi, \operatorname{curl} \psi), \mathcal{S}(v,C)\rangle \\ &\qquad\qquad\qquad\qquad \forall (v,C) \in H_0^1(\Omega) \times H_\nu^1(\Omega), \end{aligned} \tag{2.5a}$$

$$b((\breve{u}, \breve{B}), q) = 0 \quad \forall q \in L_0^2(\Omega), \tag{2.5b}$$

$$(\breve{u}, \breve{B})(0, \cdot) = (0,0), \tag{2.5c}$$

where $\check{r} \in L^2(0,T;L_0^2(\Omega))$, *and* $(\breve{u}, \breve{B}) \in L^\infty(0,T;H) \cap L^2(0,T;V)$.

Proof. Let $(\varphi, \operatorname{curl} \psi)$ and $(\widetilde{\varphi}, \operatorname{curl} \widetilde{\psi})$ be given in $L^2(0,T;L^2(\Omega))$ and let (u,B,p) and $(u_\lambda, B_\lambda, p_\lambda)$ denote the solutions of (2.4a)-(2.4c) with the controls $(\widetilde{\varphi}, \operatorname{curl} \widetilde{\psi})$ and $(\widetilde{\varphi}, \operatorname{curl} \widetilde{\psi}) + \lambda(\varphi, \operatorname{curl} \psi)$, respectively. Hence it is sufficient to prove that

$$\lim_{\lambda \to 0} \frac{\|(u_\lambda, B_\lambda) - (u,B) - \lambda((\breve{u}, \breve{B}))(\varphi, \operatorname{curl} \psi)\|_{L^2(0,T;V)}}{\lambda} = 0.$$

We denote $(\breve{u}, \breve{B}, \hat{p}) = (u_\lambda, B_\lambda, p_\lambda) - (u,B,p) - \lambda((\breve{u}, \breve{B}, \check{r}))(\varphi, \operatorname{curl} \psi)$, which then satisfies the equation

$$\begin{aligned} &\langle \frac{d}{dt}(\breve{u}, \breve{B}), \mathcal{S}(v,C)\rangle + a((\breve{u}, \breve{B}), (v,C)) + c((\breve{u}, \breve{B}), (u,B), (v,C)) \\ &\quad + c((u,B), (\breve{u}, \breve{B}), (v,C)) + b((v,C), \hat{p}) \\ &\quad = c((u_\lambda, B_\lambda) - (u,B), (u,B) - (u_\lambda, B_\lambda), (v,C)) \quad \forall (v,C) \in H_0^1(\Omega) \times H_\nu^1(\Omega), \\ &b((\breve{u}, \breve{B}), q) = 0 \quad \forall q \in L_0^2(\Omega), \\ &(\breve{u}, \breve{B})(0, \cdot) = (0,0). \end{aligned}$$

Using the skew adjoint property (2.3a), interpolations, the continuity property (2.3b), and the Grönwall inequality we get

$$\int_0^T \|(\breve{u}, \breve{B})\|_1^2 dt \le \mathcal{C} \int_0^T \|(u_\lambda, B_\lambda) - (u,B)\|_0^2 \|(u_\lambda, B_\lambda) - (u,B)\|_1^2 dt. \tag{2.6}$$

Because $(\bar{u}, \bar{B}, \bar{p}) = (u_\lambda, B_\lambda, p_\lambda) - (u,B,p)$ satisfies

$$\begin{aligned} &\langle \frac{d}{dt}(\bar{u}, \bar{B}), \mathcal{S}(v,C)\rangle + a((\bar{u}, \bar{B}), (v,C)) + c((\bar{u}, \bar{B}), (u,B), (v,C)) \\ &\quad + c((u,B), (\bar{u}, \bar{B}), (v,C)) + c((\bar{u}, \bar{B}), (\bar{u}, \bar{B}), (v,C)) + b((v,C), \bar{p}) \\ &\qquad = \lambda((\varphi, \operatorname{curl} \psi), \mathcal{S}(v,C)) \quad \forall (v,C) \in H_0^1(\Omega) \times H_\nu^1(\Omega), \\ &b((\bar{u}, \bar{B}), q) = 0 \quad \forall q \in L_0^2(\Omega), \\ &(\bar{u}, \bar{B})(0, \cdot) = (0,0), \end{aligned}$$

using the same argument as above we obtain that $(\bar{u}, \bar{B}) \in L^\infty(0,T;H)$ and

$$\int_0^T \|(u_\lambda, B_\lambda) - (u, B)\|_1^2 \, dt \le \mathcal{C}\lambda^2 \int_0^T \|(\varphi, \operatorname{curl}\psi)\|_0^2 \, dt.$$

From (2.6) we have then

$$\int_0^T \|(\breve{u}, \breve{B})\|_1^2 dt \le \mathcal{C}\lambda^4,$$

and therefore our claim. The regularity of $(\breve{u}, \breve{B})$ follows from the regularity of $(\varphi, \operatorname{curl}\psi)$. ☐

Because the Gâteaux derivative of the functional $\mathcal{J}$ should vanish at the optimal solution, now we show that the optimal control $(\hat{\varphi}, \operatorname{curl}\hat{\psi})$ must be proportional to the solution of the linear adjoint system.

THEOREM 2.3. *Let* $(u_0, B_0) \in V$ *and let* $(\hat{u}, \hat{B}, \hat{p}, \hat{\varphi}, \operatorname{curl}\hat{\psi})$ *be a solution of the optimal control problem. Let* $(w, D, r) \in L^\infty(0,T;H) \cap L^2(0,T;V)$ *be a solution of the adjoint problem*

$$\begin{aligned} &-\langle \frac{d}{dt}(w,D), \mathcal{S}(v,C)\rangle + a((w,D),(v,C)) + c((v,C),(\hat{u},\hat{B}),(w,D)) \qquad (2.7a)\\ &\qquad + c((\hat{u},\hat{B}),(v,C),(w,D)) + b((v,C),r) \\ &\qquad = \langle (\alpha_1(\hat{u}-u_d), \alpha_2(\hat{B}-B_d)), \mathcal{S}(v,C)\rangle \quad \forall (v,C) \in H_0^1(\Omega)\times H_\nu^1(\Omega), \\ &b((w,D),q) = 0 \quad \forall q \in L_0^2(\Omega), \qquad (2.7b)\\ &(w,D)(T,\cdot) = (0,0). \qquad (2.7c) \end{aligned}$$

Then

$$\hat{\varphi} = -\frac{1}{\beta_1} w, \quad \operatorname{curl}\hat{\psi} = -\frac{1}{\beta_2} D. \qquad (2.8)$$

Proof. Let $(\hat{u}, \hat{B}, \hat{p}, \hat{\varphi}, \operatorname{curl}\hat{\psi})$ be a solution of the optimal control problem. The derivative of the cost functional $\mathcal{J}(\hat{u}, \hat{B}, \hat{p}, \hat{\varphi}, \operatorname{curl}\hat{\psi})$ in the direction $(\varphi, \operatorname{curl}\psi)$ is then

$$\begin{aligned} &\frac{d\mathcal{J}(\hat{u},\hat{B},\hat{\varphi},\operatorname{curl}\hat{\psi})}{d(\hat{\varphi},\operatorname{curl}\hat{\psi})} \cdot (\varphi, \operatorname{curl}\psi) \\ &= \int_0^T \int_\Omega \Big(\alpha_1(\hat{u}-u_d)\cdot\breve{u} + \alpha_2 S(\hat{B}-B_d)\cdot\breve{B} + \beta_1\hat{\varphi}\cdot\varphi + \beta_2 S\operatorname{curl}\hat{\psi}\cdot\operatorname{curl}\psi\Big) dx\, dt \end{aligned}$$

where $(\breve{u}, \breve{B}, \breve{r}, \varphi, \operatorname{curl}\psi)$ is the solution of the system (2.5a)-(2.5c). Since $(\hat{u}, \hat{B}, \hat{p}, \hat{\varphi}$, $\operatorname{curl}\hat{\psi})$ is a optimal solution and the Gâteaux derivative of $\mathcal{J}$ exists, the latter must be zero on all directions $(\varphi, \operatorname{curl}\psi) \in L^2(0,T;L^2(\Omega))$. By taking $(v,C) := (w(t), D(t))$ in (2.5a), also $(v,C) := (\breve{u}, \breve{B})$ in (2.7a), and integrate by parts we obtain

$$\int_0^T \int_\Omega \Big(\alpha_1(\hat{u}-u_d)\cdot\breve{u} + \alpha_2(\hat{B}-B_d)\cdot S\breve{B}\Big) dx\, dt = \int_0^T \int_\Omega (\varphi\cdot w + \operatorname{curl}\psi\cdot SD) dx\, dt.$$

Therefore

$$\int_0^T \int_\Omega \Big((\beta_1\hat{\varphi} + w)\cdot\varphi + S(\beta_2\operatorname{curl}\hat{\psi} + D)\cdot\operatorname{curl}\psi\Big) dx\, dt = 0 \qquad \forall (\varphi, \operatorname{curl}\psi) \in L^2(0,T;L^2(\Omega)),$$

which finally implies (2.8). ☐

2.4. The optimality system. We have seen that the solutions of the optimal control problem are among the solutions of an optimality system, consisting in the forward MHD equation

$$\langle \frac{d}{dt}(u,B), \mathcal{S}(v,C)\rangle + a((u,B),(v,C)) + c((u,B),(u,B),(v,C)) + b((v,C),p) \quad (2.9)$$
$$= \langle (\varphi, \operatorname{curl}\psi), \mathcal{S}(v,C)\rangle \quad \forall (v,C) \in H_0^1(\Omega) \times H_\nu^1(\Omega),$$
$$b((u,B),q) = 0 \quad \forall q \in L_0^2(\Omega), \quad (2.10)$$

with initial data $(u,B)(0,\cdot) = (u_0,B_0)(\cdot)$, the backward in time adjoint system

$$-\langle \frac{d}{dt}(w,D), \mathcal{S}(v,C)\rangle + a((w,D),(v,C)) + c((v,C),(u,B),(w,D)) \quad (2.11)$$
$$+ c((u,B),(v,C),(w,D)) + b((v,C),r)$$
$$= \langle (\alpha_1(u-u_d), \alpha_2(B-B_d)), \mathcal{S}(v,C)\rangle \quad \forall (v,C) \in H_0^1(\Omega) \times H_\nu^1(\Omega),$$
$$b((w,D),q) = 0 \quad \forall q \in L_0^2(\Omega), \quad (2.12)$$

with final condition $(w,D)(T,\cdot) = (0,0)$, and the optimality condition

$$\varphi = -\frac{1}{\beta_1} w, \quad \operatorname{curl}\psi = -\frac{1}{\beta_2} D. \quad (2.13)$$

The above system of equations is a weak formulation of the system

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u - \frac{1}{\mathrm{Re}}\Delta u - S\operatorname{curl} B \times B = \varphi,$$
$$\frac{\partial B}{\partial t} + \frac{1}{\mathrm{Re_m}}\operatorname{curl}(\operatorname{curl})B - \operatorname{curl}(u \times B) = \operatorname{curl}\psi,$$
$$\operatorname{div} u = 0, \quad \operatorname{div} B = 0,$$
$$-\frac{\partial w}{\partial t} - (u\cdot\nabla)w + (\nabla u)^T w - \frac{1}{\mathrm{Re}}\Delta w - \operatorname{curl} D \times B + \nabla\pi = \alpha_1(u-u_d),$$
$$-\frac{\partial D}{\partial t} + S\operatorname{curl} B \times w + \operatorname{curl} D \times u - S\operatorname{curl}(B \times w) + \frac{1}{\mathrm{Re_m}}\operatorname{curl}(\operatorname{curl} D)$$
$$= \alpha_2(B-B_d),$$
$$\operatorname{div} w = 0, \operatorname{div} D = 0,$$
$$\varphi = -\frac{1}{\beta_1}w, \operatorname{curl}\psi = -\frac{1}{\beta_2}D,$$

in $\Omega \times (0,T)$ with the same initial, final, and boundary conditions.

In order to solve this system and determine an optimal solution, we need to discretize the problem in time and space.

3. Semidiscrete-in-time approximations.

3.1. Semidiscretization in time of the optimality system. Let $\sigma_N = \{t_n\}_{n=0}^N$ be a partition of $[0,T]$ into equal intervals $\Delta t = T/N$ with $t_0 = 0$ and $t_N = T$. For each fixed Δt and for every quantity $q(t,x)$ we associate the corresponding set $\{q^{(n)}(x)\}_{n=0}^N$ and a continuous piecewise linear function $q^N = q^N(t,x)$ such that $q^N(t_n,x) = q^{(n)}(x)$ for all $n = 0,1,\ldots,N$. We will denote by $\mathbf{q}$ the vector $(q^{(1)}, q^{(2)}, \ldots, q^{(N)})$ of functions belonging to $\mathbf{X} = X^N$ and defined discretely with respect to time. We define the discrete target $(u_d^{(n)}(x), B_d^{(n)}(x)) = (u_d(t_n,x), B_d(t_n,x))$

for $n = 0, 1, \ldots, N$ whenever $(u_d, B_d) \in \mathcal{U}_{ad}$. The state variables $(u^{(n)}, B^{(n)}) \in H_0^1(\Omega) \times H_\nu^1(\Omega)$ and $p^{(n)} \in L_0^2(\Omega)$ are constrained to satisfy the semidiscrete MHD equations

$$\frac{1}{\Delta t}((u^{(n)} - u^{(n-1)}, B^{(n)} - B^{(n-1)}), \mathcal{S}(v, C)) + a((u^{(n)}, B^{(n)}), (v, C)) \tag{3.1a}$$
$$+ c((u^{(n)}, B^{(n)}), (u^{(n)}, B^{(n)}), (v, C)) + b((v, C), p^{(n)})$$
$$= \langle(\varphi^{(n)}, \operatorname{curl}\psi^{(n)}), \mathcal{S}(v, C)\rangle \quad \forall (v, C) \in H_0^1(\Omega) \times H_\nu^1(\Omega), \quad \text{for } n = 1, \ldots, N,$$
$$b((u^{(n)}, B^{(n)}), q) = 0 \quad \forall q \in L_0^2(\Omega), \quad \text{for } n = 1, 2, \ldots, N, \tag{3.1b}$$
$$u^{(n)} = 0, B^{(n)} \cdot \nu = 0 \quad \text{on } \Gamma, \quad \text{for } n = 1, 2, \ldots, N, \tag{3.1c}$$
$$(u^{(0)}, B^{(0)}) = (u_0(x), B_0(x)) \in V, \tag{3.1d}$$

obtained from a backward Euler time discretization. Optimality is achieved by means of the minimization of the discretized in time functional

$$\mathcal{J}^N(\mathbf{u}, \mathbf{B}, \boldsymbol{\varphi}, \operatorname{curl}\boldsymbol{\psi}) = \frac{\alpha_1 \Delta t}{2} \sum_{n=1}^{N} \|u^{(n)} - u_d^{(n)}\|_0^2 + \frac{\alpha_2 S \Delta t}{2} \sum_{n=1}^{N} \|B^{(n)} - B_d^{(n)}\|_0^2 \tag{3.2}$$
$$+ \frac{\beta_1 \Delta t}{2} \sum_{n=1}^{N} \|\varphi^{(n)}\|_0^2 + \frac{\beta_2 S \Delta t}{2} \sum_{n=1}^{N} \|\operatorname{curl}\psi^{(n)}\|_0^2.$$

We also introduce the admissibility set

$$\mathcal{A}_{ad}^N = \{(\mathbf{u}, \mathbf{B}, \boldsymbol{\varphi}, \operatorname{curl}\boldsymbol{\psi}) \in \mathbf{H}_0^1(\Omega) \times \mathbf{H}_\nu^1(\Omega) \times \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega)$$
$$\text{such that } \mathcal{J}^N(\mathbf{u}, \mathbf{B}, \boldsymbol{\varphi}, \operatorname{curl}\boldsymbol{\psi}) \leq \infty,$$
$$\exists \mathbf{p} \in \mathbf{L}_0^2(\Omega) \text{ such that } (3.1a) - (3.1d) \text{ are satisfied } \}.$$

The formulation of the semidiscrete-in-time optimal control problem is then given by

given $\Omega, T, (u_0, B_0) \in V$, *and* $(u_d, B_d) \in \mathcal{U}_{ad}$,
find $(\mathbf{u}, \mathbf{B}, \boldsymbol{\varphi}, \operatorname{curl}\boldsymbol{\psi}) \in \mathcal{A}_{ad}^N$ *such that the functional* (3.2) *is minimized.*

The value of $(\varphi^{(0)}, \operatorname{curl}\psi^{(0)})$ is not defined in this formulation and it can be arbitrarily chosen as an extension of the corresponding continuous linear functions $(\varphi^N(t, x), \operatorname{curl}\psi^N(t, x)) \in C([0, T]; \mathbf{L}^2(\Omega))$.

3.2. Existence and consistency of the semidiscrete optimal control problem. We now state and prove the existence of solutions of the semidiscrete optimal control problem on an open, bounded, two-dimensional domain Ω with Lipschitz-continuous boundary Γ.

THEOREM 3.1. *Given* T, $\Delta t = T/N$, $(u_0, B_0) \in V$, *and* $(u_d, B_d) \in \mathcal{U}_{ad}$, *there exists at least one solution* $(\mathbf{u}, \mathbf{B}, \mathbf{p}, \boldsymbol{\varphi}, \operatorname{curl}\boldsymbol{\psi}) \in \mathbf{H}_0^1(\Omega) \times \mathbf{H}_\nu^1(\Omega) \times \mathbf{L}_0^2(\Omega) \times \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega)$ *of the semidiscrete-in-time optimal control problem.*

Proof. The admissible set $\mathcal{A}_{ad}^N$ is bounded and nonempty, see e.g. [11]. Given N, let $\{(\boldsymbol{\varphi}_k, \operatorname{curl}\boldsymbol{\psi}_k)\}_{k=1}^{\infty}$ be a minimizing sequence, i.e.,

$$\inf_{\mathcal{A}_{ad}^N} \mathcal{J}^N \le \mathcal{J}^N(\mathbf{u}_k, \mathbf{B}_k, \boldsymbol{\varphi}_k, \operatorname{curl}\boldsymbol{\psi}_k) \le \inf_{\mathcal{A}_{ad}^N} \mathcal{J}^N + \frac{1}{k}, \tag{3.3a}$$

$$\begin{aligned}&\frac{1}{\Delta t}((u_k^{(n)} - u_k^{(n-1)}, B_k^{(n)} - B_k^{(n-1)}), \mathcal{S}(v, C)) + a((u_k^{(n)}, B_k^{(n)}), (v, C)) \\ &\quad + c((u_k^{(n)}, B_k^{(n)}), (u_k^{(n)}, B_k^{(n)}), (v, C)) + b((v, C), p_k^{(n)}) \\ &\quad = ((\varphi_k^{(n)}, \operatorname{curl}\psi_k^{(n)}), \mathcal{S}(v, C)) \quad \forall (v, C) \in H_0^1(\Omega) \times H_\nu^1(\Omega), \quad n = 1, 2 \ldots, N,\end{aligned} \tag{3.3b}$$

$$b((u_k^{(n)}, B_k^{(n)}), q) = 0 \quad \forall q \in L_0^2(\Omega), \qquad \text{for } n = 1, 2, \ldots, N, \tag{3.3c}$$

$$u_k^{(n)} = 0, B_k^{(n)} \cdot \nu = 0 \quad \text{ on } \Gamma, \quad \text{ for } n = 1, 2, \ldots, N, \tag{3.3d}$$

$$(u_k^{(0)}, B_k^{(0)}) = (u_0(x), B_0(x)) \in V. \tag{3.3e}$$

By (3.3a) it follows that $\{(\boldsymbol{\varphi}_{\boldsymbol{k}}, \operatorname{curl}\boldsymbol{\psi}_k)\}$ is bounded in $\mathbf{L}^2(\Omega)^2$. Now taking $(v, C) = 2\Delta t(u_k^{(n)}, B_k^{(n)})$ in (3.3b), using (3.3c), Poincaré and Young inequalities we obtain

$$\begin{aligned}&\|u_k^{(n)}\|_0^2 + \|u_k^{(n)} - u_k^{(n-1)}\|_0^2 + S\|B_k^{(n)}\|_0^2 + S\|B_k^{(n)} - B_k^{(n-1)}\|_0^2 + \frac{\Delta t}{\mathrm{Re}}\|\nabla u_k^{(n)}\|_0^2 \\ &+ \frac{S\Delta t}{\mathrm{Re_m}}\|\operatorname{curl} B_k^{(n)}\|_0^2 \;\le\; \|u_k^{(n-1)}\|_0^2 + S\|B_k^{(n-1)}\|_0^2 + \mathcal{C}\Delta t\left(\|\varphi_k^{(n)}\|_0^2 + \|\operatorname{curl}\psi_k^{(n)}\|_0^2\right)\end{aligned}$$

for $n = 1, 2, \ldots, N$. Summing from $n = 1$ to N we get

$$\|u_k^{(N)}\|_0^2 + S\|B_k^{(N)}\|_0^2 + \Delta t \sum_{n=1}^{N}\left(\frac{1}{\mathrm{Re}}\|\nabla u_k^{(n)}\|_0^2 + \frac{S}{\mathrm{Re_m}}\|\operatorname{curl} B_k^{(n)}\|_0^2\right) \le \mathcal{C}.$$

Now we can extract a weakly convergent subsequence and show that this subsequence converges to the solution of the optimal control problem in the semidiscrete approximation. We can write

$$\begin{aligned}(\varphi_k^{(n)}, \operatorname{curl}\psi_k^{(n)}) &\to (\varphi^{(n)}, \operatorname{curl}\psi^{(n)}) \quad \text{weakly in } L^2(\Omega)^2 \\ (u_k^{(n)}, B_k^{(n)}) &\to (u^{(n)}, B^{(n)}) \quad \text{weakly in } H_0^1(\Omega) \times H_\nu^1,\end{aligned}$$

for all $n = 1, 2, \ldots, N$. Using the fact that the injection $H^1(\Omega) \subset L^2(\Omega)$ is compact, the subsequence $(u_k^{(n)}, B_k^{(n)})$ converges strongly. The lower semicontinuity of the functional in (3.2) allows $(\mathbf{u}, \mathbf{B}, \boldsymbol{\varphi}, \operatorname{curl}\boldsymbol{\psi})$ to minimize the functional. Since we can pass to the limit in the linear and nonlinear terms, there exists $p^{(n)} \in L^2(\Omega)$ such that $(u^{(n)}, B^{(n)}, p^{(n)}, \varphi^{(n)}, \operatorname{curl}\psi^{(n)})$ also satisfies the semidiscrete MHD system (3.1a)-(3.1d). ☐

Now we can prove the convergence of the semidiscrete optimal control problem.

THEOREM 3.2. *Given $\Delta t = T/N$, $(u_d, B_d) \in \mathcal{U}_{ad}$ and $(u_0, B_0) \in V$. For $\Delta t \to 0$, the solution $\{(u^{(n)}, B^{(n)}, p^{(n)}, \varphi^{(n)}, \operatorname{curl}\psi^{(n)})\}_{n=1}^N$ of the semidiscrete-in-time optimal control problem tends to the solution $(\hat{u}, \hat{B}, \hat{p}, \hat{\varphi}, \operatorname{curl}\hat{\psi})$ of the corresponding continuous optimal control problem.*

Proof. Let $u'^{(n)} = (u^{(n)} - u^{(n-1)})/\Delta t$, $B'^{(n)} = (B^{(n)} - B^{(n-1)})/\Delta t$, and let (u'^N, B'^N) denote the corresponding piecewise linear functions.

If $(\mathbf{u}, \mathbf{B}, \mathbf{p}, \boldsymbol{\varphi}, \operatorname{curl} \boldsymbol{\psi})$ is a solution of the semidiscrete-in-time optimal control problem, then $\mathcal{J}(\mathbf{u}, \mathbf{B}, \boldsymbol{\varphi}, \operatorname{curl} \boldsymbol{\psi}) \leq \mathcal{J}(\mathbf{u}(\mathbf{0}), \mathbf{B}(\mathbf{0}), \mathbf{0}, \mathbf{0})$, i.e.,

$$\frac{\Delta t}{2} \sum_{n=1}^{N} \Big(\alpha_1 \|u^{(n)} - u_d^{(n)}\|_0^2 + \alpha_2 S \|B^{(n)} - B_d^{(n)}\|_0^2 + \beta_1 \|\varphi^{(n)}\|_0^2 + \beta_2 S \|\operatorname{curl} \psi^{(n)}\|_0^2 \Big) \leq \mathcal{C}. \tag{3.4}$$

Therefore

$$\|\varphi^N\|^2_{L^2(0,T;L^2(\Omega))} + \|\operatorname{curl} \psi^N\|^2_{L^2(0,T;L^2(\Omega))} = \Delta t \sum_{n=1}^{N} \Big(\beta_1 \|\varphi^{(n)}\|_0^2 + \beta_2 \|\operatorname{curl} \psi^{(n)}\|_0^2 \Big) \leq \mathcal{C}. \tag{3.5}$$

With an argument similar to the one used for the Navier-Stokes case in [28], we get that the sequences $\{(u^N, B^N)\}_{N=1}^{\infty}$, $\{({u'}^N, {B'}^N)\}_{N=1}^{\infty}$ are uniformly bounded in $L^2(0,T;V) \cap L^\infty(0,T;H)$, $L^2(0,T;V')$ respectively. Hence on subsequences we have

$$\begin{aligned} (u^N, B^N) \to (\hat{u}, \hat{B}) \qquad & \text{weakly in } L^2(0,T;V), \ \text{weak-* in } L^\infty(0,T;H), \\ (\varphi^N, \operatorname{curl} \psi^N) \to (\hat{\varphi}, \operatorname{curl} \hat{\psi}) \qquad & \text{weakly in } L^2(0,T;L^2(\Omega))^2, \\ \frac{d(u^N, B^N)}{dt} \to (\hat{u}', \hat{B}') \qquad & \text{weakly in } L^2(0,T;V'). \end{aligned}$$

Since the inclusion $V \subset H$ is compact, we also get that (u^N, B^N) converges strongly in $L^2(0,T;H)$. Now we can pass to the limit in the system of equations (3.1a)-(3.1d) and in the cost functional (3.2). Since the sequence (u^N, B^N) converges weakly in $L^2(0,T;V)$ and strongly in $L^2(0,T;H)$, we can pass to the limit in the nonlinear term, as previously. Therefore when $\Delta t \to 0$ $(N \to \infty)$, the solution of the semidiscrete optimal control problem solves the continuous optimal control problem (1.1)-(1.4). □

3.3. First-order necessary conditions. Let

$$\begin{aligned} X_1 &= \mathbf{H}_0^1(\Omega) \times \mathbf{H}_\nu^1(\Omega) \times \mathbf{L}_0^2(\Omega) \times \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega) \\ X_2 &= \mathbf{H}^{-1}(\Omega) \times \mathbf{H}_\nu^1(\Omega)' \times \mathbf{L}_0^2(\Omega). \end{aligned}$$

We define the nonlinear mappings $M : X_1 \to X_2$ as $M(\mathbf{u}, \mathbf{B}, \mathbf{p}, \boldsymbol{\varphi}, \operatorname{curl} \boldsymbol{\psi}) = (\boldsymbol{\Phi}, \operatorname{curl} \boldsymbol{\Psi}, \mathbf{z})$ if and only if

$$\frac{1}{\Delta t} \Big((u^{(n)}, B^{(n)}) - (u^{(n-1)}, B^{(n-1)}), \mathcal{S}(v, C) \Big) + a((u^{(n)}, B^{(n)}), (v, C)) \tag{3.6a}$$

$$+ c((u^{(n)}, B^{(n)}), (u^{(n)}, B^{(n)}), (v, C)) + b((u^{(n)}, B^{(n)}), p^{(n)})$$

$$= \Big((\varphi^{(n)}, \operatorname{curl} \psi^{(n)}) + (\Phi^{(n)}, \operatorname{curl} \Psi^{(n)}), \mathcal{S}(v, C) \Big), \tag{3.6b}$$

$$\forall (v, C) \in H_0^1(\Omega) \times H_\nu^1(\Omega), n = 1, 2, \ldots, N$$

$$b((u^{(n)}, B^{(n)}), q) = (z^{(n)}, q), \qquad \forall q \in L_0^2(\Omega), n = 1, 2, \ldots, N \tag{3.6c}$$

$$u^{(n)} = 0, B^{(n)} \cdot \nu = 0 \qquad \text{on } \Gamma, n = 1, 2, \ldots, N \tag{3.6d}$$

$$(u^{(0)}, B^{(0)}) = (u_0(x), B_0(x)) \in V, \tag{3.6e}$$

and $Q : X_1 \to \mathbb{R} \times X_2$ by $Q(\mathbf{u}, \mathbf{B}, \mathbf{p}, \boldsymbol{\varphi}, \operatorname{curl} \boldsymbol{\psi}) = (\epsilon, (\boldsymbol{\Phi}, \operatorname{curl} \boldsymbol{\Psi}, \mathbf{z}))$ if and only if

$$\begin{pmatrix} \mathcal{J}^N(\mathbf{u}, \mathbf{B}, \boldsymbol{\varphi}, \operatorname{curl} \boldsymbol{\psi}) - \mathcal{J}^N(\hat{\mathbf{u}}, \hat{\mathbf{B}}, \hat{\boldsymbol{\varphi}}, \operatorname{curl} \hat{\boldsymbol{\psi}}) \\ M(\mathbf{u}, \mathbf{B}, \mathbf{p}, \boldsymbol{\varphi}, \operatorname{curl} \boldsymbol{\psi}) \end{pmatrix} = \begin{pmatrix} \epsilon \\ (\boldsymbol{\Phi}, \operatorname{curl} \boldsymbol{\Psi}, \mathbf{z}) \end{pmatrix} \tag{3.7}$$

for any fixed $(\hat{\boldsymbol{\varphi}}, \operatorname{curl}\hat{\boldsymbol{\psi}}) \in [\mathbf{L}^2(\Omega)]^2$ and $(\hat{\mathbf{u}}, \hat{\mathbf{B}}) \in \mathbf{V}$. Thus the constraints in the semidiscrete-in-time optimal control problem can be expressed as $M(\mathbf{u}, \mathbf{B}, \mathbf{p}, \boldsymbol{\varphi}, \operatorname{curl}\boldsymbol{\psi}) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$ and the optimal control problem can be formulated as

find $\{(\hat{\mathbf{u}}, \hat{\mathbf{B}}, \hat{\mathbf{p}}, \hat{\boldsymbol{\varphi}}, \operatorname{curl}\hat{\boldsymbol{\psi}})\}$ and $\epsilon \leq 0$ such that the equation
$Q(\mathbf{u}, \mathbf{B}, \mathbf{p}, \boldsymbol{\varphi}, \operatorname{curl}\boldsymbol{\psi}) = (\epsilon, \mathbf{0}, \mathbf{0}, \mathbf{0})$ is satisfied for all $(\mathbf{u}, \mathbf{B}, \mathbf{p}, \boldsymbol{\varphi}, \operatorname{curl}\boldsymbol{\psi}) \in X_1$.

We also define the directional derivatives $M'(\mathbf{u},\mathbf{B},\mathbf{p},\boldsymbol{\varphi},\operatorname{curl}\boldsymbol{\psi})$ and $Q'(\mathbf{u},\mathbf{B},\mathbf{p},\boldsymbol{\varphi},\operatorname{curl}\boldsymbol{\psi})$ of these nonlinear operators. Given $(\mathbf{u}, \mathbf{B}, \mathbf{p}, \boldsymbol{\varphi}, \operatorname{curl}\boldsymbol{\psi})$, let $M'(\mathbf{u}, \mathbf{B}, \mathbf{p}, \boldsymbol{\varphi}, \operatorname{curl}\boldsymbol{\psi}) : X_1 \to X_2$ be defined by $M'(\mathbf{u}, \mathbf{B}, \mathbf{p}, \boldsymbol{\varphi}, \operatorname{curl}\boldsymbol{\psi}) \cdot (\widetilde{\mathbf{w}}, \widetilde{\mathbf{D}}, \widetilde{\mathbf{r}}, \widetilde{\boldsymbol{f}}, \operatorname{curl}\widetilde{\boldsymbol{g}}) = (\bar{\boldsymbol{\Phi}}, \operatorname{curl}\bar{\boldsymbol{\Psi}}, \bar{\mathbf{z}})$ if and only if

$$\begin{aligned}
&\frac{1}{\Delta t}\Big((\widetilde{w}^{(n)}, \widetilde{D}^{(n)}) - (\widetilde{w}^{(n-1)}, \widetilde{D}^{(n-1)}), \mathcal{S}(v, C)\Big) + a((\widetilde{w}^{(n)}, \widetilde{D}^{(n)}), (v, C)) && \text{(3.8a)}\\
&\quad + c((\widetilde{w}^{(n)}, \widetilde{D}^{(n)}), (u^{(n)}, B^{(n)}), (v, C)) + c((u^{(n)}, B^{(n)}), (\widetilde{w}^{(n)}, \widetilde{D}^{(n)}), (v, C))\\
&\quad + b((v, C), \widetilde{r}^{(n)}) = \Big((\widetilde{f}^{(n)}, \operatorname{curl}\widetilde{g}^{(n)}) + (\bar{\Phi}^{(n)}, \operatorname{curl}\bar{\Psi}^{(n)}), \mathcal{S}(v, C)\Big),\\
&\quad \forall (v, C) \in H_0^1(\Omega) \times H_\nu^1(\Omega), \quad n = 1, 2, \ldots, N\\
&b((\widetilde{w}^{(n)}, \widetilde{D}^{(n)}), q) = (\bar{z}^{(n)}, q), \qquad \forall q \in L_0^2(\Omega), \quad n = 1, 2, \ldots, N && \text{(3.8b)}\\
&\widetilde{w}^{(n)} = 0, \widetilde{D}^{(n)} \cdot \nu = 0 \qquad \text{on } \Gamma, \quad n = 1, 2, \ldots, N && \text{(3.8c)}\\
&\widetilde{w}^{(0)} = 0, \widetilde{D}^{(0)} = 0, && \text{(3.8d)}
\end{aligned}$$

and $Q'(\mathbf{u}, \mathbf{B}, \mathbf{p}, \boldsymbol{\varphi}, \operatorname{curl}\boldsymbol{\psi}) : X_1 \to \mathbb{R} \times X_2$ as $Q'(\mathbf{u}, \mathbf{B}, \mathbf{p}, \boldsymbol{\varphi}, \operatorname{curl}\boldsymbol{\psi}) \cdot (\widetilde{\mathbf{w}}, \widetilde{\mathbf{D}}, \widetilde{\mathbf{r}}, \widetilde{\boldsymbol{f}}, \operatorname{curl}\widetilde{\boldsymbol{g}}) = (\bar{\epsilon}, (\bar{\boldsymbol{\Phi}}, \operatorname{curl}\bar{\boldsymbol{\Psi}}, \bar{\mathbf{z}}))$ if and only if

$$\begin{pmatrix} (\mathcal{J}^N)'(\mathbf{u}, \mathbf{B}, \boldsymbol{\varphi}, \operatorname{curl}\boldsymbol{\psi}) \cdot (\widetilde{\mathbf{w}}, \widetilde{\mathbf{D}}, \widetilde{\boldsymbol{f}}, \operatorname{curl}\widetilde{\boldsymbol{g}}) \\ M'(\mathbf{u}, \mathbf{B}, \mathbf{p}, \boldsymbol{\varphi}, \operatorname{curl}\boldsymbol{\psi}) \cdot (\widetilde{\mathbf{w}}, \widetilde{\mathbf{D}}, \widetilde{\mathbf{r}}, \widetilde{\boldsymbol{f}}, \operatorname{curl}\widetilde{\boldsymbol{g}}) \end{pmatrix} = \begin{pmatrix} \bar{\epsilon} \\ (\bar{\boldsymbol{\Phi}}, \operatorname{curl}\bar{\boldsymbol{\Psi}}, \bar{\mathbf{z}}) \end{pmatrix},$$

where

$$\begin{aligned}
&(\mathcal{J}^N)'(\mathbf{u}, \mathbf{B}, \boldsymbol{\varphi}, \operatorname{curl}\boldsymbol{\psi}) \cdot (\widetilde{\mathbf{w}}, \widetilde{\mathbf{D}}, \widetilde{\boldsymbol{f}}, \operatorname{curl}\widetilde{\boldsymbol{g}}) = \alpha_1 \Delta t \sum_{n=1}^{N} \int_\Omega (u^{(n)} - u_d^{(n)}) \widetilde{w}^{(n)} dx\\
&+ \Delta t \sum_{n=1}^{N} \int_\Omega \Big(\alpha_2 S(B^{(n)} - B_d^{(n)}) \widetilde{D}^{(n)} + \beta_1 \varphi^{(n)} \widetilde{f}^{(n)} + \beta_2 S \operatorname{curl}\psi^{(n)} \operatorname{curl}\widetilde{g}^{(n)}\Big) dx.
\end{aligned}$$

THEOREM 3.3. *Given $\Delta t = T/N$, $(u_0, B_0) \in V$ and $(u_d, B_d) \in \mathcal{U}_{ad}$.*
If $(\hat{\mathbf{u}}, \hat{\mathbf{B}}, \hat{\mathbf{p}}, \hat{\boldsymbol{\varphi}}, \operatorname{curl}\hat{\boldsymbol{\psi}})$ is a solution of the semidiscrete-in-time optimal control problem, then we have
(i) the operator $M'(\hat{\mathbf{u}}, \hat{\mathbf{B}}, \hat{\mathbf{p}}, \hat{\boldsymbol{\varphi}}, \operatorname{curl}\hat{\boldsymbol{\psi}})$ is onto X_2;
(ii) the operator $Q'(\hat{\mathbf{u}}, \hat{\mathbf{B}}, \hat{\mathbf{p}}, \hat{\boldsymbol{\varphi}}, \operatorname{curl}\hat{\boldsymbol{\psi}})$ has closed range in $\mathbb{R} \times X_2$;
(iii) the operator $Q'(\hat{\mathbf{u}}, \hat{\mathbf{B}}, \hat{\mathbf{p}}, \hat{\boldsymbol{\varphi}}, \operatorname{curl}\hat{\boldsymbol{\psi}})$ is not onto $\mathbb{R} \times X_2$;
(iv) there exists a nonzero Lagrange multiplier $(\mathbf{w}, \mathbf{D}, \mathbf{r}) \in \mathbf{H}_0^1(\Omega) \times \mathbf{H}_\nu^1(\Omega) \times \mathbf{L}_0^2(\Omega)$ satisfying the Euler equations

$$\begin{aligned}
&(\mathcal{J}^N)'(\hat{\mathbf{u}}, \hat{\mathbf{B}}, \hat{\boldsymbol{\varphi}}, \operatorname{curl}\hat{\boldsymbol{\psi}}) \cdot (\widetilde{\mathbf{w}}, \widetilde{\mathbf{D}}, \widetilde{\boldsymbol{f}}, \operatorname{curl}\widetilde{\boldsymbol{g}}) && \text{(3.9)}\\
&\qquad - \langle (\mathbf{w}, \mathbf{D}, \mathbf{r}), M'(\hat{\mathbf{u}}, \hat{\mathbf{B}}, \hat{\mathbf{p}}, \hat{\boldsymbol{\varphi}}, \operatorname{curl}\hat{\boldsymbol{\psi}}) \cdot (\widetilde{\mathbf{w}}, \widetilde{\mathbf{D}}, \widetilde{\mathbf{r}}, \widetilde{\boldsymbol{f}}, \operatorname{curl}\widetilde{\boldsymbol{g}}) \rangle = 0
\end{aligned}$$

for all $(\widetilde{\mathbf{w}}, \widetilde{\mathbf{D}}, \widetilde{\mathbf{r}}, \widetilde{\boldsymbol{f}}, \operatorname{curl} \widetilde{\boldsymbol{g}}) \in X_1$*, where* $\langle \cdot, \cdot \rangle$ *denotes the duality pairing between* X_2 *and* X_2^**.*

Proof. Let set

$$\widetilde{a}((\widetilde{w}^{(n)}, \widetilde{D}^{(n)}), (v, C)) = a((\widetilde{w}^{(n)}, \widetilde{D}^{(n)}), (v, C)) + \frac{1}{\Delta t}((\widetilde{w}^{(n)}, \widetilde{D}^{(n)}), \mathcal{S}(v, C)),$$

$$((\widetilde{\Phi}^{(n)}, \operatorname{curl} \widetilde{\Psi}^{(n)}), (v, C)) = ((\bar{\Phi}^{(n)}, \operatorname{curl} \bar{\Psi}^{(n)}), (v, C)) + \frac{1}{\Delta t}((\widetilde{w}^{(n-1)}, \widetilde{D}^{(n-1)}), (v, C)),$$

so (3.8a)-(3.8d) can be rewritten as

$$\begin{aligned}
&\widetilde{a}((\widetilde{w}^{(n)}, \widetilde{D}^{(n)}), (v, C)) + c((\widetilde{w}^{(n)}, \widetilde{D}^{(n)}), (\hat{u}^{(n)}, \hat{B}^{(n)}), (v, C)) &&(3.10a)\\
&\quad + c((\hat{u}^{(n)}, \hat{B}^{(n)}), (\widetilde{w}^{(n)}, \widetilde{D}^{(n)}), (v, C)) + b((v, C), \widetilde{r}^{(n)})\\
&\quad = \Big((\widetilde{f}^{(n)}, \operatorname{curl} \widetilde{g}^{(n)}) + (\widetilde{\Phi}^{(n)}, \operatorname{curl} \widetilde{\Psi}^{(n)}), \mathcal{S}(v, C) \Big), &&(3.10b)\\
&\qquad\qquad \forall (v, C) \in H_0^1(\Omega) \times H_\nu^1(\Omega), \quad n = 1, 2, \ldots, N\\
&b((\widetilde{w}^{(n)}, \widetilde{D}^{(n)}), q) = (\bar{z}^{(n)}, q), \qquad \forall q \in L_0^2(\Omega), \quad n = 1, 2, \ldots, N &&(3.10c)\\
&\widetilde{w}^{(n)} = 0,\ \widetilde{D}^{(n)} \cdot \nu = 0 \qquad \text{on } \Gamma, \quad n = 1, 2, \ldots, N &&(3.10d)\\
&\widetilde{w}^{(0)} = 0,\ \widetilde{D}^{(0)} = 0. &&(3.10e)
\end{aligned}$$

Since $(\bar{\Phi}^{(n)}, \operatorname{curl} \bar{\Psi}^{(n)}) \in H^{-1}(\Omega) \times H_\nu^1(\Omega)'$, we have that $(\widetilde{\Phi}^{(n)}, \operatorname{curl} \widetilde{\Psi}^{(n)}) \in H^{-1}(\Omega) \times H_\nu^1(\Omega)'$ as well. So (i) follows if it can be shown that (3.10a)-(3.10e) has a solution $(\widetilde{\mathbf{w}}, \widetilde{\mathbf{D}}, \widetilde{\mathbf{r}}) \in X_1$ for all $(\widetilde{\boldsymbol{\Phi}}, \operatorname{curl} \widetilde{\boldsymbol{\Psi}}, \bar{\mathbf{z}}) \in X_2$. Since (3.10a)-(3.10e) is a steady-state system, one can prove this using Lemmas 4.1 - 4.5 in [7].

Now let define the linear operator $\mathcal{T} \in \mathcal{L}(X_2, X_1)$ by

$$\mathcal{T}(\widetilde{\boldsymbol{\Phi}}, \operatorname{curl} \widetilde{\boldsymbol{\Psi}}, \bar{\mathbf{z}}) = (\widetilde{\mathbf{w}}, \widetilde{\mathbf{D}}, \widetilde{\mathbf{r}}, \widetilde{\boldsymbol{f}}, \operatorname{curl} \widetilde{\boldsymbol{g}})$$

for $(\widetilde{\mathbf{w}}, \widetilde{\mathbf{D}}, \widetilde{\mathbf{r}}, \widetilde{\boldsymbol{f}}, \operatorname{curl} \widetilde{\boldsymbol{g}}) \in X_1$ and $(\widetilde{\boldsymbol{\Phi}}, \operatorname{curl} \boldsymbol{\Psi}, \mathbf{z}) \in X_2$ if and only if

$$\begin{aligned}
&\widetilde{a}((\widetilde{w}^{(n)}, \widetilde{D}^{(n)}), (v, C)) + b((v, C), \widetilde{r}^{(n)}) = ((\widetilde{f}^{(n)}, \operatorname{curl} \widetilde{g}^{(n)}) + (\widetilde{\Phi}^{(n)}, \operatorname{curl} \widetilde{\Psi}^{(n)}), \mathcal{S}(v, C))\\
&\qquad\qquad \forall (v, C) \in H_0^1(\Omega) \times H_\nu^1(\Omega), \quad n = 1, 2, \ldots, N\\
&b((\widetilde{w}^{(n)}, \widetilde{D}^{(n)}), q) = (\bar{z}^{(n)}, q) \qquad \forall q \in L_0^2(\Omega), \quad n = 1, 2, \ldots, N.
\end{aligned}$$

$\mathcal{T}$ is a well defined continuous linear operator with a continuous inverse. Let also $\mathcal{R} \in \mathcal{L}(X_1, X_2)$ be defined by

$$\begin{aligned}
\mathcal{R}(\widetilde{\mathbf{w}}, \widetilde{\mathbf{D}}, \widetilde{\mathbf{r}}, \widetilde{\boldsymbol{f}}, \operatorname{curl} \widetilde{\boldsymbol{g}}) = \Big\{ \Big(&(\hat{u}^{(n)} \cdot \nabla) \widetilde{w}^{(n)} + (\widetilde{w}^{(n)} \cdot \nabla) \hat{u}^{(n)}\\
&- S(\operatorname{curl} \hat{B}^{(n)} \times \widetilde{D}^{(n)} + \operatorname{curl} \widetilde{D}^{(n)} \times \hat{B}^{(n)}), -S \operatorname{curl} (\hat{u}^{(n)} \times \widetilde{D}^{(n)} + \widetilde{w}^{(n)} \times \hat{B}^{(n)}), 0 \Big) \Big\}_{n=1}^{N}
\end{aligned}$$

for all $(\widetilde{\mathbf{w}}, \widetilde{\mathbf{D}}, \widetilde{\mathbf{r}}, \widetilde{\boldsymbol{f}}, \operatorname{curl} \widetilde{\boldsymbol{g}}) \in X_1$. It can be easily verified using the definition of the trilinear form $c(\cdot, \cdot, \cdot)$ that

$$\begin{aligned}
&\mathcal{R}(\widetilde{\mathbf{w}}, \widetilde{\mathbf{D}}, \widetilde{\mathbf{r}}, \widetilde{\boldsymbol{f}}, \operatorname{curl} \widetilde{\boldsymbol{g}})^{(n)} \cdot (v, C, q)\\
&\quad = c((\hat{u}^{(n)}, \hat{B}^{(n)}), (\widetilde{w}^{(n)}, \widetilde{D}^{(n)}), (v, C)) + c((\widetilde{w}^{(n)}, \widetilde{D}^{(n)}), (\hat{u}^{(n)}, \hat{B}^{(n)}), (v, C))\\
&\forall (v, C, q) \in H_0^1(\Omega) \times H_\nu^1(\Omega) \times L_0^2(\Omega), \quad n = 1, 2, \ldots, N.
\end{aligned}$$

Thus (3.8a)-(3.8d) are equivalent to

$$(\widetilde{\mathbf{w}},\widetilde{\mathbf{D}},\widetilde{\mathbf{r}},\widetilde{\boldsymbol{f}},\operatorname{curl}\widetilde{\boldsymbol{g}}) = \mathcal{T}\Big[(\widetilde{\mathbf{\Phi}},\operatorname{curl}\widetilde{\mathbf{\Psi}},\bar{\mathbf{z}}) - \mathcal{R}(\widetilde{\mathbf{w}},\widetilde{\mathbf{D}},\widetilde{\mathbf{r}},\widetilde{\boldsymbol{f}},\operatorname{curl}\widetilde{\boldsymbol{g}})\Big]$$

or

$$(\widetilde{\mathbf{\Phi}},\operatorname{curl}\widetilde{\mathbf{\Psi}},\bar{\mathbf{z}}) = \mathcal{T}^{-1}\Big[(\widetilde{\mathbf{w}},\widetilde{\mathbf{D}},\widetilde{\mathbf{r}},\widetilde{\boldsymbol{f}},\operatorname{curl}\widetilde{\boldsymbol{g}}) + \mathcal{T}\mathcal{R}(\widetilde{\mathbf{w}},\widetilde{\mathbf{D}},\widetilde{\mathbf{r}},\widetilde{\boldsymbol{f}},\operatorname{curl}\widetilde{\boldsymbol{g}})\Big].$$

This shows that

$$M'(\hat{\mathbf{u}},\hat{\mathbf{B}},\hat{\mathbf{p}},\hat{\boldsymbol{\varphi}},\operatorname{curl}\hat{\boldsymbol{\psi}}) = \mathcal{T}^{-1}(I+\mathcal{T}\mathcal{R}).$$

From the Sobolev embeddings we have that the operator $\mathcal{R}$ is compact from X_1 to X_2 so that $\mathcal{T}\mathcal{R}$ is compact from X_1 to X_2. Therefore the image of X_1 under $(I+\mathcal{T}\mathcal{R})$ is closed with finite codimension, which shows that the image of X_1 under the mapping $M'(\hat{\mathbf{u}},\hat{\mathbf{B}},\hat{\mathbf{p}},\hat{\boldsymbol{\varphi}},\operatorname{curl}\hat{\boldsymbol{\psi}})$ is closed. By the continuity of the bilinear and trilinear forms in the definition of $M'(\hat{\mathbf{u}},\hat{\mathbf{B}},\hat{\mathbf{p}},\hat{\boldsymbol{\varphi}},\operatorname{curl}\hat{\boldsymbol{\psi}})$ we have that this operator belongs to $\mathcal{L}(X_1,X_2)$, and therefore the kernel of $M'(\hat{\mathbf{u}},\hat{\mathbf{B}},\hat{\mathbf{p}},\hat{\boldsymbol{\varphi}},\operatorname{curl}\hat{\boldsymbol{\psi}})$ is a closed subspace.

(*ii*) Now $(J^N)'(\hat{\mathbf{u}},\hat{\mathbf{B}},\hat{\boldsymbol{\varphi}},\operatorname{curl}\hat{\boldsymbol{\psi}})$ acting on the kernel of $M'(\hat{\mathbf{u}},\hat{\mathbf{B}},\hat{\mathbf{p}},\hat{\boldsymbol{\varphi}},\operatorname{curl}\hat{\boldsymbol{\psi}})$ is either identically zero or onto $\mathbb{R}$, as a linear functional on a Banach space. Hence $(J^N)'(\hat{\mathbf{u}},\hat{\mathbf{B}},\hat{\boldsymbol{\varphi}},\operatorname{curl}\hat{\boldsymbol{\psi}})$ acting on the kernel of $M'(\hat{\mathbf{u}},\hat{\mathbf{B}},\hat{\mathbf{p}},\hat{\boldsymbol{\varphi}},\operatorname{curl}\hat{\boldsymbol{\psi}})$ has a closed range. Now we recall that if $\mathcal{A}: X_1 \to \mathbb{R}$ and $\mathcal{B}: X_1 \to X_2$ are linear continuous operators, with the range of $\mathcal{B}$ closed and the subspace $\mathcal{A}\ker(\mathcal{B})$ closed in $\mathbb{R}$, then the operator $(\mathcal{A},\mathcal{B}): X_1 \to \mathbb{R}\times X_2$ has a closed range in $\mathbb{R}\times X_2$. If we take $\mathcal{A} := (\mathcal{J}^N)'(\hat{\mathbf{u}},\hat{\mathbf{B}},\hat{\boldsymbol{\varphi}},\operatorname{curl}\hat{\boldsymbol{\psi}})$ and $\mathcal{B} := M'(\hat{\mathbf{u}},\hat{\mathbf{B}},\hat{\mathbf{p}},\hat{\boldsymbol{\varphi}},\operatorname{curl}\hat{\boldsymbol{\psi}})$ we deduce that $Q'(\hat{\mathbf{u}},\hat{\mathbf{B}},\hat{\mathbf{p}},\hat{\boldsymbol{\varphi}},\operatorname{curl}\hat{\boldsymbol{\psi}})$ has a closed range in $\mathbb{R}\times X_2$.

(*iii*) The operator $Q'(\hat{\mathbf{u}},\hat{\mathbf{B}},\hat{\mathbf{p}},\hat{\boldsymbol{\varphi}},\operatorname{curl}\hat{\boldsymbol{\psi}})$ is not onto because if it were, by the Implicit Function Theorem we would have $(\mathbf{u},\mathbf{B},\mathbf{p},\boldsymbol{\varphi},\operatorname{curl}\boldsymbol{\psi}) \in \mathcal{A}_{ad}^N$ such that $\mathcal{J}^N(\mathbf{u},\mathbf{B},\boldsymbol{\varphi},\operatorname{curl}\boldsymbol{\psi}) < \mathcal{J}^N(\hat{\mathbf{u}},\hat{\mathbf{B}},\hat{\boldsymbol{\varphi}},\operatorname{curl}\hat{\boldsymbol{\psi}})$, contradicting the hypothesis that $(\hat{\mathbf{u}},\hat{\mathbf{B}},\hat{\mathbf{p}},\hat{\boldsymbol{\varphi}},\operatorname{curl}\hat{\boldsymbol{\psi}})$ is an optimal solution.

(*iv*) Then, the Hahn-Banach theorem implies that there exists a nonzero element of $\mathbb{R}\times X_2^* = \mathbb{R}\times H_0^1(\Omega)\times H_\nu^1(\Omega)\times L_0^2(\Omega)$ that annihilates the range of $Q'(\hat{\mathbf{u}},\hat{\mathbf{B}},\hat{\mathbf{p}},\hat{\boldsymbol{\varphi}},\operatorname{curl}\hat{\boldsymbol{\psi}})$, i.e., there exists $(\epsilon,\mathbf{w},\mathbf{D},\mathbf{r}) \in \mathbb{R}\times X_2^*$ such that

$$\begin{gathered}\langle(\epsilon,\mathbf{w},\mathbf{D},\mathbf{r}),(\bar{\epsilon},\bar{\mathbf{\Phi}},\operatorname{curl}\bar{\mathbf{\Psi}},\bar{\mathbf{z}})\rangle = 0, \\ \forall(\bar{\epsilon},\bar{\mathbf{\Phi}},\operatorname{curl}\bar{\mathbf{\Psi}},\bar{\mathbf{z}}) \text{ in the range of } Q'(\hat{\mathbf{u}},\hat{\mathbf{B}},\hat{\mathbf{p}},\hat{\boldsymbol{\varphi}},\operatorname{curl}\hat{\boldsymbol{\psi}}),\end{gathered} \tag{3.11}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\mathbb{R}\times X_2$ and its dual $(\mathbb{R}\times X_2)^*$. We note that $\epsilon \neq 0$ since otherwise we would have $\langle(\bar{\mathbf{\Phi}},\operatorname{curl}\bar{\mathbf{\Psi}},\bar{\mathbf{z}}),(\mathbf{w},\mathbf{D},\mathbf{r})\rangle = 0$ for all $(\bar{\mathbf{\Phi}},\operatorname{curl}\bar{\mathbf{\Psi}},\bar{\mathbf{z}}) \in X_2$, so that $(\mathbf{w},\mathbf{D},\mathbf{r}) = (\mathbf{0},\mathbf{0},\mathbf{0})$, contradicting the fact that $(\epsilon,\mathbf{w},\mathbf{D},\mathbf{r}) \neq 0$. We may set $\epsilon = -1$, and from the definition of $Q'(\hat{\mathbf{u}},\hat{\mathbf{B}},\hat{\mathbf{p}},\hat{\boldsymbol{\varphi}},\operatorname{curl}\hat{\boldsymbol{\psi}})$ and (3.11) we obtain (3.9). □

3.4. The optimality system. From the first-order necessary conditions of optimality (3.9) we derive an optimality system from which semidiscrete-in-time optimal control states and controls may be determined.

THEOREM 3.4. *Given $\Delta t = T/N$, $(u_0,B_0) \in V$ and $(u_d,B_d) \in \mathcal{U}_{ad}$. Let $(\hat{\mathbf{u}},\hat{\mathbf{B}},\hat{\mathbf{p}},\hat{\boldsymbol{\varphi}},\operatorname{curl}\hat{\boldsymbol{\psi}})$ denote a solution of the semidiscrete-in-time optimal control problem. Then we have*

$$\beta_1\hat{\varphi}^{(n)} = -w^{(n-1)}, \quad \beta_2\operatorname{curl}\hat{\psi}^{(n)} = -D^{(n-1)} \qquad \textit{for } n = 1,2,\ldots,N, \tag{3.12}$$

where $(\mathbf{w},\mathbf{D},\mathbf{r})\in\mathbf{H}^1_0(\Omega)\times\mathbf{H}^1_\nu(\Omega)\times\mathbf{L}^2_0(\Omega)$ *satisfies the adjoint problem*

$$\begin{aligned}
&-\frac{1}{\Delta t}\Big((w^{(n)},D^{(n)})-(w^{(n-1)},D^{(n-1)}),\mathcal{S}(v,C)\Big) && \text{(3.13a)}\\
&\quad+a((v,C),(w^{(n-1)},D^{(n-1)}))+c((v,C),(\hat{u}^{(n)},\hat{B}^{(n)}),(w^{(n-1)},D^{(n-1)}))\\
&\quad+c((\hat{u}^{(n)},\hat{B}^{(n)}),(v,C),(w^{(n-1)},D^{(n-1)}))+b((v,C),r^{(n-1)})\\
&\quad=\Big((\alpha_1(\hat{u}^{(n)}-u_d^{(n)}),\alpha_2(\hat{B}^{(n)}-B_d^{(n)})),\mathcal{S}(v,C)\Big)\\
&\qquad\forall(v,C)\in H^1_0(\Omega)\times H^1_\nu(\Omega),\quad n=1,\ldots,N,\\
&b((w^{(n-1)},D^{(n-1)}),q)=0\qquad\forall q\in L^2_0(\Omega),\quad n=1,2,\ldots,N, && \text{(3.13b)}\\
&w^{(n-1)}=0,D^{(n-1)}\cdot\nu=0\qquad \textit{on }\Gamma,\quad n=1,2,\ldots,N, && \text{(3.13c)}\\
&(w^{(N)},D^{(N)})=(0,0). && \text{(3.13d)}
\end{aligned}$$

Proof. The first-order necessary condition (3.9) is equivalent to

$$\begin{aligned}
&\Delta t\sum_{n=1}^{N}\Big[\alpha_1(\hat{u}^{(n)}-u_d^{(n)})\cdot\widetilde{w}^{(n)}+\alpha_2 S(\hat{B}^{(n)}-B_d^{(n)})\cdot\widetilde{D}^{(n)}+\beta_1\hat{\varphi}^{(n)}\cdot\widetilde{f}^{(n)}\\
&\qquad+\beta_2 S\operatorname{curl}\hat{\psi}^{(n)}\cdot\operatorname{curl}\widetilde{g}^{(n)}\Big]\\
&=\Delta t\sum_{n=1}^{N}\Big[\frac{1}{\Delta t}\Big((\widetilde{w}^{(n)},\widetilde{D}^{(n)})-(\widetilde{w}^{(n-1)},\widetilde{D}^{(n-1)}),\mathcal{S}(w^{(n-1)},D^{(n-1)})\Big)\\
&+a((\widetilde{w}^{(n)},\widetilde{D}^{(n)}),(w^{(n-1)},D^{(n-1)}))+c((\widetilde{w}^{(n)},\widetilde{D}^{(n)}),(\hat{u}^{(n)},\hat{B}^{(n)}),(w^{(n-1)},D^{(n-1)}))\\
&+c((\hat{u}^{(n)},\hat{B}^{(n)}),(\widetilde{w}^{(n)},\widetilde{D}^{(n)}),(w^{(n-1)},D^{(n-1)}))+b((w^{(n-1)},D^{(n-1)}),\widetilde{r}^{(n)})\\
&-((\widetilde{f}^{(n)},\operatorname{curl}\widetilde{g}^{(n)}),\mathcal{S}(w^{(n-1)},D^{(n-1)}))+b((\widetilde{w}^{(n)},\widetilde{D}^{(n)}),r^{(n-1)})\Big].
\end{aligned}$$

Now using (3.10e), we see that

$$\begin{aligned}
&\sum_{n=1}^{N}\Big((\widetilde{w}^{(n)},\widetilde{D}^{(n)})-(\widetilde{w}^{(n-1)},\widetilde{D}^{(n-1)}),\mathcal{S}(w^{(n-1)},D^{(n-1)})\Big)\\
&=-\sum_{n=1}^{N}\Big(\mathcal{S}(\widetilde{w}^{(n)},\widetilde{D}^{(n)}),(w^{(n)},D^{(n)})-(w^{(n-1)},D^{(n-1)})\Big)+\Big(\mathcal{S}(\widetilde{w}^{(N)},\widetilde{D}^{(N)}),(w^{(N)},D^{(N)})\Big).
\end{aligned}$$

Combining the above equations and choosing $(\widetilde{w}^{(n)},\widetilde{D}^{(n)})=(0,0)$ and $\widetilde{r}^{(n)}=0$ we obtain from above that

$$\sum_{n=1}^{N}\int_\Omega\Big[(\beta_1\hat{\varphi}^{(n)}+w^{(n-1)})\cdot\widetilde{f}^{(n)}+S(\beta_2\operatorname{curl}\hat{\psi}^{(n)}+D^{(n-1)})\cdot\operatorname{curl}\widetilde{g}^{(n)}\Big]\,dx=0.$$

As the variations $(\widetilde{f}^{(n)},\operatorname{curl}\widetilde{g}^{(n)})$ are arbitrary in $\mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Omega)$, we obtain (3.12).

Choosing $(\widetilde{w}^{(n)},\widetilde{D}^{(n)})=(0,0)$ and $(\widetilde{f}^{(n)},\operatorname{curl}\widetilde{g}^{(n)})=(0,0)$ we obtain in a similar manner $b((w^{(n-1)},D^{(n-1)}),q)=0$ for all $q\in L^2_0(\Omega)$.

Finally, choosing $(\widetilde{f}^{(n)}, \mathrm{curl}\, \widetilde{g}^{(n)}) = (0,0)$ and $\widetilde{r}^{(n)} = 0$ we have

$$
\begin{aligned}
&\Delta t \sum_{n=1}^{N} \Big[\alpha_1(\hat{u}^{(n)} - u_d^{(n)}) \cdot \widetilde{w}^{(n)} + \alpha_2(\hat{B}^{(n)} - B_d^{(n)}) \cdot S\widetilde{D}^{(n)} \Big] \\
&= \Big(\mathcal{S}(\widetilde{w}^{(N)}, \widetilde{D}^{(N)}), (w^{(N)}, D^{(N)}) \Big) \\
&+ \Delta t \sum_{n=1}^{N} \Big[-\frac{1}{\Delta t} \Big(\mathcal{S}(\widetilde{w}^{(n)}, \widetilde{D}^{(n)}), (w^{(n)}, D^{(n)}) - (w^{(n-1)}, D^{(n-1)}) \Big) \\
&+ a((\widetilde{w}^{(n)}, \widetilde{D}^{(n)}), (w^{(n-1)}, D^{(n-1)})) + c((\widetilde{w}^{(n)}, \widetilde{D}^{(n)}), (\hat{u}^{(n)}, \hat{B}^{(n)}), (w^{(n-1)}, D^{(n-1)})) \\
&+ c((\hat{u}^{(n)}, \hat{B}^{(n)}), (\widetilde{w}^{(n)}, \widetilde{D}^{(n)}), (w^{(n-1)}, D^{(n-1)})) + b((\widetilde{w}^{(n)}, \widetilde{D}^{(n)}), r^{(n-1)}) \Big]
\end{aligned}
$$

from which (3.13a) and (3.13d) follow. The equation (3.13c) holds trivially since $(\mathbf{w}, \mathbf{D}) \in \mathbf{H}_0^1(\Omega) \times \mathbf{H}_\nu^1(\Omega)$. □

Thus, in order to solve the semidiscrete-in-time optimal control problem, one has to solve the semidiscrete MHD system (3.1a)-(3.1d), the semidiscrete adjoint system (3.13a)-(3.13d), and the optimality condition (3.12).

4. Fully discrete time-space approximation.

4.1. Preliminaries. We consider only conforming finite element approximations. Let $X^h \subset H^1(\Omega)$, $Y^h(\Omega) \subset H^1(\Omega)$, and $S^h(\Omega) \subset L^2(\Omega)$ be three families of finite dimensional spaces parametrized by h, such that $0 < h < 1$. We also define $X_0^h = X^h \cap H_0^1(\Omega)$, $Y_\nu^h = Y^h \cap H_\nu^1$ with corresponding norms induced by the norm on $H^1(\Omega)$, $S_0^h = S^h \cap L_0^2(\Omega)$, and $Z^h = \big\{ u^h \in X_0^h;\ \int_\Omega q^h \nabla \cdot u^h dx = 0\ \forall q \in S_0^h \big\}$. Next we define the product spaces $V^h = X^h \times Y^h$, $V_{0\nu}^h = X_0^h \times Y_\nu^h$, with the corresponding norms induced by the norm on $H^1(\Omega) \times H^1(\Omega)$.

In order to have stable and accurate approximations, we assume that X_0^h and S_0^h have been chosen so that the following inf-sup condition is satisfied on the finite-dimensional spaces, i.e., there exists a constant $\gamma^h > 0$ such that

$$
\inf_{p_h \in S_0^h} \sup_{(u_h, B_h) \in V_{0\nu}^h} \frac{b((u_h, B_h), p_h)}{\|(u_h, B_h)\|_1 \|p_h\|_0} \geq \gamma^h. \tag{4.1}
$$

This constraint is the same with the one necessary for the analogous discretization of the Navier-Stokes equations to yield meaningful approximations. For pairs of finite element spaces that satisfy (4.1) see [7], [12]. There is no constraint on the spaces Y_ν^h, so in order to approximate the velocity and pressure we can use the spaces used traditionally for the Navier-Stokes equations. To approximate the magnetic field we can use any appropriate subspace of $H^1(\Omega)$. In order to get error estimates it may be convenient to choose $Y^h = X^h$, i.e., the underlying finite element spaces for the magnetic and velocity fields are the same.

Since in general $Z^h \not\subset Z = \big\{ u \in H_0^1(\Omega);\ \nabla \cdot u = 0 \big\}$, the form (2.2c) does not satisfy the antisymmetry property (2.3a), whenever $u_1 \in Z^h$. In order to preserve this property over the subspaces, we introduce the form

$$\begin{aligned}
\widetilde{c}((u_1,B_1),(u_2,B_2),(u_3,B_3)) &= \int_\Omega ((u_1\cdot\nabla)u_2\cdot u_3 - (u_1\cdot\nabla)u_3\cdot u_2)\,dx \\
&- S\int_\Omega (\operatorname{curl} B_2\times B_1\cdot u_3 - \operatorname{curl} B_3\times B_1\cdot u_2)\,dx \\
&- S\int_\Omega (u_2\times B_1 \operatorname{curl} B_3 - u_3\times B_1\operatorname{curl} B_2)\,dx. \\
&= \frac{1}{2}[c((u_1,B_1),(u_2,B_2),(u_3,B_3)) - c((u_1,B_1),(u_3,B_3),(u_2,B_2))].
\end{aligned}$$

Using the divergence theorem it is easily verified that

$$\widetilde{c}((u_1,B_1),(u_2,B_2),(u_3,B_3)) = c((u_1,B_1),(u_2,B_2),(u_3,B_3))$$

on $(Z\times H^1(\Omega))\times((H^1(\Omega)\times H^1(\Omega))^2$. In addition we now have that

$$\widetilde{c}((u_1,B_1),(u_2,B_2),(u_3,B_3)) = -\,\widetilde{c}((u_1,B_1),(u_3,B_3),(u_2,B_2))$$

on all $((H^1(\Omega)\times H^1(\Omega))^3$, and in particular, this antisymmetry property holds on the finite-dimensional subspaces, e.g., even when $u_1\in X^h$.

4.2. Formulation of the fully discrete optimal control approximation. Let $\sigma_N=\{t_n\}_{n=0}^N$ be a partition of $[0,T]$ into equal intervals $\Delta t=T/N$ with $t_0=0$ and $t_N=T$. On the finite element spaces $V^h_{0\nu}\subset H^1_0(\Omega)\times H^1_\nu(\Omega)$ and $S^h_0\subset L^2_0(\Omega)$ we assume that hypothesis (4.1) holds. For each fixed Δt (or N) and for every quantity $q(t,x)$ we associate the corresponding set $\{q_h^{(n)}\}_{n=1}^N$. We will denote the vector $(q_h^{(1)},q_h^{(2)},\dots,q_h^{(N)})$ as $\mathbf{q}_h$ and the space W^N as $\mathbf{W}$. We also define the continuous piecewise linear function $q_h^N(t,x)$ by the conditions $q_h^N(t_n,x)=q_h(t_n,x)\forall n=1,2,\dots,N$.

Given $\Delta=T/N$, $(\boldsymbol{\varphi}_h,\operatorname{curl}\boldsymbol{\psi}_h)\in\mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Omega)$ and $(u_0,B_0)\in V$, $(\mathbf{u}_h,\mathbf{B}_h)$ is called a generalized solution of the fully discrete time-space approximation of the MHD equations if $(u_h^{(n)},B_h^{(n)})\in V^h_{0\nu}$ and $p_h^{(n)}\in S^h_0$ and $(u_h^{(n)},B_h^{(n)},p_h^{(n)})$ satisfies the system

$$\begin{aligned}
&\frac{1}{\Delta t}((u_h^{(n)}-u_h^{(n-1)},B_h^{(n)}-B_h^{(n-1)}),\mathcal{S}(v_h,C_h)) + a((u_h^{(n)},B_h^{(n)}),(v_h,C_h)) \qquad (4.2a)\\
&\qquad + \widetilde{c}((u_h^{(n)},B_h^{(n)}),(u_h^{(n)},B_h^{(n)}),(v_h,C_h)) + b((v_h,C_h),p_h^{(n)}) \\
&\qquad = \langle(\varphi_h^{(n)},\operatorname{curl}\psi_h^{(n)}),\mathcal{S}(v_h,C_h)\rangle \quad \forall (v_h,C_h)\in V^h_{0\nu}, \\
&b((u_h^{(n)},B_h^{(n)}),q_h)=0 \quad \forall q_h\in S^h_0, \qquad (4.2b)
\end{aligned}$$

for $n=1,2,\dots,N$, with initial data $(u_h^{(0)},B_h^{(0)})=\Pi^h(u_0(x),B_0(x))$, where Π^h denotes the projection of $(u_0(x),B_0(x))$ onto $V^h_{0\nu}$.

The discrete functional used in the optimal control problem is given by

$$\begin{aligned}
\mathcal{J}_h^N(\mathbf{u}_h,\mathbf{B}_h,\boldsymbol{\varphi}_h,\operatorname{curl}\boldsymbol{\psi}_h) =& \frac{\alpha_1\Delta t}{2}\sum_{n=1}^N \|u_h^{(n)}-u_{hd}^{(n)}\|_0^2 + \frac{\alpha_2\mathcal{S}\Delta t}{2}\sum_{n=1}^N\|B_h^{(n)}-B_{hd}^{(n)}\|_0^2 \\
&+ \frac{\beta_1\Delta t}{2}\sum_{n=1}^N\|\varphi_h^{(n)}\|_0^2 + \frac{\beta_2\mathcal{S}\Delta t}{2}\sum_{n=1}^N\|\operatorname{curl}\psi_h^{(n)}\|_0^2. \qquad (4.3)
\end{aligned}$$

The formulation of the fully discrete optimal control problem is given by

given $\Omega, \Delta t = T/N, (u_0, B_0) \in V$, *and* $(u_d, B_d) \in \mathcal{U}_{ad}$,
find $(\hat{\mathbf{u}}_h, \hat{\mathbf{B}}_h, \hat{\mathbf{p}}_h, \hat{\boldsymbol{\varphi}}_h, \operatorname{curl}\hat{\boldsymbol{\psi}}_h) \in V^h_{0\nu} \times S^h_0 \times S^h \times S^h$
such that (4.2a) − (4.2b) *are satisfied and the functional* (4.3) *is minimized.*

The existence and convergence of the solution of the fully discrete optimal control problem can be proved in the same way as for the semidiscrete case if we limit our analysis to conforming finite element approximations. With the same technique used in the case of the fully discrete approximations of the Navier-Stokes equations (see [28]) it can be proved that the optimal solution $(\hat{\mathbf{u}}_h, \hat{\mathbf{B}}_h, \hat{\mathbf{p}}_h, \hat{\boldsymbol{\varphi}}_h, \operatorname{curl}\hat{\boldsymbol{\psi}}_h)$ converges to the optimal solution $(\hat{u}, \hat{B}, \hat{p}, \hat{\varphi}, \operatorname{curl}\hat{\psi})$ of the continuous problem as $\Delta t \to 0$ and $h \to 0$. The necessary conditions of optimality can be found using the same techniques that were used for the semidiscrete-in-time case. For completeness we state the theorem that gives the control as a solution of an adjoint problem.

THEOREM 4.1. *Given* $\Delta t = T/N$, $(u_0, B_0) \in V$ *and* $(u_d, B_d) \in \mathcal{U}_{ad}$. *Let* $(\hat{\mathbf{u}}_h, \hat{\mathbf{B}}_h, \hat{\mathbf{p}}_h, \hat{\boldsymbol{\varphi}}_h, \operatorname{curl}\hat{\boldsymbol{\psi}}_h)$ *denote a solution of the fully discrete optimal control problem. Then we have*

$$\beta_1 \hat{\varphi}_h^{(n)} = -w_h^{(n-1)}, \quad \beta_2 \operatorname{curl}\hat{\psi}_h^{(n)} = -D_h^{(n-1)} \qquad \forall n = 1, 2, \ldots, N, \tag{4.4}$$

where the functions $(w_h^{(n)}, D_h^{(n)}) \in V^h_{0\nu}$, $r_h^{(n)} \in S^h_0$, $n = 0, 1, \ldots, N$, *satisfy the adjoint system*

$$\begin{aligned}
&-\frac{1}{\Delta t}\Big((w_h^{(n)}, D_h^{(n)}) - (w_h^{(n-1)}, D_h^{(n-1)}), \mathcal{S}(v_h, C_h)\Big) \qquad (4.5a)\\
&\quad + a((v_h, C_h), (w_h^{(n-1)}, D_h^{(n-1)})) + \widetilde{c}((v_h, C_h), (\hat{u}_h^{(n)}, \hat{B}_h^{(n)}), (w_h^{(n-1)}, D_h^{(n-1)}))\\
&\quad + \widetilde{c}((\hat{u}_h^{(n)}, \hat{B}_h^{(n)}), (v_h, C_h), (w_h^{(n-1)}, D_h^{(n-1)})) + b((v_h, C_h), r_h^{(n-1)})\\
&\quad = \Big((\alpha_1(\hat{u}_h^{(n)} - u_d^{(n)}), \alpha_2(\hat{B}_h^{(n)} - B_d^{(n)})), \mathcal{S}(v_h, C_h)\Big) \qquad \forall (v_h, C_h) \in V^h_{0\nu},
\end{aligned}$$

$$b((w_h^{(n-1)}, D_h^{(n-1)}), q_h) = 0 \qquad \forall q_h \in S^h_0(\Omega), \tag{4.5b}$$

for $n = 1, 2, \ldots, N$ *along with the terminal condition* $(w_h^{(N)}, D_h^{(N)}) = (0, 0)$.

5. A gradient method. Since the discrete MHD system (4.2a)-(4.2b) marches forward in time starting from an initial condition, and the adjoint system (4.5a)-(4.5b) marches backward in time from a terminal condition, any practical algorithm would involve a split of the optimality system into two parts.

We consider now a gradient method for the solution of the fully discrete optimal control problem. At each iteration the method requires sequential solution of (4.2a)-(4.2b) and (4.5a)-(4.5b).

Let $\mathcal{J}_h^N(k) = \mathcal{J}_h^N(\mathbf{u}_h(k), \mathbf{B}_h(k), \boldsymbol{\varphi}_h(k), \operatorname{curl}\boldsymbol{\psi}_h(k))$, where $\mathcal{J}_h^N(\cdot,\cdot,\cdot,\cdot)$ is given by (4.3), k is the iteration counter of the gradient algorithm, and $(\boldsymbol{\varphi}_h(k), \operatorname{curl}\boldsymbol{\psi}_h(k))$ is the kth iterate for the optimal control. We will use a prescribed tolerance ε to test the convergence of the functional.

The Gradient Algorithm

(a) initialization:

(i) choose ε and $(\boldsymbol{\varphi}_h(0), \operatorname{curl}\boldsymbol{\psi}_h(0))$; set $k = 0$ and $\lambda = 1$;

(ii) get $(\mathbf{u}_h(0), \mathbf{B}_h(0))$ by solving (4.2a)-(4.2b) with

$$(\boldsymbol{\varphi}_h, \operatorname{curl}\boldsymbol{\psi}_h) = (\boldsymbol{\varphi}_h(0), \operatorname{curl}\boldsymbol{\psi}_h(0))$$

(iii) evaluate $J^N(0)$;

(b) main loop:

(iv) set $k = k + 1$;

(v) get $(\mathbf{w}_h(k), \mathbf{D}_h(k))$ from (4.5a)-(4.5b) with

$$(\mathbf{u}_h, \mathbf{B}_h) = (\mathbf{u}_h(k-1), \mathbf{B}_h(k-1));$$

(vi) set $\begin{cases} \boldsymbol{\varphi}_h(k) = \boldsymbol{\varphi}_h(k-1) - \lambda(\beta_1 \boldsymbol{\varphi}_h(k-1) + \mathbf{w}_h(k)), \\ \operatorname{curl} \boldsymbol{\psi}_h(k) = \operatorname{curl} \boldsymbol{\psi}_h(k-1)) - \lambda(\beta_2 \operatorname{curl} \boldsymbol{\psi}_h(k-1) + \mathbf{D}_h(k)); \end{cases}$

(vii) get $(\mathbf{u}_h(k), \mathbf{B}_h(k))$ from (4.2a)-(4.2b) with

$$(\boldsymbol{\varphi}_h, \operatorname{curl} \boldsymbol{\psi}_h) = (\boldsymbol{\varphi}_h(k), \operatorname{curl} \boldsymbol{\psi}_h(k));$$

(viii) evaluate $\mathcal{J}_h^N(k)$;

(ix) if $\mathcal{J}_h^N(k) \geq \mathcal{J}_h^N(k-1)$, set $\lambda = \lambda/2$ and go to (vi); otherwise continue;

(x) if $|\mathcal{J}_h^N(k) - \mathcal{J}_h^N(k-1)|/|\mathcal{J}_h^N(k)| > \varepsilon$, set $\lambda = 1.5\lambda$ and go to (iv); otherwise stop.

The bulk of the computational costs are found in the backward-in-time solution of the discrete adjoint system in step (v) and in the forward-in-time solution of the discrete state system in step (vii).

The convergence property of the gradient algorithm is given in the following result.

THEOREM 5.1. *Let* $(\mathbf{u}_h(k), \mathbf{B}_h(k), \mathbf{p}_h(k), \mathbf{w}_h(k), \mathbf{D}_h(k), \mathbf{r}_h(k), \boldsymbol{\varphi}_h(k), \operatorname{curl} \boldsymbol{\psi}_h(k))$ *denote the k-th iterate of the gradient algorithm and let* $(\hat{\mathbf{u}}_h, \hat{\mathbf{B}}_h, \hat{\mathbf{p}}_h, \hat{\mathbf{w}}_h, \hat{\mathbf{D}}_h, \hat{\mathbf{r}}_h, \hat{\boldsymbol{\varphi}}_h, \operatorname{curl} \hat{\boldsymbol{\psi}}_h)$ *denote the solution of the fully discrete optimality system (4.2a)-(4.2b) and (4.5a)-(4.5b). If* $\min\{\beta_1, \beta_2\}/\max\{\alpha_1, \alpha_2\}$ *is large enough, then there exists a ball* $\mathcal{B}$ *in* $L^2(0,T;V_{0\nu}^h)$ *whose radius depends on* $\alpha_1, \alpha_2, \beta_1, \beta_2$*, such that if* $(\boldsymbol{\varphi}_h(0), \operatorname{curl} \boldsymbol{\psi}_h(0)) \in \mathcal{B}$*, then*

$$(\mathbf{u}_h(k), \mathbf{B}_h(k), \mathbf{p}_h(k), \mathbf{w}_h(k), \mathbf{D}_h(k), \mathbf{r}_h(k), \boldsymbol{\varphi}_h(k), \operatorname{curl} \boldsymbol{\psi}_h(k)) \longrightarrow (\hat{\mathbf{u}}_h, \hat{\mathbf{B}}_h, \hat{\mathbf{p}}_h, \hat{\mathbf{w}}_h, \hat{\mathbf{D}}_h, \hat{\mathbf{r}}_h, \hat{\boldsymbol{\varphi}}_h, \operatorname{curl} \hat{\boldsymbol{\psi}}_h)$$

as $k \to \infty$.

Proof. First we recall the following classical result [1, 5]. Let X be a Hilbert space with norm $\|\cdot\|$, and $\mathcal{J}$ a real-valued functional on X. Suppose that $\mathcal{J}$ is of class $\mathcal{C}^2$, that $\hat{x}$ is a local minimizer of $\mathcal{J}$, and there exist two real numbers c_1 and c_2 and a ball $\mathcal{B} \subset X$ centered in $\hat{x}$ such that for all $x \in \mathcal{B}$ and $\delta x_1, \delta x_2 \in X$ we have that

$$\mathcal{J}''(x)(\delta x_1, \delta x_2) \leq c_1 \|\delta x_1\| \|\delta x_2\| \qquad \text{and} \qquad c_2 \|\delta x_1\|^2 \leq \mathcal{J}''(x)(\delta x_1, \delta x_1), \tag{5.1}$$

where $\mathcal{J}''(x)(\delta x_1, \delta x_2)$ is the bilinear form associated with the second derivatives of $\mathcal{J}$. Then the iterates of the gradient algorithm converge to $\hat{x}$ for any initial iterate $x(0) \in \mathcal{B}$.

Now let $\Delta t = T/N$. For each $(\boldsymbol{\varphi}_h, \operatorname{curl} \boldsymbol{\psi}_h) \in L^2(0,T;V_{0\nu}^h)$, let $(\mathbf{u}_h, \mathbf{B}_h) = ((\mathbf{u}_h(\boldsymbol{\varphi}_h, \operatorname{curl} \boldsymbol{\psi}_h),\ \mathbf{B}_h(\boldsymbol{\varphi}_h, \operatorname{curl} \boldsymbol{\psi}_h))$ denote the solution of (4.2a)-(4.2b), for $n = 1, 2, \ldots, N$ and initial data $(u_h^{(0)}, B_h^{(0)})$. Then the first variations

$$(\mathbf{u}_{ih}, \mathbf{B}_{ih}) = (D(\mathbf{u}_h, \mathbf{B}_h)/D(\boldsymbol{\varphi}_h, \operatorname{curl} \boldsymbol{\psi}_h)) \cdot \delta(\boldsymbol{\varphi}_{ih}, \operatorname{curl} \boldsymbol{\psi}_{ih}), \quad i = 1, 2,$$

are solutions of $(u_{ih}^{(0)}, B_{ih}^{(0)}) = (0,0)$ and

$$\begin{aligned}
&\frac{1}{\Delta t}\Big((u_{ih}^{(n)} - u_{ih}^{(n-1)}, B_{ih}^{(n)} - B_{ih}^{(n-1)}), \mathcal{S}(v_h, C_h)\Big) + a((u_{ih}^{(n)}, B_{ih}^{(n)}), (v_h, C_h)) \qquad (5.2)\\
&\quad + \widetilde{c}((u_{ih}^{(n)}, B_{ih}^{(n)}), (u_h^{(n)}, B_h^{(n)}), (v_h, C_h)) + \widetilde{c}((u_h^{(n)}, B_h^{(n)}), (u_{ih}^{(n)}, B_{ih}^{(n)}), (v_h, C_h))\\
&\quad + b((v_h, C_h), p_{ih}^n) = \Big(\delta(\varphi_{ih}^{(n)}, \operatorname{curl}\psi_{ih}^{(n)}), \mathcal{S}(v_h, C_h)\Big) \quad \forall (v_h, C_h) \in V_{0\nu}^h,\\
&b((u_{ih}^{(n)}, B_{ih}^{(n)}), q_h) = 0 \quad \forall q_h \in S_0^h,
\end{aligned}$$

for $n = 1, 2, \ldots, N$, respectively.
The second variation $(\tilde{\mathbf{u}}_h, \tilde{\mathbf{B}}_h) = (D^2(\mathbf{u}_h, \mathbf{B}_h)/D(\boldsymbol{\varphi}_h, \operatorname{curl}\boldsymbol{\psi}_h)^2) \cdot \delta(\boldsymbol{\varphi}_{1h}, \operatorname{curl}\boldsymbol{\psi}_{1h}) \cdot \delta(\boldsymbol{\varphi}_{2h}, \operatorname{curl}\boldsymbol{\psi}_{2h})$, is a solution of $(\tilde{u}_h^{(0)}, \tilde{B}_h^{(0)}) = (0,0)$ and

$$\begin{aligned}
&\frac{1}{\Delta t}\Big((\tilde{u}_h^{(n)} - \tilde{u}_h^{(n-1)}, \tilde{B}_h^{(n)} - \tilde{B}_h^{(n-1)}), \mathcal{S}(v_h, C_h)\Big) + a((\tilde{u}_h^{(n)}, \tilde{B}_h^{(n)}), (v_h, C_h)) \qquad (5.3)\\
&\quad + \widetilde{c}((\tilde{u}_h^{(n)}, \tilde{B}_h^{(n)}), (u_h^{(n)}, B_h^{(n)}), (v_h, C_h)) + \widetilde{c}((u_h^{(n)}, B_h^{(n)}), (\tilde{u}_h^{(n)}, \tilde{B}_h^{(n)}), (v_h, C_h))\\
&\quad + b((v_h, C_h), \tilde{p}_h^n) = -\widetilde{c}((u_{1h}^{(n)}, B_{1h}^{(n)}), (u_{2h}^{(n)}, B_{2h}^{(n)}), (v_h, C_h))\\
&\quad - \widetilde{c}((u_{2h}^{(n)}, B_{2h}^{(n)}), (u_{1h}^{(n)}, B_{1h}^{(n)}), (v_h, C_h)) \quad \forall (v_h, C_h) \in V_{0\nu}^h,\\
&b((\tilde{u}_h^{(n)}, \tilde{B}_h^{(n)}), q_h) = 0 \quad \forall q_h \in S_0^h
\end{aligned}$$

for $n = 1, 2, \ldots, N$.
Hence the second Fréchet derivative of $\mathcal{J}_h^N(\mathbf{u}_h, \mathbf{B}_h, \boldsymbol{\varphi}_h, \operatorname{curl}\boldsymbol{\psi}_h)$ is given by

$$\begin{aligned}
&D^2\mathcal{J}_h^N(\mathbf{u}_h, \mathbf{B}_h, \boldsymbol{\varphi}_h, \operatorname{curl}\boldsymbol{\psi}_h) \cdot \delta(\boldsymbol{\varphi}_{1h}, \operatorname{curl}\boldsymbol{\psi}_{1h}) \cdot \delta(\boldsymbol{\varphi}_{2h}, \operatorname{curl}\boldsymbol{\psi}_{2h})\\
&\quad = \Delta t \sum_{n=1}^{N} \int_\Omega \Big(\alpha_1 u_{1h}^{(n)} \cdot u_{2h}^{(n)} + \alpha_2 S B_{1h}^{(n)} \cdot B_{2h}^{(n)} + \beta_1\, \delta\varphi_{1h}^{(n)} \cdot \delta\varphi_{2h}^{(n)}\\
&\qquad + \beta_2 S \delta\operatorname{curl}\psi_{1h}^{(n)} \cdot \delta\operatorname{curl}\psi_{2h}^{(n)}\Big)\, dx\\
&\quad + \Delta t \sum_{n=1}^{N} \int_\Omega \Big(\alpha_1 (u_h^{(n)} - u_d^{(n)}) \cdot \tilde{u}_h^{(n)} + \alpha_2 S (B_h^{(n)} - B_d^{(n)}) \cdot \tilde{B}_h^{(n)}\Big)\, dx.
\end{aligned}$$

Also denote by $(w_h^{(n)}, D_h^{(n)}) \in V_{0\nu}^h$, $r_h^{(n)} \in S_0^h$ the solution of the adjoint system (4.5a)-(4.5b) with the terminal condition $(w_h^{(N)}, D_h^{(N)}) = (0,0)$. Then

$$\begin{aligned}
&D^2\mathcal{J}_h^N(\mathbf{u}_h, \mathbf{B}_h, \boldsymbol{\varphi}_h, \operatorname{curl}\boldsymbol{\psi}_h) \cdot \delta(\boldsymbol{\varphi}_{1h}, \operatorname{curl}\boldsymbol{\psi}_{1h}) \cdot \delta(\boldsymbol{\varphi}_{2h}, \operatorname{curl}\boldsymbol{\psi}_{2h}) \qquad (5.4)\\
&\quad = \Delta t \sum_{n=1}^{N} \Big(\alpha_1 (u_{1h}^{(n)}, u_{2h}^{(n)}) + \alpha_2 (S B_{1h}^{(n)}, B_{2h}^{(n)}) + \beta_1 (\delta\varphi_{1h}^{(n)}, \delta\varphi_{2h}^{(n)})\\
&\quad + \beta_2 (S \delta\operatorname{curl}\psi_{1h}^{(n)}, \delta\operatorname{curl}\psi_{2h}^{(n)}) - \widetilde{c}((u_{1h}^{(n)}, B_{1h}^{(n)}), (u_{2h}^{(n)}, B_{2h}^{(n)}), (w_h^{(n-1)}, D_h^{(n-1)}))\\
&\quad - \widetilde{c}((u_{2h}^{(n)}, B_{2h}^{(n)}), (u_{1h}^{(n)}, B_{1h}^{(n)}), (w_h^{(n-1)}, D_h^{(n-1)}))\Big).
\end{aligned}$$

Let $(\boldsymbol{\varphi}_h, \operatorname{curl}\boldsymbol{\psi}_h)$ be the initial guess in the ball $\mathcal{B}$ of radius ξ, i.e., $\|(\hat{\boldsymbol{\varphi}}_h, \operatorname{curl}\hat{\boldsymbol{\psi}}_h) - (\boldsymbol{\varphi}_h, \operatorname{curl}\boldsymbol{\psi}_h)\| = \rho \le \xi$. Now we have to show that there exists a ξ such that $\forall \rho \le \xi$, (5.1) is satisfied for some c_1 and c_2.

If in (4.2a) we take $(v_h, C_h) = (u^{(n)}, B^{(n)})$, multiply by $2\Delta t$, we obtain

$$\begin{aligned}&\|u_h^{(n)}\|_0^2 + S\|B_h^{(n)}\|_0^2 + \Delta t\frac{1}{\mathrm{Re}}\|u_h^{(n)}\|_1^2 + \Delta t\frac{S}{\mathrm{Re_m}}\|\mathrm{curl}\, B_h^{(n)}\|_0^2\\ &\quad\le \|u_h^{(n-1)}\|_0^2 + S\|B_h^{(n-1)}\|_0^2 + \Delta t\lambda_1^{-1}\mathrm{Re}\|\varphi_h^{(n)}\|_0^2 + \Delta t\lambda_2^{-1}\mathrm{Re_m}S\|\mathrm{curl}\,\psi_h^{(n)}\|_0^2.\end{aligned}$$

When we summ from 1 to n we get

$$\begin{aligned}&\|u_h^{(n)}\|_0^2 + S\|B_h^{(n)}\|_0^2 + \Delta t\frac{1}{\mathrm{Re}}\sum_{m=1}^{n}\|u_h^{(m)}\|_1^2 + \Delta t\frac{S}{\mathrm{Re_m}}\sum_{m=1}^{n}\|\mathrm{curl}\, B_h^{(m)}\|_0^2\\ &\quad\le \|u_h^{(0)}\|_0^2 + S\|B_h^{(0)}\|_0^2 + \Delta t\lambda_1^{-1}\mathrm{Re}\sum_{m=1}^{n}\|\varphi_h^{(m)}\|_0^2 + \Delta t\lambda_2^{-1}\mathrm{Re_m}S\sum_{m=1}^{n}\|\mathrm{curl}\,\psi_h^{(m)}\|_0^2\end{aligned} \tag{5.5}$$

and without any restrictions on Δt we have

$$\frac{1}{\mathrm{Re}}\|\mathbf{u}_h\|_1^2 + \frac{S}{\mathrm{Re_m}}\|\mathrm{curl}\,\mathbf{B}_h\|_0^2 \le f_1(\rho),$$

where $f_1(\cdot)$ is a continuous function. If in (5.2) we choose $(v_h, C_h) = (u_{ih}^{(n)}, B_{ih}^{(n)})$ we get

$$\begin{aligned}&\|u_{ih}^{(n)}\|_0^2 + S\|B_{ih}^{(n)}\|_0^2 + \Delta t\frac{1}{\mathrm{Re}}\|u_{ih}^{(n)}\|_1^2 + \Delta t\frac{S}{\mathrm{Re_m}}\|\mathrm{curl}\, B_{ih}^{(n)}\|_0^2\\ &\quad\le \|u_{ih}^{(n-1)}\|_0^2 + S\|B_{ih}^{(n-1)}\|_0^2 + 2\Delta t\mathrm{Re}\lambda_1^{-1}\|\delta\varphi_{ih}^{(n)}\|_0^2 + 2\Delta t\mathrm{Re_m}S\lambda_2^{-1}\|\delta\mathrm{curl}\,\psi_{ih}^{(n)}\|_0^2\\ &\quad\quad + 2\Delta t K_2\left(\|u_{ih}^{(n)}\|_0^2 + S\|B_{ih}^{(n)}\|_0^2\right)\left(\frac{1}{\mathrm{Re}}\|u_h^{(n)}\|_1^2 + \frac{S}{\mathrm{Re_m}}\|\mathrm{curl}\, B_h^{(n)}\|_0^2\right)\end{aligned}$$

where $K_1 = K_0(\max\{1, 1/S\})^{1/2}(\max\{1, \mathrm{Re}, \mathrm{Re_m}/S\})^{1/2}$, $K_2 := K_1^2$.
If Δt satisfies the restriction

$$\Delta t \le \frac{1}{4K_2 f_1(\rho)},$$

then

$$\begin{aligned}&\frac{1}{\mathrm{Re}}\|\mathbf{u}_{ih}\|_1^2 + \frac{S}{\mathrm{Re_m}}\|\mathrm{curl}\,\mathbf{B}_{ih}\|_0^2\\ &\quad\le 2e^{4K_2\left(\frac{1}{\mathrm{Re}}\|\mathbf{u}_h\|_1^2 + \frac{S}{\mathrm{Re_m}}\|\mathrm{curl}\,\mathbf{B}_h\|_0^2\right)}\left(\mathrm{Re}\lambda_1^{-1}\|\delta\boldsymbol{\varphi}_{ih}\|_0^2 + \mathrm{Re_m}S\lambda_2^{-1}\|\delta\mathrm{curl}\,\boldsymbol{\psi}_{ih}\|_0^2\right).\end{aligned} \tag{5.6}$$

By taking $(v_h, C_h) = (w_h^{(n-1)}, D_h^{(n-1)})$ in (4.5a), in the same manner as above we obtain

$$\begin{aligned}&\frac{1}{\mathrm{Re}}\|\mathbf{w}_h\|_1^2 + \frac{S}{\mathrm{Re_m}}\|\mathrm{curl}\,\mathbf{D}_h\|_0^2\\ &\quad\le 2e^{4K_2\left(\frac{1}{\mathrm{Re}}\|\mathbf{u}_h\|_1^2 + \frac{S}{\mathrm{Re_m}}\|\mathrm{curl}\,\mathbf{B}_h\|_0^2\right)}\left(\mathrm{Re}\,\alpha_1^2\lambda_1^{-1}\|\mathbf{u} - \mathbf{u}_d\|_0^2 + \mathrm{Re_m}S\alpha_2^2\lambda_2^{-1}\|\mathbf{B}_h - \mathbf{B}_d\|_0^2\right).\end{aligned} \tag{5.7}$$

Now from (5.4) we obtain

$$
\begin{aligned}
&D^2\mathcal{J}_h^N(\mathbf{u}_h,\mathbf{B}_h,\boldsymbol{\varphi}_h,\operatorname{curl}\boldsymbol{\psi}_h)\,(\delta(\boldsymbol{\varphi}_{1h},\operatorname{curl}\boldsymbol{\psi}_{1h}),\delta(\boldsymbol{\varphi}_{2h},\operatorname{curl}\boldsymbol{\psi}_{2h}))\\
&\le \left(\alpha_1\|\mathbf{u}_{1h}\|_0^2+\alpha_2 S\|\mathbf{B}_{1h}\|_0^2\right)^{1/2}\left(\alpha_2\|\mathbf{u}_{2h}\|_0^2+\alpha_2 S\|\mathbf{B}_{2h}\|_0^2\right)^{1/2}\\
&+\left(\beta_1\|\delta\boldsymbol{\varphi}_{1h}\|_0^2+\beta_2 S\|\delta\operatorname{curl}\boldsymbol{\psi}_{1h}\|_0^2\right)^{1/2}\cdot\left(\beta_1\|\delta\boldsymbol{\varphi}_{2h}\|_0^2+\beta_2 S\|\delta\operatorname{curl}\boldsymbol{\psi}_{2h}\|_0^2\right)^{1/2}\\
&+\Delta t K_1\left[\sum_{n=1}^N\left(\|u_{1h}^{(n)}\|_0^2+S\|B_{1h}^{(n)}\|_0^2\right)\left(\frac{1}{\mathrm{Re}}\|u_{1h}^{(n)}\|_1^2+\frac{S}{\mathrm{Re_m}}\|\operatorname{curl}B_{1h}^{(n)}\|_0^2\right)\right]^{1/4}\\
&\qquad\cdot\left[\sum_{n=1}^N\left(\|u_{2h}^{(n)}\|_0^2+S\|B_{2h}^{(n)}\|_0^2\right)\left(\frac{1}{\mathrm{Re}}\|u_{2h}^{(n)}\|_1^2+\frac{S}{\mathrm{Re_m}}\|\operatorname{curl}B_{2h}^{(n)}\|_0^2\right)\right]^{1/4}\\
&\qquad\cdot\left[\sum_{m=1}^N\left(\frac{1}{\mathrm{Re}}\|w_h^{(n-1)}\|_1^2+\frac{S}{\mathrm{Re_m}}\|\operatorname{curl}D_h^{(n-1)}\|_0^2\right)\right]^{1/2}.
\end{aligned}
$$

From (4.3), (5.5), (5.6), and (5.7) we get then

$$
\begin{aligned}
&D^2J_h^N(\mathbf{u}_h,\mathbf{B}_h,\boldsymbol{\varphi}_h,\operatorname{curl}\boldsymbol{\psi}_h)\,(\delta(\boldsymbol{\varphi}_{1h},\operatorname{curl}\boldsymbol{\psi}_{1h}),\delta(\boldsymbol{\varphi}_{2h},\operatorname{curl}\boldsymbol{\psi}_{2h}))\\
&\qquad\le c_1\|\delta\left(\boldsymbol{\varphi}_{1h},\operatorname{curl}\boldsymbol{\psi}_{1h}\right)\|_0\|\delta\left(\boldsymbol{\varphi}_{2h},\operatorname{curl}\boldsymbol{\psi}_{2h}\right)\|_0.
\end{aligned}
$$

On the other hand we have

$$
\begin{aligned}
&D^2J_h^N(\mathbf{u}_h,\mathbf{B}_h,\boldsymbol{\varphi}_h,\operatorname{curl}\boldsymbol{\psi}_h)\,(\delta(\boldsymbol{\varphi}_{1h},\operatorname{curl}\boldsymbol{\psi}_{1h}),\delta(\boldsymbol{\varphi}_{1h},\operatorname{curl}\boldsymbol{\psi}_{1h}))\\
&\ge\Delta t\sum_{n=1}^N\left[\alpha_1\|u_{1h}^{(n)}\|_0^2+\alpha_2 S\|B_{1h}^{(n)}\|_0^2+\beta_1\|\delta\varphi_{1h}^{(n)}\|_0^2+\beta_2 S\|\delta\operatorname{curl}\psi_{1h}^{(n)}\|_0^2\right]\\
&-2K_1\left[\sum_{n=1}^N\left(\|u_{1h}^{(n)}\|_0^2+S\|B_{1h}^{(n)}\|_0^2\right)\Delta t\left(\frac{1}{\mathrm{Re}}\|u_{1h}^{(n)}\|_1^2+\frac{S}{\mathrm{Re_m}}\|\operatorname{curl}B_{1h}^{(n)}\|_0^2\right)\right]^{1/2}\\
&\qquad\cdot\left[\Delta t\sum_{n=1}^N\left(\frac{1}{\mathrm{Re}}\|w_h^{(n-1)}\|_1^2+\frac{S}{\mathrm{Re_m}}\|\operatorname{curl}D_h^{(n-1)}\|_0^2\right)\right]^{1/2},
\end{aligned}
$$

and then by (5.6) and (5.7) we obtain

$$
\begin{aligned}
&D^2J_h^N(\mathbf{u}_h,\mathbf{B}_h,\boldsymbol{\varphi}_h,\operatorname{curl}\boldsymbol{\psi}_h)\,(\delta(\boldsymbol{\varphi}_{1h},\operatorname{curl}\boldsymbol{\psi}_{1h}),\delta(\boldsymbol{\varphi}_{1h},\operatorname{curl}\boldsymbol{\psi}_{1h}))\\
&\quad\ge\|\delta\boldsymbol{\varphi}_{1h}\|_0^2\Big[\beta_1-2^{5/2}K_1\mathrm{Re}\lambda_1^{-1}e^{6K_2f_1(\rho)}\left(\mathrm{Re}\alpha_1^2\lambda_1^{-1}\|\mathbf{u}_h-\mathbf{u}_d\|_0^2\right.\\
&\qquad\qquad\left.+\mathrm{Re_m}S\alpha_2^2\lambda_2^{-1}\|\mathbf{B}_h-\mathbf{B}_d\|^2\right)^{1/2}\Big]\\
&\quad+S\|\delta\operatorname{curl}\boldsymbol{\psi}_{1h}\|_0^2\Big[\beta_2-2^{5/2}K_1\mathrm{Re_m}\lambda_2^{-1}e^{6K_2f_1(\rho)}\left(\mathrm{Re}\alpha_1^2\lambda_1^{-1}\|\mathbf{u}_h-\mathbf{u}_d\|_0^2\right.\\
&\qquad\qquad\left.+\mathrm{Re_m}S\alpha_2^2\lambda_2^{-1}\|\mathbf{B}_h-\mathbf{B}_d\|^2\right)^{1/2}\Big]\\
&\quad\ge\|\delta(\boldsymbol{\varphi}_{1h},\operatorname{curl}\boldsymbol{\psi}_{1h})\|_0^2\left[\min\{\beta_1,\beta_2\}-\max\{\alpha_1,\alpha_2\}f_2(\rho)\right],
\end{aligned}
$$

where $f_2(\cdot)$ is a continuous function. It follows that, for small enough values of the ratio $\max\{\alpha_i\}/\min\{\beta_i\}$, we can choose ξ such that $\forall\|(\hat{\boldsymbol{\varphi}}_h,\operatorname{curl}\hat{\boldsymbol{\psi}}_h)-(\boldsymbol{\varphi}_h,\operatorname{curl}\boldsymbol{\psi}_h)\|=\rho\le\xi$ there exists $c_2>0$ satisfying

$$
\begin{aligned}
&D^2J_h^N(\mathbf{u}_h,\mathbf{B}_h,\boldsymbol{\varphi}_h,\operatorname{curl}\boldsymbol{\psi}_h)\,(\delta(\boldsymbol{\varphi}_{1h},\operatorname{curl}\boldsymbol{\psi}_{1h}),\delta(\boldsymbol{\varphi}_{1h},\operatorname{curl}\boldsymbol{\psi}_{1h}))\\
&\qquad\ge c_2\|\delta\left(\boldsymbol{\varphi}_{1h},\operatorname{curl}\boldsymbol{\psi}_{1h}\right)\|_0^2.
\end{aligned}\tag{5.8}
$$

□

6. Computational experiments. The optimality system presents a formidable computational challenge even for relatively simple geometries and relatively coarse grids. It involves the forward-in-time MHD system coupled with the backward-in-time adjoint system.

One can split the optimality system into two constituent systems of equations. At any step of the iteration, the MHD system can be solved by marching forward in time from the initial condition; at any time the body force must be determined, through the gradient algorithm, from the adjoint solution obtained in the previous step. At every time step, the nonlinearity of the MHD system is treated with the Newton method. The adjoint equations can then be solved marching backward in time, starting from the final condition, which along with the right-hand side and coefficients, are determined from the MHD solution. Thus, both the state and adjoint variables must be available and stored over the entire space-time domain. The iterative procedure, i.e., the forward MHD sweeps followed by the backward adjoints sweeps, is repeated until convergence is achieved. The projected gradient algorithm with variable step, described in the previous section, is a slowly convergent method and many iterations are necessary, even for simple test cases discussed below.

The solution globally consists of fourteen fields: five for the state of the fluid, five for the adjoint state, and four for the control. As mentioned above, this solution must be available over the whole time-space domain and cannot be stored in the computer memory, even for simple test cases that use a 20×20 spatial mesh and 100 time steps. Therefore, these fields must be stored out of the core and be accessible during the matrix and right-hand side assemblies.

We consider the unit square domain $(0,1) \times (0,1) \subset \mathbb{R}^2$. We assume that the time interval $[0,1]$ is divided into equal intervals of duration $\Delta t = 1/N$. The finite element spaces are chosen to be continuous piecewise biquadratic polynomials for the velocity and magnetic field, and piecewise bilinear polynomials for the pressure, i.e., the Taylor-Hood finite element pair, with respect to rectangular mesh. The mesh size is denoted by h, and calculations with varying mesh sizes have been performed. All the vector plots are normalized by the maximum values.

Test 1. In the first simulation we used only the current control to match our goal, with more emphasis on tracking the velocity field.

A simple stationary target velocity $u_d(x,y) = (u_d^1(x,y), u_d^2(x,y))$ and magnetic field target $B_d(x,y) = (B_d^1(x,y), B_d^2(x,y))$ are chosen; e.g.,

$$u_d^1 = 10\frac{d}{dy}\big(\phi(0.4,x)\phi(0.4,y)\big), \quad u_d^2 = -10\frac{d}{dx}\big(\phi(0.4,x)\phi(0.4,y)\big),$$
$$B_d^1 = \sin(\pi x)\cos(\pi y), \quad B_d^2 = \cos(\pi x)\sin(\pi y),$$

where $\phi(t,z) = (1-z)^2(1-\cos(2\pi t z))$. For the initial velocity we choose

$$u_0 = -u_d \quad \text{and} \quad v_0 = -v_d$$

so that the initial flow rotates in an opposite sense from the target flow; we choose a zero initial magnetic field.

The parameters in (4.3) are $\alpha_1 = 10, \alpha_2 = 1, \beta_1 = 0, \beta_2 = 0.01, h = 1/16$, $\Delta t = 0.005$, $Re = 200$, and $Re_m = 1$. The evolutions of the velocity and magnetic field are given in Figure 6.1 and Figure 6.2, respectively. In each figure the time

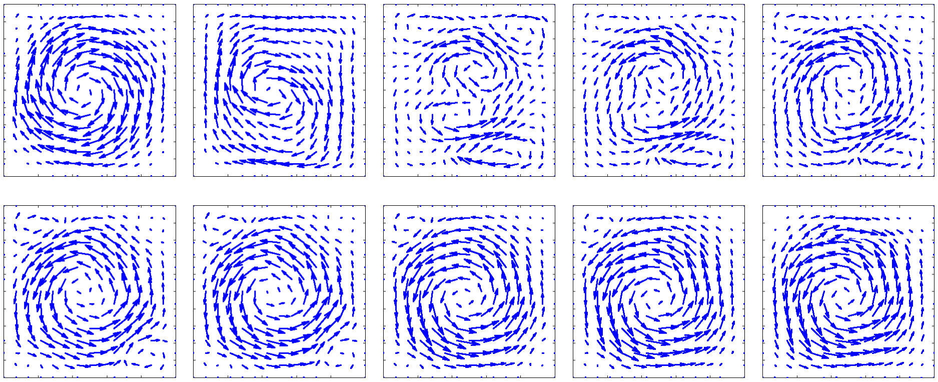

Fig. 6.1: Controlled velocity u at $t = 0, .06, .1, .13, .2, .4, .5, .75, .9$ and target velocity u_d

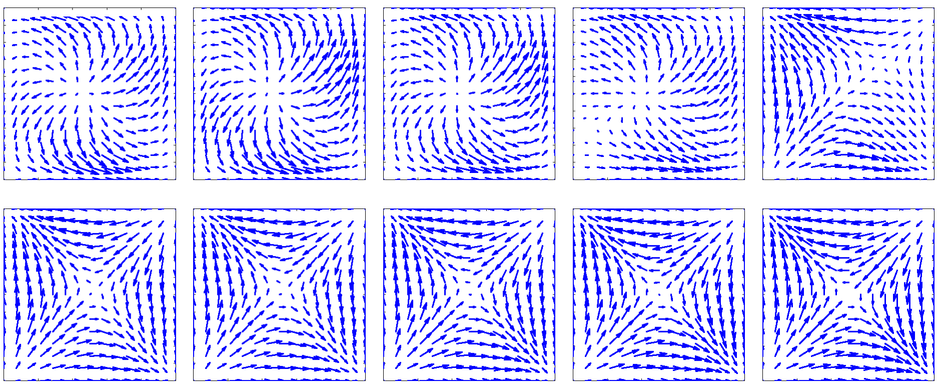

Fig. 6.2: Controlled magnetic field B at $t = .01, .06, .1, .13, .2, .4, .5, .75, .9$ and target B_d

evolution of the controlled fluid is on the top line and first four snapshots on the second line, and the steady desired flow is on the fifth position on the bottom line.

Test 2. In the second simulation we used both the distributed force controls and the current to track the velocity and magnetic field.

The target velocity is chosen (u_d^1, u_d^2) to be

$$u_d^1(t,x,y) = a(1,.4,x,y) + a(2,t,x,y)/(4\pi t + 1)$$
$$u_d^2(t,x,y) = b(1,.4,x,y) + b(2,t,x,y)/(4\pi t + 1)$$

where

$$\phi(\kappa,t,z) = (1-z)^2(1-\cos(2\kappa\pi t z)),$$
$$a(\kappa,t,x,y) = \frac{d}{dy}\left(\phi(\kappa,t,x)\phi(\kappa,t,y)\right),$$
$$b(\kappa,t,x,y) = -\frac{d}{dx}\left(\phi(\kappa,t,x)\phi(\kappa,t,y)\right).$$

This velocity field is a superposition of two flows, one having a vortex at the center of the domain and another with four vortices. Each of these flows prevails at different times of the evolution. The initial velocity for the controlled flow is chosen to be

$$u_0(x,y) = -u_d(.25,x,y) \qquad \text{and} \qquad v_0(x,y) = -v_d(.25,x,y)$$

so it has an opposite rotational sense and magnitude from that of the target flow. The magnetic field target and initial data are the same as in Test 1. The parameters are $\alpha_1 = 30$, $\alpha_2 = 1$, $\beta_1 = 1$, $\beta_2 = 0.001$, $h = 1/16$, $\Delta t = 0.01$, $Re = 200$, and $Re_m = 1$. The evolutions of the controlled velocity u and target velocity u_d are given in Figure 6.3, on the first and third line, respectively second and fourth line.

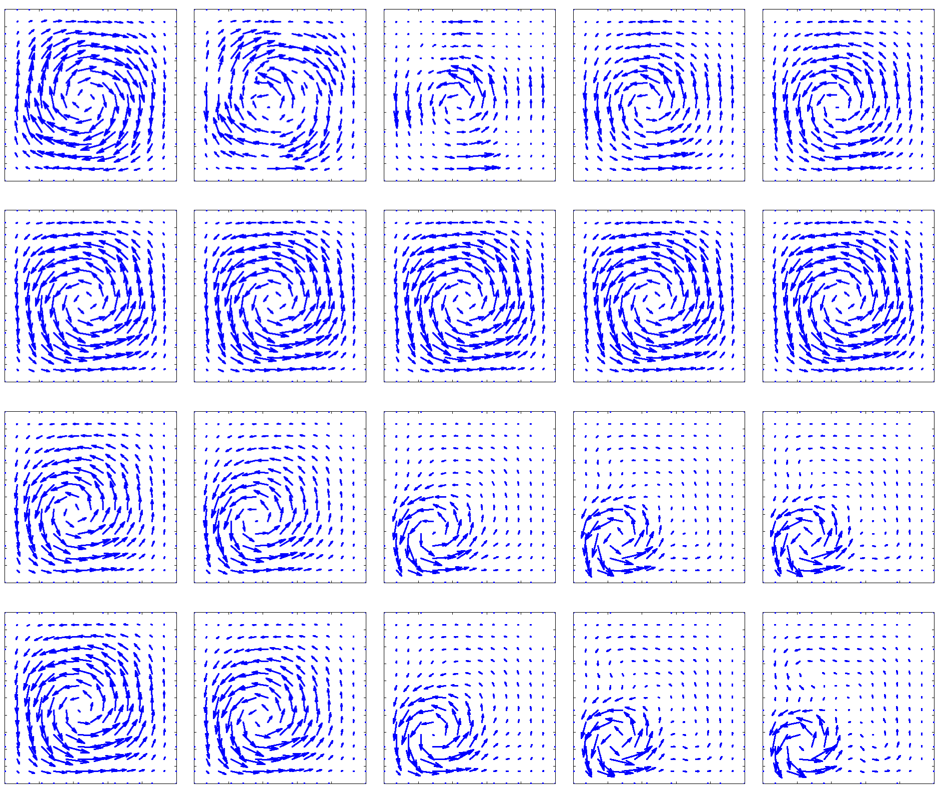

Fig. 6.3: Controlled velocity u and target velocity u_d at time steps $t = 0, .05, .1, .15, .2, .35, .5, .75, .9, 1$

REFERENCES

[1] F. Abergel and R. Temam, *On some control problems in fluid mechanics*, Theor. Comput. Fluid Dyn., **1** (1990), 303–325.

[2] R. Adams, *Sobolev spaces*, Academic Press, New York, London, 1975.

[3] G. Ambrosino and R. Albanese, *Magnetic control of plasma current, position, and shape in tokamaks: A survey of modeling and control approaches*, Control Syst. Mag. **25** (2005), no. 5, 76–92.

[4] T. Berger, J. Kim, C. Lee, and J. Lim, *Turbulent boundary layer control utilizing Lorentz force*, Phys. Fluids **12** (2000), 631649.

[5] P. Ciarlet, *Introduction to numerical linear algebra and optimization*, Cambridge University Press, Cambridge, UK, 1989.

[6] A. Gailitis and O. Lielausis, *On a possibility to reduce the hydrodynamical resistance of a plate in an electrolyte*, (in Russian), Applied Magnetohydrodynamics. Reports of the Physics Institute **12** (1961), 143–146.
[7] V. Girault and P.-A. Raviart, *Finite element method for Navier-Stokes equations*, Springer, Berlin, 1986.
[8] R. Glowinski, A. Kearsley, T. Pan, and J. Periaux, *Numerical simulation and optimal shape for viscous flow by a fictitious domain method*, Internat. J. Numer. Methods Fluids **20** (1995), no. 8-9, 695–711.
[9] R. Griesse and K. Kunisch, *Optimal control for a stationary MHD system in velocity-current formulation*, SIAM J. Control Optim. **45** (2006), no. 5, 1822–1845.
[10] M.D. Gunzburger, *Perspectives in Flow Control and Optimization*, Adv. Des. Control, vol. 5, SIAM, Philadelphia, 2002.
[11] M.D. Gunzburger, A.J. Meir, and J. Peterson, *On the existence, uniqueness, and finite element approximation of solutions of the equations of stationary, incompressible magnetohydrodynamics*, Math. Comp. **56** (1991), no. 194, 523–563.
[12] M.D. Gunzburger and J. Peterson, *On conforming finite element methods for the inhomogeneous stationary Navier-Stokes equations*, Numer. Math. **42** (1991), 173–194.
[13] M.D. Gunzburger and C. Trenchea, *Analysis and discretization of an optimal control problem for the time-periodic MHD equations*, J. Math. Anal. Appl. **308** (2005), no. 2, 440–466.
[14] ———, *Analysis of an optimal control problem for the three-dimensional coupled modified Navier-Stokes and Maxwell equations*, J. Math. Anal. Appl. **333** (2007), 295–310.
[15] C. Henoch and J. Stace, *Experimental investigations of a salt water turbulent boundary layer modified by an applied streamwise magnetohydrodynamic body force*, Phys. Fluids **7** (1995), 1371–1381.
[16] L. S. Hou and Meir A. J., *Boundary optimal control of MHD flows*, Appl. Math. Optim. **32** (1995), no. 2, 143–162.
[17] L. S. Hou and Peterson J., *Boundary optimal control for an electrically conducting fluid using boundary electrical potential controls*, Nonlinear Anal. **24** (1995), no. 6, 857–874.
[18] L. S. Hou and S. S. Ravindran, *Computations of boundary optimal control problems for an electrically conducting fluid*, J. Comput. Phys. **128** (1996), 319–330.
[19] A. Jameson, *Computational aerodynamics for aircraft design*, Science **245** (1989), 361–371.
[20] L. Landau and E. Lifschitz, *Électrodynamique des milieux continus*, Physique théorique, vol. tome VIII, MIR, Moscow, 1969.
[21] K. McManus, T. Poinsot, and S. Candel, *Review of active control of combustion instabilities*, Prog. Energy Comb. Sci. **19** (1993), 1–29.
[22] A. J. Meir and P. G. Schmidt, *A velocity-current formulation for stationary MHD flow*, Appl. Math. Comp. **65** (1994), 95–109.
[23] ———, *Variational methods for stationary MHD flow under natural interface conditions*, Nonlinear Anal. **26** (1996), 659–689.
[24] ———, *Analysis and numerical approximation of a stationary MHD flow problem with non-ideal boundary*, SIAM J. Numer. Anal. **36** (1999), no. 4, 1304–1332.
[25] S.S. Ravindran, *Real-time computational algorithm for optimal control of an MHD flow system*, SIAM J. Sci. Comput **26** (2005), no. 4, 1369–1388.
[26] M. Sermange and R. Temam, *Some mathematical questions related to the MHD equations*, Comm. Pure Appl. Math. **36** (1983), 635–664.
[27] N.S. Singh and P.R. Bandyopandhyay, *Linear feedback control on boundary layer using electromagnetic microtiles*, Trans. ASME J. Fluid Engrg. **119** (1997), 852–858.
[28] R. Temam, *Navier-Stokes Equations. Theory and Numerical Analysis*, North-Holland, Amsterdam, 1979.
[29] V. Tikhomirov, *Fundamental principles of the theory of extremal problems*, Wiley, Chichester, 1986.
[30] A. Tsinober, *MHD flow drag reduction*, Viscous Drag Reduction in Boundary Layers, (D.M. Busnell and J.N. Hefner, eds.), Progress in Astronautics and Aeronautics, vol. 123, American Institute of Astronautics and Aeronautics, Reston, VA, 1990, pp. 327–349.
[31] A. B. Tsinober and A. G. Shtern, *Possibility of increasing the flow stability in a boundary layer by means of crossed electric and magnetic fields*, Magnetohydrodynamics **3** (1967), no. 2, 152–154.
[32] J. C. Turner Jr., *Modelling control of crystal growth processes*, Comput. Math. Appl. **48** (2004), no. 7-8, 1231–1243.