# Stability of partitioned methods for magnetohydrodynamics flows at small magnetic Reynolds number

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Abstract. MHD flows are governed by the Navier-Stokes equations coupled with the Maxwell equations. Broadly, MHD flows in astrophysics occur at large magnetic Reynolds numbers while those in terrestrial applications, such as liquid metals, occur at small magnetic Reynolds numbers, the case considered herein. The physical processes of fluid flows and electricity and magnetism are quite different and numerical simulations of non-model problems can require different meshes, time steps and methods. We introduce implicit-explicit (IMEX) methods where the MHD equations can be evolved in time by calls to the NSE and Maxwell codes, each possibly optimized for the subproblem's respective physics.

## 1. Introduction

The MHD equations describe the motion of electrically conducting, incompressible flows in the presence of a magnetic field. If an electrically conducting fluid moves in a magnetic field, the magnetic field exerts forces which may substantially modify the flow. Conversely, the flow itself gives rise to a second, induced field and thus modifies the magnetic field. Initiated by Alfven in 1942 [1], MHD models occur in astrophysics, geophysics as well as engineering. Understanding these flows is central to many important applications, e.g., liquid metal cooling of nuclear reactors [2, 8, 19], sea water propulsion [12], process metallurgy [3].

The magnetic Reynolds number $R_m$ is an important parameter in MHD, being indicative of the relative strength of induced magnetic field and imposed magnetic field:

$$R_m = \frac{\text{Induced field}}{\text{Applied field}} = \mu\sigma u L.$$

Here $\mu$ is the permeability of free space, $\sigma$ is the electrical conductivity, $u$ and $L$ are the characteristic velocity and length scale correspondingly. In most industrial and laboratory flows, it is impossible to reach large values of velocity and length scale. Consequently, MHD flows in terrestrial applications typically occur at small magnetic Reynolds number. While the magnetic field considerably alters the fluid motion, the induced field is usually found to be negligible by comparison with

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the imposed field [4, 9, 17]. Neglecting the induced magnetic field reduces MHD models to the system (RMHD) below, which is studied herein.

Let $\Omega$ be a bounded, Lipschitz domain in $\mathbb{R}^d$ $(d = 3)$. With $\mathbf{f}$ and $\mathbf{B}$ known, the electrically conducting flow can be completely described in terms of the fluid velocity $\mathbf{u}$, pressure $p$ and electric potential $\phi$. The reduced MHD (RMHD) equations are given by, see, e.g., [16, 7, 21]: Given $\mathbf{f}$, $\mathbf{B}$ and time $T > 0$, find $\mathbf{u}$, $p$ and $\phi$ such that:

$$\tag{RMHD} \begin{aligned} \frac{1}{N}(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) - \frac{1}{M^2}\Delta\mathbf{u} + \nabla p &= \mathbf{f} + (\mathbf{B}\times\nabla\phi + \mathbf{B}\times(\mathbf{B}\times\mathbf{u}))\,, \\ \Delta\phi &= \nabla\cdot(\mathbf{u}\times\mathbf{B}), \text{ and } \nabla\cdot\mathbf{u} = 0. \end{aligned}$$

Here $M$, $N$ are the Hartman number and interaction parameter given by

$$M = BL\sqrt{\frac{\sigma}{\rho\nu}}, \quad N = \sigma B^2\frac{L}{\rho u}$$

where $B$ is the magnetic field, $\rho$ is the density, and $\nu$ is the kinematic viscosity, all assumed constant. (RMHD) is supplemented by the homogeneous Dirichlet boundary conditions

$$\mathbf{u} = 0,\ \phi = 0 \text{ on } \partial\Omega\times[0,T]$$

and the initial data

$$\tag{1.1} \mathbf{u}(x,0) = \mathbf{u}_0(x),\ \phi(x,0) = \phi_0(x)\ \forall x\in\Omega.$$

Constant parameters and simple boundary conditions allows us to focus on the uncoupling of RMHD into physical subprocesses.

MHD flows involve different physical processes: the motion of fluid is governed by hydrodynamics equations and the electric potential is governed by electrodynamics equations. One approach to any coupled problem is monolithic methods. In these methods, the globally coupled problem is assembled at each time step and then solved iteratively. Partitioned methods, which solve the coupled problem by successively solving the sub-physics problems, are another attractive and promising approach for solving RMHD system. These allow us to employ the best NSE codes and best Ohm's law codes, each highly optimized for the sub-problems' respective physics.

The two partitioned methods we study herein include a first order, one step scheme and a second order, two step schemes, both of which consist of implicit discretization of the subproblem terms and explicit discretization of coupling terms. We prove that these methods are uniformly stable over $0 \le t \le \infty$. This stability result is surprising in that in combining implicit and explicit time discretizations one often sees the stability of the combination governed by that of the explicit method used.

For results on the steady-state MHD problems see [20] (2D), [16] (small $R_m$) and [7] (perfectly conducting walls). [13, 14, 11, 15] studied more boundary conditions that account for the electromagnetic interaction of the fluid with the outside world. For further discussions, see [6, 5]. For evolutionary MHD, see Schmidt [18] and for stability of fully coupled time discretization schemes, we refer to [21].

## 2. Notation and preliminaries

We denote the $L^2(\Omega)$ norms and corresponding inner products by $\|\cdot\|$ and $(\cdot,\cdot)$. Let $H^{-1}(\Omega)$ denote the dual space of $H_0^1(\Omega)$. The velocity, pressure and potential spaces are $X = (H_0^1(\Omega))^d$, $Q = L_0^2(\Omega)$ and $S = H_0^1(\Omega)$, respectively. The space of divergence free functions is given by

$$V = \{\mathbf{v} \in X : \ (\nabla \cdot \mathbf{v}, q) = 0 \ \ \forall q \in Q\}\,.$$

A weak formulation of (RMHD) is: Find $\mathbf{u} : [0,T] \to X$, $p : [0,T] \to Q$ and $\phi : [0,T] \to S$ for a.e. $t \in (0,T]$ satisfying

$$\begin{aligned}
\frac{1}{N}((\mathbf{u}_t,\mathbf{v}) + (\mathbf{u}\cdot\nabla\mathbf{u},\mathbf{v})) &+ \frac{1}{M^2}(\nabla\mathbf{u},\nabla\mathbf{v}) - (p,\nabla\cdot\mathbf{v}) \\
&+(\mathbf{u}\times\mathbf{B},\mathbf{v}\times\mathbf{B}) - (\nabla\phi,\mathbf{v}\times\mathbf{B}) = (\mathbf{f},\mathbf{v}) \quad \forall \mathbf{v} \in X, \\
(\nabla\cdot\mathbf{u}, q) &= 0 \quad \forall q \in Q, \\
-(\nabla\phi,\nabla\psi) + (\mathbf{u}\times\mathbf{B},\nabla\psi) &= 0 \quad \forall \psi \in S,
\end{aligned} \tag{2.1}$$

with the initial condition (1.1) a.e. in $\Omega$. Note that, setting $\mathbf{v} = \mathbf{u}$, $\psi = \phi$ and adding, the coupling terms exactly cancel in the monolithic sum and one verifies the stability of the continuous problem.

To make a spatial discretization of the RMHD system by the finite element method, we select finite element spaces

$$\text{velocity: } X^h \subset X, \text{ pressure: } Q^h \subset Q, \text{ and potential: } S^h \subset S$$

which are built on a conforming, edge to edge triangulation with maximum triangle parameter denoted by a subscript "$h$". We assume that $X^h \times Q^h$ satisfies the usual discrete inf-sup condition for the stability of the discrete pressure and $X^h, Q^h, S^h$ satisfy approximation properties of piecewise polynomials on quasi-uniform meshes of local degrees $k, k-1, k$ respectively. The discretely divergence free velocity space is denoted by

$$V^h := X^h \cap \{\mathbf{v}_h : (q_h, \nabla\cdot\mathbf{v}_h) = 0, \ \text{ for all } q_h \in Q^h\}.$$

Also define the usual, explicitly skew symmetrized trilinear form

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = \frac{1}{2}((\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w}) - (\mathbf{u}\cdot\nabla\mathbf{w},\mathbf{v})).$$

The monolithic, semi-discrete approximation of (2.1) are maps $(\mathbf{u}_h, p_h, \phi_h) : [0,T] \to X^h \times Q^h \times S^h$ satisfying for all $\mathbf{v}_h \in X^h, q_h \in Q^h, \psi_h \in S^h$

$$\begin{aligned}
\frac{1}{N}((\mathbf{u}_{h,t},\mathbf{v}_h) + b(\mathbf{u}_h,\mathbf{u}_h,\mathbf{v}_h)) &+ \frac{1}{M^2}(\nabla\mathbf{u}_h,\nabla\mathbf{v}_h) - (p_h,\nabla\cdot\mathbf{v}_h) \\
&+(\mathbf{u}_h\times\mathbf{B},\mathbf{v}_h\times\mathbf{B}) - (\nabla\phi_h,\mathbf{v}_h\times\mathbf{B}) = (\mathbf{f},\mathbf{v}_h), \\
&(\nabla\cdot\mathbf{u}_h, q_h) = 0, \\
-(\nabla\phi_h,\nabla\psi_h) + (\mathbf{u}_h\times\mathbf{B},\nabla\psi_h) &= 0.
\end{aligned} \tag{2.2}$$

**2.1. The implicit-explicit partitioned schemes.** The methods we propose and analyze herein have the coupling terms lagged or extrapolated, thus the system uncouples into two subproblem solves.

**Algorithm 2.1** (First order IMEX scheme)**.** *Given* $\mathbf{u}_h^n \in X^h, p_h^n \in Q^h, \phi_h^n \in S^h$, *find* $\mathbf{u}_h^{n+1} \in X^h, p_h^{n+1} \in Q^h, \phi_h^{n+1} \in S^h$ *satisfying*

$$
\begin{aligned}
&\frac{1}{N}\left(\left(\frac{\mathbf{u}_h^{n+1}-\mathbf{u}_h^n}{\Delta t}, \mathbf{v}_h\right) + b(\mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1}, \mathbf{v}_h)\right) + \frac{1}{M^2}(\nabla \mathbf{u}_h^{n+1}, \nabla \mathbf{v}_h) \\
&\qquad -(p_h^{n+1}, \nabla\cdot\mathbf{v}_h) + (\mathbf{u}_h^{n+1}\times\mathbf{B}, \mathbf{v}_h\times\mathbf{B}) - (\nabla\phi_h^n, \mathbf{v}_h\times\mathbf{B}) = (\mathbf{f}^{n+1}, \mathbf{v}_h), \\
&\qquad (\nabla\cdot\mathbf{u}_h^{n+1}, q_h) = 0, \\
&\qquad -(\nabla\phi_h^{n+1}, \nabla\psi_h) + (\mathbf{u}_h^n\times\mathbf{B}, \nabla\psi_h) = 0,
\end{aligned}
\tag{IMEX1}
$$

*for all* $\mathbf{v}_h \in X^h, q_h \in Q^h$ *and* $\psi_h \in S^h$.

The second scheme we consider employs second order, three level BDF discretization for the subproblem terms. The coupling terms are treated by two step extrapolation in Navier-Stokes equation and by implicit method in Ohm's law. Since one needs the updated value of $u_h$ at current time level to compute $\phi_h$, this method is uncoupled but sequential: $\phi_h^n \to \mathbf{u}_h^{n+1} \to \phi_h^{n+1}$. Nevertheless, solving the subproblems sequentially does not take considerably longer time, since for RMHD system, computing time for the nonlinear equation of $\mathbf{u}_h$ would dominate that for the linear equation of $\phi_h$.

**Algorithm 2.2** (Second order IMEX scheme)**.** *Given* $\mathbf{u}_h^{n-1}, \mathbf{u}_h^n \in X^h, p_h^{n-1}, p_h^n \in Q^h, \phi_h^{n-1}, \phi_h^n \in S^h$, *find* $\mathbf{u}_h^{n+1} \in X^h, p_h^{n+1} \in Q^h, \phi_h^{n+1} \in S^h$ *satisfying*

$$
\begin{aligned}
&\frac{1}{N}\left(\left(\frac{3\mathbf{u}_h^{n+1}-4\mathbf{u}_h^n+\mathbf{u}_h^{n-1}}{2\Delta t}, \mathbf{v}_h\right) + b(\mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1}, \mathbf{v}_h)\right) + \frac{1}{M^2}(\nabla \mathbf{u}_h^{n+1}, \nabla \mathbf{v}_h) \\
&\quad -(p_h^{n+1}, \nabla\cdot\mathbf{v}_h) + (\mathbf{u}_h^{n+1}\times\mathbf{B}, \mathbf{v}_h\times\mathbf{B}) - (\nabla(2\phi_h^n - \phi^{n-1}), \mathbf{v}_h\times\mathbf{B}) = (\mathbf{f}^{n+1}, \mathbf{v}_h), \\
&\qquad (\nabla\cdot\mathbf{u}_h^{n+1}, q_h) = 0, \\
&\qquad -(\nabla\phi_h^{n+1}, \nabla\psi_h) + (\mathbf{u}_h^{n+1}\times\mathbf{B}, \nabla\psi_h) = 0,
\end{aligned}
\tag{IMEX2}
$$

*for all* $\mathbf{v}_h \in X^h, q_h \in Q^h$ *and* $\psi_h \in S^h$.

## 3. Stability of the two partitioned methods

In this section, we establish stability of the approximations in Algorithms IMEX1 and IMEX2.

**Theorem 3.1** (Unconditional stability of Algorithm IMEX1)**.** *Let* $(\mathbf{u}_h^n, p_h^n, \phi_h^n) \in X^h \times Q^h \times S^h$ *satisfy* (IMEX1) *for each* $n \in \{1, 2, ..., \frac{T}{\Delta t}\}$. *Then*

$$
\begin{aligned}
&\frac{1}{N}\|\mathbf{u}_h^n\|^2 + \frac{1}{N}\sum_{j=0}^{n-1}\|\mathbf{u}_h^{j+1}-\mathbf{u}_h^j\|^2 + \Delta t\|\nabla\phi_h^n\|^2 + \Delta t\|\mathbf{B}\times\mathbf{u}_h^n\|^2 + \frac{\Delta t}{M^2}\sum_{j=0}^{n-1}\|\nabla\mathbf{u}_h^{j+1}\|^2 \\
&\quad + \Delta t\sum_{j=0}^{n-1}\left(\|-\nabla\phi_h^j + \mathbf{u}_h^{j+1}\times\mathbf{B}\|^2 + \|-\nabla\phi_h^{n+1} + \mathbf{u}_h^n\times\mathbf{B}\|^2\right) \\
&\qquad \le \frac{1}{N}\|\mathbf{u}_h^0\|^2 + \Delta t\|\nabla\phi_h^0\|^2 + \Delta t\|\mathbf{B}\times\mathbf{u}_h^0\|^2 + M^2\Delta t\sum_{j=0}^{n-1}\|\mathbf{f}^{j+1}\|_{-1}^2.
\end{aligned}
$$

PROOF. In (IMEX1), setting $\mathbf{v}_h = \mathbf{u}_h^{j+1}, q_h = p_h^{j+1}, \psi_h = \phi_h^{j+1}$, we have

$$\frac{1}{2\Delta t}\cdot\frac{1}{N}\left(\|\mathbf{u}_h^{j+1}\|^2 - \|\mathbf{u}_h^j\|^2 + \|\mathbf{u}_h^{j+1}-\mathbf{u}_h^j\|^2\right) + \frac{1}{M^2}\|\nabla\mathbf{u}_h^{j+1}\|^2 + \|\mathbf{B}\times\mathbf{u}_h^{j+1}\|^2$$
$$= \left(\mathbf{B}\times\nabla\phi_h^j, \mathbf{u}_h^{j+1}\right) + (\mathbf{f}^{j+1}, \mathbf{u}_h^{j+1}),$$
$$\|\nabla\phi_h^{j+1}\|^2 = \left(\mathbf{u}_h^j, \mathbf{B}\times\nabla\phi_h^{j+1}\right).$$

Adding, multiplying by $2\Delta t$ and summing from $j=0$ to $n-1$ give

$$\frac{1}{N}\|\mathbf{u}_h^n\|^2 + \frac{1}{N}\sum_{j=0}^{n-1}\|\mathbf{u}_h^{j+1}-\mathbf{u}_h^j\|^2 + \Delta t\|\nabla\phi_h^n\|^2 + \Delta t\|\mathbf{B}\times\mathbf{u}_h^n\|^2$$
$$+\frac{2\Delta t}{M^2}\sum_{j=0}^{n-1}\|\nabla\mathbf{u}_h^{j+1}\|^2 + \Delta t\sum_{j=0}^{n-1}\left(\|-\nabla\phi_h^j + \mathbf{u}_h^{j+1}\times\mathbf{B}\|^2 + \|-\nabla\phi_h^{n+1} + \mathbf{u}_h^n\times\mathbf{B}\|^2\right)$$
$$= \frac{1}{N}\|\mathbf{u}_h^0\|^2 + \Delta t\|\nabla\phi_h^0\|^2 + \Delta t\|\mathbf{B}\times\mathbf{u}_h^0\|^2 + 2\Delta t\sum_{j=0}^{n-1}(\mathbf{f}^{j+1}, \mathbf{u}_h^{j+1}).$$

Applying Young's inequality yields the result. □

Algorithm IMEX2 is stable over $0 \le t < \infty$ with a condition on time step.

**Theorem 3.2** (Stability of Algorithm IMEX2)**.** *Let $(\mathbf{u}_h^n, p_h^n, \phi_h^n) \in X^h\times Q^h\times S^h$ satisfy* (IMEX2) *for each $n\in\{1,2,...,\frac{T}{\Delta t}\}$. Under the time step restriction*

$$\Delta t < \frac{1}{2N\|\mathbf{B}\|_{L^\infty}^2(M^2C_P^2\|\mathbf{B}\|_{L^\infty}^2+1)}$$

*Algorithm IMEX2 is uniformly in time stable*

$$\frac{1}{2N}\|\mathbf{u}_h^n\|^2 + \frac{1}{2N}\|2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|^2 + \frac{\Delta t}{2M^2}\sum_{j=1}^{n-1}\|\nabla\mathbf{u}_h^{j+1}\|^2$$
$$+\Delta t\sum_{j=1}^{n-1}\|-\nabla(2\phi_h^j - \phi_h^{j-1}) + \mathbf{u}_h^{j+1}\times\mathbf{B}\|^2$$
$$+\Delta t\sum_{j=1}^{n-1}\|-\nabla(2\phi_h^j - \phi_h^{j-1}) + (2\mathbf{u}_h^j - \mathbf{u}_h^{j-1})\times\mathbf{B}\|^2$$
$$\le \frac{1}{2N}\|\mathbf{u}_h^1\|^2 + \frac{1}{2N}\|2\mathbf{u}_h^1 - \mathbf{u}_h^0\|^2 + 2\Delta t M^2\sum_{j=1}^{n-1}\|\mathbf{f}^{j+1}\|_{-1}^2.$$

For a detailed proof we refer to [10].

## 4. Numerical experiment

First, using the exact solutions introduced in [21], we verify the convergence rates of our method. Second, we consider large values of $M$ and $N$ to test the uniform in time stability.

**4.1. Test 1.** The domain is $\Omega = [0,\pi]^2$ and true solution $(\mathbf{u}, p, \phi)$ from [21] is given by

$$\mathbf{u}(x,y,t) = (2\cos(2x)\sin(2y), -2\sin(2x)\cos(2y))e^{-5t},$$
$$p(x,y,t) = 0,$$
$$\phi(x,y,t) = (\cos(2x)\cos(2y) + x^2 - y^2)e^{-5t}.$$

Take the time interval $0 \le t \le 1$ and set $M = 20, N = 16$. The imposed magnetic field is $\mathbf{B} = (0,0,1)$. We utilize piecewise quadratic for velocity and piecewise linear for pressure for the Navier-Stokes equation and continuous piecewise quadratic finite elements for the Ohm's law. The boundary condition on the problem is inhomogeneous Dirichlet: $\mathbf{u}_h = \mathbf{u}$ on $\partial\Omega$. The initial data and source terms are chosen to correspond the exact solution.

We denote $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(0,T;L^2(\Omega))}$ and $\|\cdot\|_2 = \|\cdot\|_{L^2(0,T;L^2(\Omega))}$. From the tables, IMEX1 is first order and IMEX2 is second order.

| $h$ | $\Delta t$ | $\\|\mathbf{u}-\mathbf{u}_h\\|_\infty$ | $\\|\nabla\mathbf{u}-\nabla\mathbf{u}_h\\|_2$ | $\\|\nabla\cdot(p-p_h)\\|_\infty$ | $\\|\phi-\phi_h\\|_\infty$ | $\\|\nabla\phi-\nabla\phi_h\\|_2$ |
|---|---|---|---|---|---|---|
| 1/5 | 1/40 | 1.047$e$+0 | 2.921$e$+0 | 1.956$e$+0 | 5.760$e$-1 | 9.764$e$-1 |
| 1/10 | 1/80 | 7.406$e$-1 | 2.062$e$+0 | 1.005$e$+0 | 3.913$e$-1 | 6.764$e$-1 |
| 1/20 | 1/160 | 4.338$e$-1 | 1.214$e$+0 | 5.094$e$-1 | 2.277$e$-1 | 3.952$e$-1 |
| 1/40 | 1/320 | 2.348$e$-1 | 6.522$e$-1 | 2.564$e$-1 | 1.237$e$-1 | 2.137$e$-1 |
| 1/80 | 1/640 | 1.223$e$-1 | 3.374$e$-1 | 1.286$e$-1 | 6.459$e$-2 | 1.113$e$-1 |
| Rate of conv. | | 0.7853 | 0.7889 | 0.9825 | 0.7975 | 0.7928 |

Table 1: The convergence performance for Algorithm IMEX1.

| $h$ | $\Delta t$ | $\\|\mathbf{u}-\mathbf{u}_h\\|_\infty$ | $\\|\nabla\mathbf{u}-\nabla\mathbf{u}_h\\|_2$ | $\\|\nabla\cdot(p-p_h)\\|_\infty$ | $\\|\phi-\phi_h\\|_\infty$ | $\\|\nabla\phi-\nabla\phi_h\\|_2$ |
|---|---|---|---|---|---|---|
| 1/5 | 1/40 | 3.217$e$-1 | 1.108$e$+0 | 2.837$e$-1 | 1.634$e$-1 | 3.206$e$-1 |
| 1/10 | 1/80 | 4.633$e$-2 | 3.670$e$-1 | 6.676$e$-2 | 2.374$e$-2 | 6.380$e$-2 |
| 1/20 | 1/160 | 8.973$e$-3 | 8.325$e$-2 | 1.614$e$-2 | 4.694$e$-3 | 1.519$e$-2 |
| 1/40 | 1/320 | 2.081$e$-3 | 1.533$e$-2 | 4.003$e$-3 | 1.096$e$-3 | 3.806$e$-3 |
| 1/80 | 1/640 | 5.118$e$-4 | 3.104$e$-3 | 1.001$e$-3 | 2.698$e$-4 | 9.577$e$-4 |
| Rate of conv. | | 2.1747 | 2.1541 | 2.0353 | 2.2922 | 2.0841 |

Table 2: The convergence performance for Algorithm IMEX2.

**4.2. Test 2.** Many important applications of MHD in laboratory and industry involve large Hartmann number and interaction parameter, see, e.g., [17, 4]. The time step condition for stability of IMEX2 looks pessimistic in these cases. We compare our methods for such flows, confirming the unconditional stability of IMEX1. We observe that IMEX2 is stable for much larger time steps than predicted by Theorem 3.2.

Let $\Omega = [0,10^{-1}]^2$ and $\mathbf{B} = (0,0,1)$. We consider the flow of liquid aluminium at 700°C:

$$\sigma = 4.1\cdot 10^6 \text{ mho/m}, \qquad \rho = 2400 \text{ kg/m}^3,$$
$$\nu = 6\cdot 10^{-7} \text{ m}^2/\text{s}, \qquad \eta = 1.94\cdot 10^{-1} \text{ m}^2/\text{s}.$$

We take the characteristic values of length, velocity and magnetic field to be $L = 0.1$m, $u = 0.1$m/s, $B = 1$T, typical for laboratory and industrial flows. The Reynolds number, magnetic Reynolds number, Hartmann number and interaction parameter are then $Re = 16667$, $R_m = 0.051496$, $M = 5336$, $N = 1708$ correspondingly.

We take the source term $\mathbf{f}$ and the boundary condition to be 0 and the initial condition is given by

$$\mathbf{u}_0(x,y) = (10\pi\cos(10\pi x)\sin(10\pi y), -10\pi\sin(10\pi x)\cos(10\pi y)),$$
$$\phi_0(x,y) = (\cos(10\pi x)\cos(10\pi y) + x^2 - y^2).$$

For a system lacking of external energy exchange and body forces, the system energy decays over time. The energy $E^j = \|\mathbf{u}_h^j\|^2 + \|\phi_h^j\|^2$ is computed using two different methods studied herein, on $h = 1/10$. For each algorithm, the time step is chosen purposely to give us an estimate of practical restriction on time step for the stability of the method. The results are showed in Figure 1.

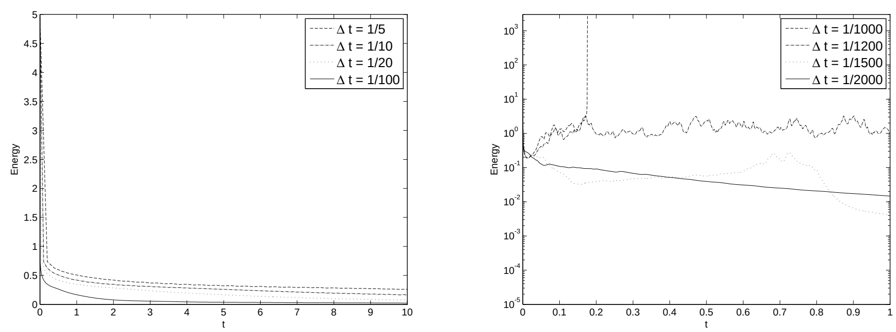


Figure 1: The decay of system energy computed by IMEX1 (left) and IMEX2 (right) with several different time steps chosen.

Figure 1 confirms the unconditional stability of IMEX1 established in Theorem 3.1. It also indicates that the experimental stability condition for IMEX2 is $\Delta t \lesssim 1/1500$, which, while restrictive, is significantly better than the time step restriction in Theorem 3.2.

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