APPROXIMATE DECONVOLUTION MODELS FOR MAGNETOHYDRODYNAMICS

A. LABOVSKY* AND C. TRENCHEA†

Abstract. We consider the family of approximate deconvolution models (ADM) for the simulation of the large eddies in turbulent viscous, incompressible, electrically conducting flows. We prove the existence and uniqueness of solutions to the ADM-MHD equations, their weak converge to the solution of the MHD equations as the averaging radii tend to zero, and derive a bound on the modeling error. We demonstrate that the energy and helicity of the models are conserved, and the models preserve the Alfvén waves. We provide the results of the computational tests, that verify the accuracy and physical fidelity of the models.

Key words. Large eddy simulation, magnetohydrodynamics, approximate deconvolution

AMS subject classifications. 65M12, 76F65, 76W05

1. Introduction.

Magnetohydrodynamics (MHD, [2]) studies the dynamics of electrically conducting fluids such as plasmas, liquid metals and salt water. The MHD model applies to astrophysics, geophysics and engineering problems.

The magnetic field induces currents in the fluid, which create Lorentz forces on the fluid and change the magnetic field itself. The dynamics of MHD flows is complex, with increased difficulties when modelling and simulating turbulent MHD.

There is a large body of literature dedicated to both experimental and theoretical investigations on the influence of electromagnetic force on flows such plasma confinement, liquid-metal cooling of nuclear reactors, electromagnetic casting, MHD sea water propulsion (see e.g., [21, 31, 32, 19, 20, 43, 15, 44, 22, 40, 5, 16, 4]). The MHD effects arising from the macroscopic interaction of liquid metals with applied currents and magnetic fields are exploited in metallurgical processes to control the flow of metallic melts: the electromagnetic stirring of molten metals [33], electromagnetic turbulence control in induction furnaces [45], electromagnetic damping of buoyancy-driven flow during solidification [35], and the electromagnetic shaping of ingots in continuous casting [37]. Although the Lorentz forces can be used (specifically in engineering) to control the flow, there are many different effects and regimes in the MHD flows, which makes the flow more complex; this also explains why the difficulties of modeling turbulent MHD flows are magnified, compared to the Navier-Stokes turbulent flows.

Direct numerical simulation of a $3d$ turbulent flow is often not computationally economical or even feasible. On the other hand, the largest structures in the flow (containing most of the flow's energy) are responsible for much of the mixing and most of the flow's momentum transport. This led to various numerical regularizations, one being the Large Eddy Simulation (LES) [36], [23], [6]. LES is based on the idea that the flow can be represented by a collection of scales with different sizes, and instead of trying to approximate all of them down to the smallest one, one defines a filter width $\delta > 0$ and computes only the scales of size bigger than δ (large scales), while the effect of the small scales on the large scales is modeled. This reduces the number

*Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, Partially supported by NSF Grant DMS-0508260

†Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260

of degrees of freedom in a simulation and represents accurately the large structures in the flow.

In [25] we considered the problem of modeling the motion of large structures in a viscous, incompressible, electrically conducting, turbulent fluid. We introduced a simple closed LES model and its performed full numerical analysis. The LES model can be also addressed as the zeroth order Approximate Deconvolution Model - referring to the family of models in [1]. In this report we consider the family of the Approximate Deconvolution Models for MagnetoHydroDynamics (ADM for MHD), perform the numerical analysis of the models, and verify their physical fidelity.

The mathematical description of the problem proceeds as follows. Assuming the fluid to be viscous and incompressible, the governing equations are the Navier-Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism, coupled via the Lorentz force and Ohm's law (see e.g. [39]). Let $\Omega = (0, L)^3$ be the flow domain, and $u(t, x), p(t, x), B(t, x)$ be the velocity, pressure, and the magnetic field of the flow, driven by the velocity body force f and magnetic field force $\operatorname{curl} g$. Then u, p, B satisfy the MHD equations:

$$
\begin{aligned}
&u_t + \nabla\cdot(uu^T) - \frac{1}{\mathrm{Re}}\Delta u + \frac{S}{2}\nabla(B\cdot B) - S\nabla\cdot(BB^T) + \nabla p = f,\\
&B_t + \frac{1}{\mathrm{Re_m}}\operatorname{curl}(\operatorname{curl} B) + \operatorname{curl}(B\times u) = \operatorname{curl} g,\\
&\nabla\cdot u = 0, \nabla\cdot B = 0,
\end{aligned}
\tag{1.1}
$$

in $Q = (0, T)\times\Omega$, with the initial data:

$$
u(0, x) = u_0(x), \quad B(0, x) = B_0(x) \quad \text{in } \Omega,
\tag{1.2}
$$

and with periodic boundary conditions (with zero mean):

$$
\Phi(t, x + L\mathrm{e}_i) = \Phi(t, x), i = 1, 2, 3, \quad \int_\Omega \Phi(t, x)dx = 0,
\tag{1.3}
$$

for $\Phi = u, u_0, p, B, B_0, f, g$.

Here Re, $\mathrm{Re_m}$, and S are nondimensional constants that characterize the flow: the Reynolds number, the magnetic Reynolds number and the coupling number, respectively. For derivation of (1.1), physical interpretation and mathematical analysis, see [7, 26, 38, 18] and the references therein.

The necessity of choosing different filtering widths has been justified computationally for coupled Navier-Stokes problems in [34]. We define the modified pressure by $P := \frac{S}{2}|B|^2 + p$. If $\overline{}^{\delta_1}$, $\overline{}^{\delta_2}$ denote two local, spacing averaging operators that commute with the differentiation, then averaging (1.1) gives the following non-closed equations for $\overline{u}^{\delta_1}$, $\overline{B}^{\delta_2}$, $\overline{P}^{\delta_1}$ in $(0, T)\times\Omega$:

$$
\begin{aligned}
&\overline{u}_t^{\delta_1} + \nabla\cdot(\overline{uu^T}^{\delta_1}) - \frac{1}{\mathrm{Re}}\Delta\overline{u}^{\delta_1} - S\nabla\cdot(\overline{BB^T}^{\delta_1}) + \nabla\overline{P}^{\delta_1} = \overline{f}^{\delta_1},\\
&\overline{B}_t^{\delta_2} + \frac{1}{\mathrm{Re_m}}\operatorname{curl}(\operatorname{curl}\overline{B}^{\delta_2}) + \nabla\cdot(\overline{Bu^T}^{\delta_2}) - \nabla\cdot(\overline{uB^T}^{\delta_2}) = \operatorname{curl}\overline{g}^{\delta_2},\\
&\nabla\cdot\overline{u}^{\delta_1} = 0, \quad \nabla\cdot\overline{B}^{\delta_2} = 0.
\end{aligned}
\tag{1.4}
$$

The usual closure problem which we study here arises because $\overline{uu^T}^{\delta_1} \neq \overline{u}^{\delta_1}\,\overline{u}^{\delta_1}$, $\overline{BB^T}^{\delta_1} \neq \overline{B}^{\delta_1}\,\overline{B}^{\delta_1}$, $\overline{uB^T}^{\delta_2} \neq \overline{u}^{\delta_1}\,\overline{B}^{\delta_2}$. To isolate the turbulence closure problem from the difficult problem of wall laws for near wall turbulence, we study (1.1) hence (1.4) subject to (1.3). (For commutator closure procedures in presence of walls, see e.g. [8, 9, 12, 13, 28].) The closure problem is to replace the tensors $\overline{uu^T}^{\delta_1}$, $\overline{BB^T}^{\delta_1}$, $\overline{uB^T}^{\delta_2}$ with tensors $\mathscr{T}(\overline{u}^{\delta_1}, \overline{u}^{\delta_1})$, $\mathscr{T}(\overline{B}^{\delta_1}, \overline{B}^{\delta_1})$, $\mathscr{T}(\overline{u}^{\delta_1}, \overline{B}^{\delta_2})$, respectively, depending only on $\overline{u}^{\delta_1}, \overline{B}^{\delta_2}$ and not u, B. There are many closure models proposed in large eddy simulation reflecting the centrality of closure in turbulence simulation. Calling w, q, W the resulting approximations to $\overline{u}^{\delta_1}, \overline{P}^{\delta_1}, \overline{B}^{\delta_2}$, we are led to considering the following model

$$
\begin{aligned}
&w_t + \nabla \cdot \mathscr{T}(w, w) - \frac{1}{\mathrm{Re}} \Delta w - S \nabla \cdot \mathscr{T}(W, W) + \nabla q = \overline{f}^{\delta_1} \\
&W_t + \frac{1}{\mathrm{Re_m}} \mathrm{curl}(\mathrm{curl}\, W) + \nabla \cdot \mathscr{T}(w, W) - \nabla \cdot \mathscr{T}(W, w) = \mathrm{curl}\, \overline{g}^{\delta_2}, \\
&\nabla \cdot w = 0, \quad \nabla \cdot W = 0.
\end{aligned}
$$

With any reasonable averaging operator, the true averages $\overline{u}^{\delta_1}, \overline{B}^{\delta_2}, \overline{p}^{\delta_1}$ are smoother than u, B, p. We consider the family of closure models, pioneered by Stolz and Adams [1]. These Approximate Deconvolution Models (ADM) use the deconvolution operators G_N^1 and G_N^2, that will be defined in Section 2. The ADM for the MHD reads

$$
w_t + \nabla \cdot \overline{(G_N^1 w)(G_N^1 w)}^{\delta_1} - \frac{1}{\mathrm{Re}} \Delta w - S \nabla \cdot \overline{(G_N^2 W)\,(G_N^2 W)}^{\delta_1} + \nabla q = \overline{f}^{\delta_1}, \tag{1.5a}
$$

$$
W_t + \frac{1}{\mathrm{Re_m}} \mathrm{curl}(\mathrm{curl}\, W) + \nabla \cdot (\overline{(G_N^2 W)(G_N^1 w)}^{\delta_2}) - \nabla \cdot (\overline{(G_N^1 w)(G_N^2 W)}^{\delta_2}) \tag{1.5b}
$$

$$
= \mathrm{curl}\, \overline{g}^{\delta_2},
$$

$$
\nabla \cdot w = 0, \quad \nabla \cdot W = 0, \tag{1.5c}
$$

subject to $w(0, x) = \overline{u}_0^{\delta_1}(x), W(0, x) = \overline{B}_0^{\delta_2}(x)$ and periodic boundary conditions (with zero means).

We shall show that the ADM MHD model (1.5) has the mathematical properties expected of a model derived from the MHD equations by an averaging operation and which are important for practical computations. Note that $N = 0$ in (1.5) leads to the model discussed in [25, 24, 27].

The model considered can be developed for quite general averaging operators, see e.g. [1]. The choice of averaging operator in (1.5) is a differential filter, defined as follows. Let the $\delta > 0$ denote the averaging radius, related to the finest computationally feasible mesh. (In this report we use different lengthscales for the Navier-Stokes and Maxwell equations). Given $\phi \in L_0^2(\Omega)$, $\overline{\phi}^{\delta} \in H^2(\Omega) \cap L_0^2(\Omega)$ is the unique solution of

$$
A_\delta \overline{\phi}^{\delta} := -\delta^2 \Delta \overline{\phi}^{\delta} + \overline{\phi}^{\delta} = \phi \quad \text{in } \Omega, \tag{1.6}
$$

subject to periodic boundary conditions. Under periodic boundary conditions, this averaging operator commutes with differentiation, and with this averaging operator, the model (1.5) has consistency $O(\delta^{2N+2})$, i.e.,

$$\begin{aligned}
\overline{uu}^{\delta_1} &= \overline{G_N^1\overline{u}^{\delta_1}\, G_N^1\overline{u}^{\delta_1}}^{\delta_1} + O({\delta_1}^{2N+2}),\\
\overline{BB}^{\delta_1} &= \overline{G_N^2\overline{B}^{\delta_2}\, G_N^2\overline{B}^{\delta_2}}^{\delta_1} + O({\delta_2}^{2N+2}),\\
\overline{uB}^{\delta_2} &= \overline{G_N^1\overline{u}^{\delta_1}\, G_N^2\overline{B}^{\delta_2}}^{\delta_2} + O({\delta_1}^{2N+2} + {\delta_2}^{2N+2}),
\end{aligned}$$

for smooth u, B. We prove that the model (1.5) has a unique, strong solution w, W that converges in the appropriate sense $w \to u$, $W \to B$, as $\delta_1, \delta_2 \to 0$.

In Section 2 we address the global existence and uniqueness of the solution for the closed MHD model. Section 3 treats the questions of limit consistency of the model and verifiability. The conservation of the kinetic energy and helicity for the approximate deconvolution model is presented in Section 4. Section 5 shows that the model preserves the Alfén waves, with the velocity tending to the velocity of Alfvén waves in the MHD, as the radii δ_1, δ_2 tend to zero. The computational results in Section 6 confirm the accuracy and the physical fidelity of the models.

2. Existence and uniqueness for the ADM MHD equations. We introduce the family of the approximate deconvolution operators G_N^1, G_N^2, which are used in the ADM models (1.5).

DEFINITION 2.1 (Approximate Deconvolution Operator). *For a fixed finite N, define the Nth approximate deconvolution operators G_N^1 and G_N^2 by*

$$G_N^i\phi = \sum_{n=0}^{N}(I - A_{\delta_i}^{-1})^n\phi, \text{ for } i = 1, 2.$$

Note that since the differential filter A_{δ_i} is self adjoint, G_N^i is also. G_N^i was shown to be an $O(\delta_i^{2N+2})$ approximate inverse to the filter operator $A_{\delta_i}^{-1}$ (see [11]). It is easy to show that since A_{δ_i} commutes with differentiation, so does G_N^i.

LEMMA 2.2. *The operator G_N^i is compact, positive, and is an asymptotic inverse to the filter $A_{\delta_i}^{-1}$, i.e., for very smooth ϕ and as $\delta_i \to 0$ satisfies*

$$\phi = G_N^i\overline{\phi}^{\delta_i} + (-1)^{N+1}\delta_i^{2N+2}\Delta^{N+1}A_{\delta_i}^{-(N+1)}\phi, \ i = 1, 2. \tag{2.1}$$

The proof of Lemma 2.2 can be found in [11].

LEMMA 2.3. *$\|\cdot\|_{G_N^i}$ defined by $\|v\|_{G_N^i} = (v, G_N^i v)$ is a norm on Ω, equivalent to the $L^2(\Omega)$ norm, and $(\cdot,\cdot)_{G_N^i}$ defined by $(v, w)_{G_N^i} = (v, G_N^i w)$ is an inner product on Ω.*

For the proof see [6].

We shall use the standard notations for function spaces in the space periodic case (see [42]). Let $H_p^m(\Omega)$ denote the space of functions (and their vector valued counterparts also) that are locally in $H^m(\mathbb{R}^3)$, are periodic of period L and have zero mean, i.e. satisfy (1.3). We recall the solenoidal spaces

$$\begin{aligned}
H &= \{(\phi,\psi) \in (H_2^0(\Omega))^2, \nabla\cdot\phi = \nabla\cdot\psi = 0 \text{ in } \mathscr{D}(\Omega)'\},\\
V &= \{(\phi,\psi) \in (H_2^1(\Omega))^2, \nabla\cdot\phi = \nabla\cdot\psi = 0 \text{ in } \mathscr{D}(\Omega)'\}.
\end{aligned}$$

We define the operator $\mathscr{A} \in \mathscr{L}(V, V')$ by setting (see e.g., [38])

$$\langle \mathscr{A}(w_1, W_1), (w_2, W_2)\rangle = \int_\Omega \left(\frac{1}{\mathrm{Re}}\nabla w_1\cdot\nabla w_2 + \frac{1}{\mathrm{Re_m}}\mathrm{curl}\, W_1\cdot \mathrm{curl}\, W_2\right)dx, \tag{2.2}$$

for all $(w_i, W_i) \in V$. The operator $\mathscr{A}$ is an unbounded operator on H, with the domain $D(\mathscr{A}) = \{(w, W) \in V; (\Delta w, \Delta W) \in H\}$ and we denote again by $\mathscr{A}$ its restriction to H.
We define also a continuous tri-linear form $\mathscr{B}_0$ on $V \times V \times V$ by setting

$$\mathscr{B}_0((w_1, W_1), (w_2, W_2), (w_3, W_3)) = \int_\Omega \Big(\nabla \cdot \overline{(G_N^1 w_2)(G_N^1 w_1)}^{\delta_1} w_3 \tag{2.3}$$

$$- S\nabla \cdot \overline{(G_N^2 W_2)(G_N^2 W_1)}^{\delta_1} w_3 + \nabla \cdot \overline{(G_N^2 W_2)(G_N^1 w_1)}^{\delta_2} W_3 - \nabla \cdot \overline{(G_N^1 w_2)(G_N^2 W_1)}^{\delta_2} W_3 dx$$

and a continuous bilinear operator $\mathscr{B}(\cdot) : V \times V \to V'$ with

$$\langle \mathscr{B}(w_1, W_1), (w_2, W_2)\rangle = \mathscr{B}_0((w_1, W_1), (w_1, W_1), (w_2, W_2))$$

for all $(w_i, W_i) \in V$.
The following properties of the trilinear form $\mathscr{B}_0$ hold (see [30, 38, 17, 14])

$$\begin{aligned} &\mathscr{B}_0((w_1, W_1), (w_2, W_2), (A_{\delta_1} G_N^1 w_2, S A_{\delta_2} G_N^2 W_2)) = 0, \\ &\mathscr{B}_0((w_1, W_1), (w_2, W_2), (A_{\delta_1} G_N^1 w_3, S A_{\delta_2} G_N^2 W_3)) \\ &\qquad = -\mathscr{B}_0((w_1, W_1), (w_3, W_3), (A_{\delta_1} G_N^1 w_2, S A_{\delta_2} G_N^2 W_2)), \end{aligned} \tag{2.4}$$

for all $(w_i, W_i) \in V$. Also

$$\begin{aligned} &|\mathscr{B}_0((w_1, W_1), (w_2, W_2), (w_3, W_3))| \\ &\quad \le C\|(G_N^1 w_1, G_N^2 W_1)\|_{m_1} \|(G_N^1 w_2, G_N^2 W_2)\|_{m_2+1} \|(\overline{w_3}^{\delta_1}, \overline{W_3}^{\delta_2})\|_{m_3} \end{aligned} \tag{2.5}$$

for all $(w_1, W_1) \in H^{m_1}(\Omega), (w_2, W_2) \in H^{m_2+1}(\Omega), (w_3, W_3) \in H^{m_3}(\Omega)$ and

$$\begin{aligned} &m_1 + m_2 + m_3 \ge \frac{d}{2}, \quad \text{if } m_i \ne \frac{d}{2} \text{ for all } i = 1, \dots, d, \\ &m_1 + m_2 + m_3 > \frac{d}{2}, \quad \text{if } m_i = \frac{d}{2} \text{ for any of } i = 1, \dots, d. \end{aligned}$$

In terms of $V, H, \mathscr{A}, \mathscr{B}(\cdot)$ we can rewrite (1.5) as

$$\begin{aligned} &\frac{d}{dt}(w, W) + \mathscr{A}(w, W) + \mathscr{B}(w, W) = (\overline{\boldsymbol{f}}^{\delta_1}, \operatorname{curl} \overline{\boldsymbol{g}}^{\delta_2}), t \in (0, T), \\ &(w, W)(0) = (\overline{u}_0^{\delta_1}, \overline{B}_0^{\delta_2}), \end{aligned} \tag{2.6}$$

where $(\boldsymbol{f}, \operatorname{curl} \boldsymbol{g}) = P(f, \operatorname{curl} g)$, and $P : L^2(\Omega) \to H$ is the Hodge projection.

THEOREM 2.4. *For any $(\overline{u_0}^{\delta_1}, \overline{B_0}^{\delta_2}) \in V$ and $(\overline{f}^{\delta_1}, curl \overline{g}^{\delta_2}) \in L^2(0, T; H)$ there exists a unique strong solution to (1.5) $(w, W) \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega))$ and $w_t, W_t \in L^2((0, T) \times \Omega)$. Moreover, the following energy equality holds:*

$$\mathscr{E}(t) + \int_0^t \varepsilon(\tau) d\tau = \mathscr{E}(0) + \int_0^t \mathscr{P}(\tau) d\tau, \qquad t \in [0, T], \tag{2.7}$$

where

$$\begin{aligned} \mathscr{E}(t) &= \frac{{\delta_1}^2}{2} \|\nabla w(t, \cdot)\|^2_{G_N^1} + \frac{1}{2}\|w(t, \cdot)\|^2_{G_N^1} + \frac{{\delta_2}^2 S}{2}\|\nabla W(t, \cdot)\|^2_{G_N^2} + \frac{S}{2}\|W(t, \cdot)\|^2_{G_N^2}, \\ \varepsilon(t) &= \frac{{\delta_1}^2}{\mathrm{Re}}\|\Delta w(t, \cdot)\|^2_{G_N^1} + \frac{1}{\mathrm{Re}}\|\nabla w(t, \cdot)\|^2_{G_N^1} + \frac{{\delta_2}^2 S}{\mathrm{Re_m}}\|\Delta W(t, \cdot)\|^2_{G_N^2} + \frac{S}{\mathrm{Re_m}}\|\nabla W(t, \cdot)\|^2_{G_N^2}, \\ \mathscr{P}(t) &= (f(t), G_N^1 w(t)) + S(curl\, g(t), G_N^2 W(t)). \end{aligned} \tag{2.8}$$

Proof. (Sketch) The proof follows from [25], using a semigroup approach and the machinery of nonlinear differential equations of accretive type in Banach spaces. The key to the model, as in MHD, is to make the nonlinear terms to vanish by an appropriate choice of test function. We observe that by (2.4)

$$\mathscr{B}_0((w,W),(w,W),(A_{\delta_1}G_N^1 w, SA_{\delta_2}G_N^2 W)) = 0,$$

thus taking the inner product of (2.6) with $(A_{\delta_1}G_N^1 w, SA_{\delta_2}G_N^2 W)$ and integrating by parts we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|w\|^2_{G_N^1} + \delta_1^2\|\nabla w\|^2_{G_N^1} + S\|W\|^2_{G_N^2} + \delta_2^2 S\|\nabla W\|^2_{G_N^2}\Big) \\
&\quad + \frac{1}{\mathrm{Re}}\Big(\|\nabla w\|^2_{G_N^1} + \delta_1^2\|\Delta w\|^2_{G_N^1}\Big) + \frac{S}{\mathrm{Re_m}}\Big(\|\nabla W\|^2_{G_N^2} + \delta_2^2 S\|\Delta W\|^2_{G_N^2}\Big) \\
&= (\boldsymbol{f}, G_N^1 w) + S(\operatorname{curl}\boldsymbol{g}, G_N^2 W).
\end{aligned}$$

Integrating both sides of this equality with respect to the time variable, leads to the energy equality (2.7).
□

The pressure is recovered from the weak solution via the classical DeRham theorem (see [29]).

THEOREM 2.5. *Let $m \in \mathbb{N}$, $(u_0, B_0) \in V \cap H^{m+1}(\Omega)$ and $(f, curl\, g) \in L^2(0,T; H^{m+1}(\Omega))$. Then there exists a unique solution w, W, q to the equation (1.5) such that*

$$\begin{aligned}
&(w,W) \in L^\infty(0,T;H^{m+1}(\Omega)) \cap L^2(0,T;H^{m+2}(\Omega)), \\
&q \in L^2(0,T;H^m(\Omega)).
\end{aligned}$$

Proof. The result is already proven when $m = 0$ in Theorem 2.4. For any $m \in \mathbb{N}^*$, we assume that

$$(w,W) \in L^\infty(0,T;H^m(\Omega)) \cap L^2(0,T;H^{m+1}(\Omega)) \tag{2.9}$$

so it remains to prove

$$(D^m w, D^m W) \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)),$$

where D^m denotes any partial derivative of total order m. We take the m^{th} derivative of (1.5) and have

$$\begin{aligned}
&D^m w_t - \frac{1}{\mathrm{Re}}\Delta D^m w + D^m\overline{(G_N^1 w\cdot\nabla G_N^1 w)}^{\delta_1} - SD^m\overline{(G_N^2 W\cdot\nabla G_N^2 W)}^{\delta_1} + \nabla D^m q = D^m\overline{f}^{\delta_1}, \\
&D^m W_t + \frac{1}{\mathrm{Re_m}}\nabla\times\nabla\times D^m W + D^m\overline{(G_N^1 w\cdot\nabla G_N^2 W)}^{\delta_2} - D^m\overline{(G_N^2 W\cdot\nabla G_N^1 w)}^{\delta_2} \\
&\qquad = \nabla\times D^m\overline{g}^{\delta_2}, \\
&\nabla\cdot D^m w = 0, \nabla\cdot D^m W = 0, \\
&D^m w(0,\cdot) = D^m\overline{u_0}^{\delta_1}, D^m W(0,\cdot) = D^m\overline{B_0}^{\delta_2},
\end{aligned}$$

with periodic boundary conditions and zero mean, and the initial conditions with zero divergence and mean. Taking $A_{\delta_1} D^m w$, $A_{\delta_2} D^m W$ as test functions we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|D^m w\|_0^2 + {\delta_1}^2\|\nabla D^m w\|_0^2 + S\|D^m W\|_0^2 + S{\delta_2}^2\|\nabla D^m W\|_0^2\right) \\
&\qquad + \frac{1}{\mathrm{Re}}\left(\|\nabla D^m w\|_0^2 + \delta_1^2\|\Delta D^m w\|_0^2\right) + \frac{1}{\mathrm{Re_m}}\left(\|\nabla D^m W\|_0^2 + \delta_2^2\|\Delta D^m W\|_0^2\right) \\
&= \int_\Omega \left(D^m f D^m w + \nabla \times D^m g D^m W\right) dx - \mathscr{X},
\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}
\mathscr{X} = &\int_\Omega \Big(D^m(G_N^1 w \cdot \nabla G_N^1 w) - S D^m(G_N^2 W \cdot \nabla G_N^2 W)\Big) D^m w \\
&\qquad + \Big(D^m(G_N^1 w \cdot \nabla G_N^2 W) - D^m(G_N^2 W \cdot \nabla G_N^1 w)\Big) D^m W \, dx.
\end{aligned}$$

Now we apply (2.5) and use the induction assumption (2.9)

$$\begin{aligned}
\mathscr{X} = &\sum_{|\alpha|\le m}\binom{m}{\alpha}\sum_{i,j=1}^{3}\int_\Omega \Big(D^\alpha G_N^1 w_i D^{m-\alpha} D_i G_N^1 w_j - S D^\alpha G_N^2 W_i D^{m-\alpha} D_i G_N^2 W_j\Big) D^m w_j \\
&\qquad + \Big(D^\alpha G_N^1 w_i D^{m-\alpha} D_i G_N^2 W_j - D^\alpha G_N^2 W_i D^{m-\alpha} D_i G_N^1 w_j\Big) D^m W_j \, dx \\
\le\ & C(m)\Big(\|G_N^1 w\|_m^{3/2}\|G_N^1 w\|_{m+1}^{1/2} + \|G_N^2 W\|_m^{3/2}\|G_N^2 W\|_{m+1}^{1/2}\Big)\|w\|_m \\
&+ \Big(\|G_N^1 w\|_m\|G_N^2 W\|_m^{1/2}\|G_N^2 W\|_{m+1}^{1/2} + \|G_N^2 W\|_m\|G_N^1 w\|_m^{1/2}\|G_N^1 w\|_{m+1}^{1/2}\Big)\|W\|_m.
\end{aligned}$$

Integrating (2.10) on $(0,T)$, using the Cauchy-Schwarz and Hölder inequalities, Lemma 2.2, 2.3 and the assumption (2.9) we obtain the desired result for w, W. We conclude the proof mentioning that the regularity of the pressure term q is obtained via classical methods, see e.g. [41, 3]. ☐

3. Accuracy of the model.

We address first the question of consistency, i.e., we show that the solution of the closed model (1.5) converges to a solution of the MHD equations (1.1) when δ_1, δ_2 tend zero.
Let $\tau_u, \tau_B, \tau_{Bu}$ denote

$$\tau_u = G_N^1\overline{u}^{\delta_1} G_N^1\overline{u}^{\delta_1} - uu,\ \ \tau_B = G_N^2\overline{B}^{\delta_2} G_N^2\overline{B}^{\delta_2} - BB,\ \ \tau_{Bu} = G_N^2\overline{B}^{\delta_2} G_N^1\overline{u}^{\delta_1} - Bu, \tag{3.1}$$

where u, B is a solution of the MHD equations obtained as a limit of a subsequence of the sequence $w_{\delta_1}, W_{\delta_2}$.

We prove in Theorem 3.2 that the model's consistency errors $\|\overline{u}^{\delta_1} - w\|_{L^\infty(0,T;L^2(Q))}$, $\|\overline{B}^{\delta_2} - W\|_{L^\infty(0,T;L^2(Q))}$ are bounded by $\|\tau_u\|_{L^2(Q_T)}$, $\|\tau_B\|_{L^2(Q_T)}$, $\|\tau_{Bu}\|_{L^2(Q_T)}$.

3.1. Limit consistency of the model.

THEOREM 3.1. *There exist two sequences $\delta_1^n, \delta_2^n \to 0$ as $n \to 0$ such that*

$$(w_{\delta_1^n}, W_{\delta_2^n}, q_{\delta_1^n}) \to (u, B, P) \quad \text{as } \delta_1^n, \delta_2^n \to 0,$$

where $(u, B, p) \in L^\infty(0,T;H) \cap L^2(0,T;V) \times L^{\frac{4}{3}}(0,T;L^2(\Omega))$ is a solution of the MHD equations (1.1), and P is the modified pressure. The sequences $\{w_{\delta_1^n}\}_{n\in\mathbb{N}}, \{W_{\delta_2^n}\}_{n\in\mathbb{N}}$

converge strongly to u, B in $L^{\frac{4}{3}}(0,T;L^2(\Omega))$ and weakly in $L^2(0,T;H^1(\Omega))$, respectively, while $\{q_{\delta_1^n}\}_{n\in\mathbb{N}}$ converges weakly to P in $L^{\frac{4}{3}}(0,T;L^2(\Omega))$.

Proof. The proof follows that of Theorem 3.1 in [25], and is an easy consequence of Theorem 3.2 and Proposition 3.3. □

3.2. Verifiability of the model.

THEOREM 3.2. *Suppose that the true solution of (1.1) satisfies the regularity condition $(u,B)\in L^4(0,T;V)$. Then the consistency errors $e=\overline{u}^{\delta_1}-w$, $E=\overline{B}^{\delta_2}-W$ satisfy*

$$\begin{aligned} &\|e(t)\|_0^2+S\|E(t)\|_0^2+\int_0^t\Big(\frac{1}{\mathrm{Re}}\|\nabla e(s)\|_0^2+\frac{S}{\mathrm{Re_m}}\|curl\,E(s)\|_0^2\Big)ds\\ &\quad\le C\Phi(t)\int_0^t\left(\mathrm{Re}\|\tau_u(s)+S\tau_B(s)\|_0^2+\mathrm{Re_m}\|\tau_{Bu}(s)-\tau_{Bu}(s)\|_0^2\right)ds, \end{aligned}\tag{3.2}$$

where $\Phi(t)=\exp\Big\{\mathrm{Re}^3\int_0^t\|\nabla u\|_0^4ds, {\mathrm{Re_m}}^3\int_0^t\|\nabla u\|_0^4ds+\mathrm{Re_m}\mathrm{Re}^2\int_0^t\|\nabla B\|_0^4\Big\}$.

Proof. The errors $e=\overline{u}^{\delta_1}-w, E=\overline{B}^{\delta_2}-W$ satisfy in variational sense

$$\begin{aligned} &e_t+\nabla\cdot(\overline{G_N^1\overline{u}^{\delta_1}G_N^1\overline{u}^{\delta_1}-G_N^1wG_N^1w}^{\delta_1})-\frac{1}{\mathrm{Re}}\Delta e+S\nabla\cdot(\overline{G_N^2\overline{B}^{\delta_2}G_N^2\overline{B}^{\delta_2}-G_N^2WG_N^2W}^{\delta_1})\\ &\quad+\nabla(\overline{p}^{\delta_1}-q)=\nabla\cdot(\overline{\tau}_u^{\delta_1}+S\overline{\tau}_B^{\delta_1}),\\ &E_t+\frac{1}{\mathrm{Re_m}}\nabla\times\nabla\times E+\nabla\cdot\overline{G_N^2\overline{B}^{\delta_2}G_N^1\overline{u}^{\delta_1}-G_N^2WG_N^1w}^{\delta_2}-\nabla\cdot\overline{G_N^1\overline{u}^{\delta_1}G_N^2\overline{B}^{\delta_2}-G_N^1wG_N^2W}^{\delta_2}\\ &\quad=\nabla\cdot(\overline{\tau}_{Bu}^{\delta_2}-\overline{\tau}_{uB}^{\delta_2}), \end{aligned}$$

and $\nabla\cdot e=\nabla\cdot E=0$, $e(0)=E(0)=0$. Taking the inner product with $(A_{\delta_1}G_N^1e, SA_{\delta_2}G_N^2E)$ we get as for (2.7) the following

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|e\|_{G_N^1}^2+S\|E\|_{G_N^2}^2+\delta_1^2\|\nabla e\|_{G_N^1}^2+\delta_2^2S\|\mathrm{curl}\,E\|_{G_N^2}^2\right)\\ &+\frac{1}{\mathrm{Re}}\|\nabla e\|_{G_N^1}^2+\frac{S}{\mathrm{Re_m}}\|\mathrm{curl\ E}\|_{G_N^2}^2+\frac{\delta_1^2}{\mathrm{Re}}\|\Delta e\|_{G_N^1}^2+\frac{\delta_2^2S}{\mathrm{Re_m}}\|\mathrm{curl\,curl}\,E\|_{G_N^2}^2\\ &+\int_\Omega\Big(\nabla\cdot(G_N^1\overline{u}^{\delta_1}G_N^1\overline{u}^{\delta_1}-G_N^1wG_N^1w)G_N^1e+S\nabla\cdot(G_N^2\overline{B}^{\delta_2}G_N^2\overline{B}^{\delta_2}-G_N^2WG_N^2W)G_N^1e\\ &\quad+S\nabla\cdot(G_N^2\overline{B}^{\delta_2}G_N^1\overline{u}^{\delta_1}-G_N^2WG_N^1w)G_N^2E-S\nabla\cdot(G_N^1\overline{u}^{\delta_1}G_N^2\overline{B}^{\delta_2}-G_N^1wG_N^2W)G_N^2E\Big)dx\\ &=-\int_\Omega\Big((\tau_u+S\tau_B)\cdot\nabla G_N^1e+S(\tau_{Bu}-\tau_{uB})\cdot\nabla G_N^2E\Big)dx\\ &\le\frac{1}{2\mathrm{Re}}\|\nabla e\|_0^2+\frac{S}{2\mathrm{Re_m}}\|\mathrm{curl}\,E\|_0^2+\frac{\mathrm{Re}}{2}\|\tau_u+S\tau_B\|_0^2+\frac{\mathrm{Re_m}}{2S}\|\tau_{Bu}-\tau_{uB}\|_0^2. \end{aligned}$$

Consider the identity $G_N^1\overline{u}^{\delta_1}G_N^1\overline{u}^{\delta_1}-G_N^1wG_N^1w=G_N^1eG_N^1\overline{u}^{\delta_1}+G_N^1wG_N^1e$, obtained by adding and subtracting the term $G_N^1wG_N^1\overline{u}^{\delta_1}$ to the left hand side and denoting $e:=\overline{u}^{\delta_1}-w$.Using also Lemmas 2.2, 2.3, the divergence free condition and (2.5) we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(\|e\|_0^2+S\|E\|_0^2+\delta_1^2\|\nabla e\|_0^2+S\delta_2^2\|\mathrm{curl}\,E\|_0^2\right)\\
&\quad+\frac{1}{\mathrm{Re}}\|\nabla e\|_0^2+\frac{S}{\mathrm{Re_m}}\|\mathrm{curl}\ \mathrm{E}\|_0^2+\frac{\delta_1^2}{\mathrm{Re}}\|\Delta e\|_0^2+\frac{\delta_2^2 S}{\mathrm{Re_m}}\|\mathrm{curl}\,\mathrm{curl}\,E\|_0^2\\
&\le\int_\Omega\Big(-G_N^1 e\cdot\nabla G_N^1\overline{u}^{\delta_1}G_N^1 e-S\nabla\cdot(G_N^2EG_N^2\overline{B}^{\delta_2})G_N^1e-S\nabla\cdot(G_N^2EG_N^1\overline{u}^{\delta_1})G_N^2E\\
&\qquad+SG_N^1e\cdot\nabla G_N^2\overline{B}^{\delta_2}G_N^2E\Big)dx+\mathrm{Re}\|\tau_u+S\tau_B\|_0^2+\mathrm{Re_m}\|\tau_{Bu}-\tau_{uB}\|_0^2\\
&\le C\Big(\|\nabla e\|_0^{3/2}\|e\|_0^{1/2}\|\nabla\overline{u}^{\delta_1}\|_0+2S\|E\|_0^{1/2}\|\nabla E\|_0^{1/2}\|\nabla\overline{B}^{\delta_2}\|_0\|\nabla e\|_0\\
&\qquad+S\|E\|_0^{1/2}\|\nabla E\|_0^{3/2}\|\nabla\overline{u}^{\delta_1}\|_0\Big)+\mathrm{Re}\|\tau_u+S\tau_B\|_0^2+\mathrm{Re_m}\|\tau_{Bu}-\tau_{uB}\|_0^2.
\end{aligned}
$$

Using $ab\le\varepsilon a^{4/3}+C\varepsilon^{-3}b^4$ we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(\|e\|_0^2+S\|E\|_0^2+\delta_1^2\|\nabla e\|_0^2+S\delta_2^2\|\mathrm{curl}\,E\|_0^2\right)\\
&\quad+\frac{1}{\mathrm{Re}}\|\nabla e\|_0^2+\frac{S}{\mathrm{Re_m}}\|\mathrm{curl}\ \mathrm{E}\|_0^2+\frac{\delta_1^2}{\mathrm{Re}}\|\Delta e\|_0^2+\frac{\delta_2^2 S}{\mathrm{Re_m}}\|\mathrm{curl}\,\mathrm{curl}\,E\|_0^2\\
&\le C\Big(\mathrm{Re}^3\|e\|_0^2\|\nabla\overline{u}^{\delta_1}\|_0^4+\mathrm{Re_m}\mathrm{Re}^2\|E\|_0^2\|\nabla\overline{B}^{\delta_2}\|_0^4+\mathrm{Re_m}^3\|E\|_0^2\|\nabla\overline{u}^{\delta_1}\|_0^4\Big)\\
&\quad+\mathrm{Re}\|\tau_u+S\tau_B\|_0^2+\mathrm{Re_m}\|\tau_{Bu}-\tau_{uB}\|_0^2
\end{aligned}
$$

and by the Gronwall inequality we deduce

$$
\begin{aligned}
&\|e(t)\|_0^2+S\|E(t)\|_0^2+\int_0^t\Big(\frac{1}{\mathrm{Re}}\|\nabla e(s)\|_0^2+\frac{S}{\mathrm{Re_m}}\|\mathrm{curl}\,E(s)\|_0^2\Big)ds\\
&\le C\Psi(t)\int_0^t\left(\mathrm{Re}\|\tau_u(s)+S\tau_B(s)\|_0^2+\mathrm{Re_m}\|\tau_{Bu}(s)-\tau_{uB}(s)\|_0^2\right)ds,
\end{aligned}
$$

where

$$
\Psi(t)=\exp\left\{\mathrm{Re}^3\int_0^t\|\nabla\overline{u}^{\delta_1}\|_0^4ds,\mathrm{Re_m}^3\int_0^t\|\nabla\overline{u}^{\delta_1}\|_0^4ds+\mathrm{Re_m}\mathrm{Re}^2\int_0^t\|\nabla\overline{B}^{\delta_2}\|_0^4ds\right\}.
$$

Using the stability bounds $\|\nabla\overline{u}^{\delta_1}\|_0\le\|\nabla u\|_0$, $\|\nabla\overline{B}^{\delta_2}\|_0\le\|\nabla B\|_0$ we conclude the proof. □

3.3. Consistency error estimate. The bounds on the errors (3.1) are given in the following proposition.

PROPOSITION 3.3. *Let*

$$
(u,B)\in L^4((0,T)\times\Omega)\cap L^4(0,T;H^{2N+2}(\Omega)),N\ge 0.
$$

Then

$$
\begin{aligned}
\|\tau_u\|_{L^2(Q)}&\le C\delta_1^{2N+2},\\
\|\tau_B\|_{L^2(Q)}&\le C\delta_2^{2N+2},\\
\|\tau_{Bu}\|_{L^2(Q)}&\le C(\delta_1^{2N+2}+\delta_2^{2N+2}),
\end{aligned}
$$

where $C=C(\|(u,B)\|_{L^4((0,T)\times\Omega)},\|(u,B)\|_{L^4(0,T;H^{2N+2}(\Omega))})$.
The proof uses Lemma 2.2 and follows the outline of the proofs in Section 3.3 of [25].

4. Conservation laws. As our model is some sort of a regularizing numerical scheme, we would like to make sure that the model inherits some of the original properties of the 3D MHD equations.

It is well known that kinetic energy and helicity are critical in the organization of the flow.

The energy $E = \frac{1}{2}\int_\Omega (v(x)\cdot v(x) + B(x)\cdot B(x))dx$, the cross helicity $H_C = \frac{1}{2}\int_\Omega (v(x)\cdot B(x))dx$ and the magnetic helicity $H_M = \frac{1}{2}\int_\Omega (\mathbb{A}(x)\cdot B(x))dx$ (where $\mathbb{A}$ is the vector potential, $B = \nabla\times\mathbb{A}$) are the three invariants of the MHD equations (1.1) in the absence of kinematic viscosity and magnetic diffusivity ($\frac{1}{\mathrm{Re}} = \frac{1}{\mathrm{Re_m}} = 0$), and in the absence of forcing, $f = g = 0$.

Introduce the characteristic quantities of the model (1.5)

$$\begin{aligned}
E_{ADM} &= \frac{1}{2}[(A_{\delta_1}w, w)_{G_N^1} + (A_{\delta_2}W, W)_{G_N^2}],\\
H_{C,ADM} &= \frac{1}{2}(A_{\delta_1}w, A_{\delta_2}W), \text{ and}\\
H_{M,ADM} &= \frac{1}{2}(A_{\delta_2}W, \overline{\mathbb{A}}^{\delta_2})_{G_N^2}, \text{ where } \overline{\mathbb{A}}^{\delta_2} = A_{\delta_2}^{-1}\mathbb{A}.
\end{aligned}$$

This section is devoted to proving that these quantities are conserved by (1.5) with the periodic boundary conditions and $\frac{1}{\mathrm{Re}} = \frac{1}{\mathrm{Re_m}} = 0$, $f = g = 0$. Also, note that

$$E_{ADM} \to E,\ H_{C,ADM} \to H_C,\ H_{M,ADM} \to H_M, \qquad \text{as } \delta_{1,2} \to 0.$$

THEOREM 4.1. *The following conservation laws hold,* $\forall T > 0$

$$E_{ADM}(T) = E_{ADM}(0), \tag{4.1}$$

$$H_{C,ADM}(T) \le H_{C,ADM}(0) + C(T)\max_{i=1,2}\delta_i^{2N+2}, \tag{4.2}$$

and

$$H_{M,ADM}(T) = H_{M,ADM}(0). \tag{4.3}$$

REMARK 4.1. *Note that the cross helicity $H_{C,ADM}$ of the model is not conserved exactly, but it possesses two important properties:*

$$H_{C,ADM} \to H_C \text{ as } \delta_{1,2} \to 0,$$

and

$$H_{C,ADM}(T) \to H_{C,ADM}(0) \text{ as } N \text{ increases.}$$

In the case of equal radii, $\delta_1 = \delta_2$, the following cross helicity is exactly conserved:

$$H_{\times,ADM}(w, W)(t) = \frac{1}{2}\left((w, W)_N + \delta^2(\nabla w, \nabla W)_N\right).$$

Proof. The proof follows the outline of the corresponding proof in [25]. Consider (1.5) with $\frac{1}{\mathrm{Re}} = \frac{1}{\mathrm{Re_m}} = 0$ and $f = g = 0$.

Start by proving (4.1). Multiply (1.5a) by $A_{\delta_1}G_N^1 w$, and multiply (1.5b) by $A_{\delta_2}G_N^2 W$. Integrate both equations over Ω; the operators A_{δ_i} are self-adjoint and

$A\bar{\phi} = \phi$; using also the fact that $\frac{1}{\mathrm{Re}} = \frac{1}{\mathrm{Re_m}} = 0$, and $f = g = 0$, gives

$$\frac{1}{2}\frac{d}{dt}(A_{\delta_1}w, w)_{G_N^1} = ((\nabla \times G_N^2 W) \times G_N^2 W, w)_{G_N^1}, \tag{4.4}$$
$$\frac{1}{2}\frac{d}{dt}(A_{\delta_2}W, W)_{G_N^2} - (G_N^2 W \cdot \nabla G_N^1 w, W)_{G_N^2} = 0. \tag{4.5}$$

Adding (4.4)-(4.5) and using the identity

$$((\nabla \times v) \times u, w) = (u \cdot \nabla v, w) - (w \cdot \nabla v, u) \tag{4.6}$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\Big[(A_{\delta_1}w, w)_{G_N^1} + (A_{\delta_2}W, W)_{G_N^2}\Big]$$
$$= (G_N^2 W \cdot \nabla G_N^2 W, G_N^1 w) - (G_N^1 w \cdot \nabla G_N^2 W, G_N^2 W) + (G_N^2 W \cdot \nabla G_N^1 w, G_N^2 W) = 0,$$

which yields (4.1).

To prove (4.2), multiply (1.5a)-(1.5b) by $A_{\delta_1}G_N^2 W$ and $A_{\delta_2}G_N^1 w$, respectively, and integrate over Ω to get

$$(\frac{\partial A_{\delta_1}w}{\partial t}, W)_{G_N^2} + (G_N^1 w \cdot \nabla G_N^1 w, W)_{G_N^2} = 0, \tag{4.7}$$
$$(\frac{\partial A_{\delta_2}W}{\partial t}, w)_{G_N^1} + (G_N^1 w \cdot \nabla G_N^2 W, w)_{G_N^1} = 0. \tag{4.8}$$

Adding (4.7) and (4.8), we obtain

$$(\frac{\partial A_{\delta_1}w}{\partial t}, G_N^2 W) + (\frac{\partial A_{\delta_2}W}{\partial t}, G_N^1 w) = 0. \tag{4.9}$$

From Corollary 2.2 it follows that

$$G_N^1 w = A_{\delta_1}w + (-1)^N \delta_1^{2N+2}\Delta^{N+1}A_{\delta_1}^{-N}w, \tag{4.10}$$
$$G_N^2 W = A_{\delta_2}W + (-1)^N \delta_2^{2N+2}\Delta^{N+1}A_{\delta_2}^{-N}W.$$

Then (4.9) gives

$$\frac{d}{dt}(A_{\delta_1}w, A_{\delta_2}W) = (\frac{\partial A_{\delta_1}w}{\partial t}, A_{\delta_2}W) + (\frac{\partial A_{\delta_2}W}{\partial t}, A_{\delta_1}w) \tag{4.11}$$
$$= (\frac{\partial A_{\delta_1}w}{\partial t}, (-1)^{N+1}\delta_2^{2N+2}\Delta^{N+1}A_{\delta_2}^{-N}W) + (\frac{\partial A_{\delta_2}W}{\partial t}, (-1)^{N+1}\delta_1^{2N+2}\Delta^{N+1}A_{\delta_1}^{-N}w).$$
$$= (-1)^{N+1}\delta_2^{2N+2}(\frac{\partial A_{\delta_1}w}{\partial t}, \Delta^{N+1}A_{\delta_2}^{-N}W) + (-1)^{N+1}\delta_1^{2N+2}(\frac{\partial A_{\delta_2}W}{\partial t}, \Delta^{N+1}A_{\delta_1}^{-N}w),$$

which proves (4.2).

Next, we prove (4.3). By multiplying (1.5b) by $A_{\delta_2}G_N^2\overline{\mathbb{A}}^{\delta_2}$, and integrating over Ω we get

$$\frac{1}{2}\frac{d}{dt}(\nabla \times A_{\delta_2}\overline{\mathbb{A}}^{\delta_2}, G_N^2\overline{\mathbb{A}}^{\delta_2}) \tag{4.12}$$
$$+ (G_N^1 w \cdot \nabla G_N^2 W, G_N^2\overline{\mathbb{A}}^{\delta_2}) - (G_N^2 W \cdot \nabla G_N^1 w, G_N^2\overline{\mathbb{A}}^{\delta_2}) = 0.$$

Since the cross-product of two vectors is orthogonal to each of them

$$((\nabla \times G_N^2 \overline{\mathbb{A}}^{\delta_2}) \times G_N^1 w, \nabla \times G_N^2 \overline{\mathbb{A}}^{\delta_2}) = 0, \tag{4.13}$$

it follows from (4.13) and (4.6) that

$$(G_N^1 w \cdot \nabla G_N^2 \overline{\mathbb{A}}^{\delta_2}, \nabla \times G_N^2 \overline{\mathbb{A}}^{\delta_2}) = ((\nabla \times G_N^2 \overline{\mathbb{A}}^{\delta_2}) \cdot \nabla G_N^2 \overline{\mathbb{A}}^{\delta_2}, G_N^1 w). \tag{4.14}$$

Since $G_N^2 W = \nabla \times G_N^2 \overline{\mathbb{A}}^{\delta_2}$, we obtain from (4.12) and (4.14) that (4.3) holds. ☐

5. Alfvén waves. In this section we prove that our model possesses a very important property of the MHD: the ability of the magnetic field to transmit transverse inertial waves - Alfvén waves. We follow the argument typically used to prove the existence of Alfvén waves in MHD, see, e.g., [10].

Using the density ρ and permeability μ, we write the equations of the model (1.5) in the form

$$w_t + \nabla \cdot (\overline{(G_N^1 w)(G_N^1 w)}^{\delta_1}) + \nabla \overline{p}^{\delta_1} = \frac{1}{\rho\mu} \overline{(\nabla \times G_N^2 W) \times G_N^2 W}^{\delta_1} - \nu \nabla \times (\nabla \times w), \tag{5.1a}$$

$$\frac{\partial W}{\partial t} = \overline{\nabla \times ((G_N^1 w) \times (G_N^2 W))}^{\delta_2} - \eta \nabla \times (\nabla \times W), \tag{5.1b}$$

$$\nabla \cdot w = 0, \quad \nabla \cdot W = 0, \tag{5.1c}$$

where $\nu = \frac{1}{\mathrm{Re}}$, $\eta = \frac{1}{\mathrm{Re_m}}$.

Assume a uniform, steady magnetic field W_0, perturbed by a small velocity field w. We denote the perturbations in current density and magnetic field by j_{model} and W_p, with

$$\nabla \times W_p = \mu j_{model}. \tag{5.2}$$

Also, the vorticity of the model is

$$\omega_{model} = \nabla \times w. \tag{5.3}$$

Since $G_N^1 w \cdot \nabla G_N^1 w$ is quadratic in the small quantity w, it can be neglected in the Navier-Stokes equation (5.1a), and therefore

$$\frac{\partial w}{\partial t} + \nabla \overline{p}^{\delta_1} = \frac{1}{\rho\mu} \overline{(\nabla \times G_N^2 W_p) \times G_N^2 W_0}^{\delta_1} - \nu \nabla \times (\nabla \times w). \tag{5.4}$$

The leading order terms in the induction equation (5.1b) are

$$\frac{\partial W_p}{\partial t} = \overline{\nabla \times (G_N^1 w \times G_N^2 W_0)}^{\delta_2} - \eta \nabla \times (\nabla \times W_p). \tag{5.5}$$

Following the argument of [25] and using the approximating result of Corollary 2.2, we obtain that in the case of a perfect fluid ($\nu = \eta = 0$) and in the case $\nu = 0$, $\eta \gg 1$ a transverse wave is recovered. The group velocity of the wave is equal to

$$\tilde{v}_a = v_a + O(\delta_1^{2N+2} + \delta_2^{2N+2}),$$

where v_a is the Alfvén velocity $W_0/\sqrt{\rho\mu}$.

We conclude that our model (1.5) preserves the Alfvén waves and the group velocity of the waves $\tilde{v}_a$ tends to the true Alfvén velocity v_a as the radii tend to zero.

6. Computational results. In this section we present computational results for the ADM models of zeroth, first and second order. The convergence rates are presented and the fidelity of the models is verified by comparing the quantities, which are conserved in the ideal inviscid case. The computations are made for the two-dimensional problems, where the energy and enstrophy of the models are compared to those of the averaged MHD. Note that since the chosen test problems are two-dimensional, we cannot compare the helicities of the model to those of the MHD; that is why we turn to the two-dimensional analogue of helicity - the enstrophy. Enstrophy of a solution u is defined as $\int_\Omega |\nabla \times u|^2 d\mathbf{x}$. One can easily verify (or find in the literature) that the enstrophy $\int_\Omega |\nabla \times u|^2 + |\nabla \times B|^2 d\mathbf{x}$ of the ideal MHD is constant.

Consider the MHD flow in $\Omega = (0.5, 1.5) \times (0.5, 1.5)$. The Reynolds number and magnetic Reynolds number are $Re = 10^5, Re_m = 10^5$, the final time is $T = 1/4$, and the averaging radii are $\delta_1 = \delta_2 = h$.

Take

$$f = \begin{pmatrix} \frac{1}{2}\pi \sin(2\pi x) e^{-4\pi^2 t/Re} - x e^{2t} \\ \frac{1}{2}\pi \sin(2\pi y) e^{-4\pi^2 t/Re} - y e^{2t} \end{pmatrix},$$

$$\nabla \times g = \begin{pmatrix} e^t(x - (\cos \pi x \sin \pi y + \pi x \sin \pi x \sin \pi y + \pi y \cos \pi x \cos \pi y) e^{-2\pi^2 t/Re}) \\ e^t(-y - (\sin \pi x \cos \pi y + \pi x \cos \pi x \cos \pi y + \pi y \sin \pi x \sin \pi y) e^{-2\pi^2 t/Re}) \end{pmatrix}.$$

The solution to this problem is

$$u = \begin{pmatrix} -\cos(\pi x)\sin(\pi y) e^{-2\pi^2 t/Re} \\ \sin(\pi x)\cos(\pi y) e^{-2\pi^2 t/Re} \end{pmatrix},$$

$$p = -\frac{1}{2}(\cos(2\pi x) + \cos(2\pi y)) e^{-4\pi^2 t/Re},$$

$$B = \begin{pmatrix} x e^t \\ -y e^t \end{pmatrix}.$$

Hence, although the theoretical results were obtained only for the periodic boundary conditions, we apply the family of ADMs to the problem with Dirichlet boundary conditions.

The results presented in the following tables are obtained by using the software $FreeFEM++$. The velocity and magnetic field are sought in the finite element space of piecewise quadratic polynomials, and the pressure in the space of piecewise linears. In order to draw conclusions about the convergence rate, we take the time step $k = h^2$. We compare the solutions (w, W), obtained by the ADM models, to the true solution (u, B) and the average of the true solution $(\bar{u}, \bar{B})$.

Note that the family of ADMs is constructed so that the model's solution approximates the averaged solutions $(\bar{u}, \bar{B})$ of the averaged MHD, and not the true solution (u, B). Therefore, the second order accuracy in approximating the true solution (u, B) is expected for ADM models of any order, whereas the accuracy in approximating the averaged solution $(\bar{u}, \bar{B})$ should increase as the order of the model increases.

The solution, computed by the zeroth order ADM, approximates both the true solution (u, B) and the average of the true solution $(\bar{u} = (-\delta_1^2\Delta + I)^{-1}u, \bar{B} = (-\delta_2^2\Delta + I)^{-1}B$ with the second order accuracy. The accuracy in approximating the averaged solution increases as the order of the model is increased.

The computational results in Tables 6.1-6.3 were obtained by using the software FreeFEM++, and therefore we were limited to using the Taylor-Hood elements. Thus, it is hard to see the increase in the models' accuracy beyond $O(h^2)$, since the modeling error is overcome by the discretization error; to see the advantage of the higher order approximate deconvolution models more clearly, one would need to employ finite elements of higher degree, although it is clear from the tables below that the convergence rate exceeds the value of two, before being reduced by the discretization error.

Table 6.1
Approximating the true solution, $Re = 10^5$, $Re_m = 10^5$, Zeroth Order ADM

h	$\|w-u\|_{L^2(0,T;L^2(\Omega))}$	*rate*	$\|W-B\|_{L^2(0,T;L^2(\Omega))}$	*rate*
1/4	0.0862904		0.0253257	
1/8	0.0515562	0.7431	0.0268628	-0.085
1/16	0.0204763	1.3322	0.0132399	1.0207
1/32	0.00611337	1.7439	0.00412013	1.6841
1/64	0.00163356	1.9039	0.001116	1.8844

Table 6.2
Approximating the true solution, $Re = 10^5$, $Re_m = 10^5$, First Order ADM

h	$\|w-u\|_{L^2(0,T;L^2(\Omega))}$	*rate*	$\|W-B\|_{L^2(0,T;L^2(\Omega))}$	*rate*
1/4	0.086748		0.0219869	
1/8	0.0504853	0.781	0.0146218	0.5885
1/16	0.0196045	1.3647	0.00401043	1.8663
1/32	0.00589278	1.7342	0.00078723	2.3489
1/64	0.00159084	1.8892	0.000170555	2.2065

Table 6.3
Approximating the true solution, $Re = 10^5$, $Re_m = 10^5$, Second Order ADM

h	$\|w-u\|_{L^2(0,T;L^2(\Omega))}$	*rate*	$\|W-B\|_{L^2(0,T;L^2(\Omega))}$	*rate*
1/4	0.0854318		0.0229699	
1/8	0.0500093	0.7726	0.0170217	0.4324
1/16	0.0194169	1.3649	0.00472331	1.8495
1/32	0.00587995	1.7234	0.000856363	2.4635
1/64	0.00159835	1.8792	0.000167472	2.3543

Table 6.4
Approximating the average solution, $Re = 10^5$, $Re_m = 10^5$, Zeroth Order ADM

h	$\|w-\bar{u}\|_{L^2(0,T;L^2(\Omega))}$	*rate*	$\|W-\bar{B}\|_{L^2(0,T;L^2(\Omega))}$	*rate*
1/4	0.0247837		0.0253257	
1/8	0.0245241	0.0152	0.0268628	-0.085
1/16	0.0131042	0.9042	0.0132399	1.0207
1/32	0.00434599	1.5923	0.00412013	1.6841
1/64	0.00120907	1.8458	0.001116	1.8844

TABLE 6.5
Approximating the average solution, $Re = 10^5$, $Re_m = 10^5$, First Order ADM

h	$\|w - \bar{u}\|_{L^2(0,T;L^2(\Omega))}$	rate	$\|W - \bar{B}\|_{L^2(0,T;L^2(\Omega))}$	rate
1/4	0.0228254		0.0219869	
1/8	0.015202	0.5864	0.0146218	0.5885
1/16	0.0043297	1.8119	0.00401043	1.8663
1/32	0.000867986	2.3185	0.00078723	2.3489
1/64	0.000192121	2.1757	0.000170555	2.2065

TABLE 6.6
Approximating the average solution, $Re = 10^5$, $Re_m = 10^5$, Second Order ADM

h	$\|w - \bar{u}\|_{L^2(0,T;L^2(\Omega))}$	rate	$\|W - \bar{B}\|_{L^2(0,T;L^2(\Omega))}$	rate
1/4	0.0236209		0.0229699	
1/8	0.0172027	0.4574	0.0170217	0.4324
1/16	0.00506669	1.7635	0.00472331	1.8495
1/32	0.000956194	2.4057	0.000856363	2.4635
1/64	0.000194768	2.2955	0.000167472	2.3543

Since the flow is not ideal (nonzero power input, nonzero viscosity/magnetic diffusivity, non-periodic boundary conditions), the energy and enstrophy are not conserved. But we expect the energy and enstrophy of the models to approximate the energy and enstrophy of the averaged MHD.

The enstrophy of the first and second order models approximates the enstrophy of the averaged MHD better than the zeroth order model's enstrophy, see Figure 6.1.

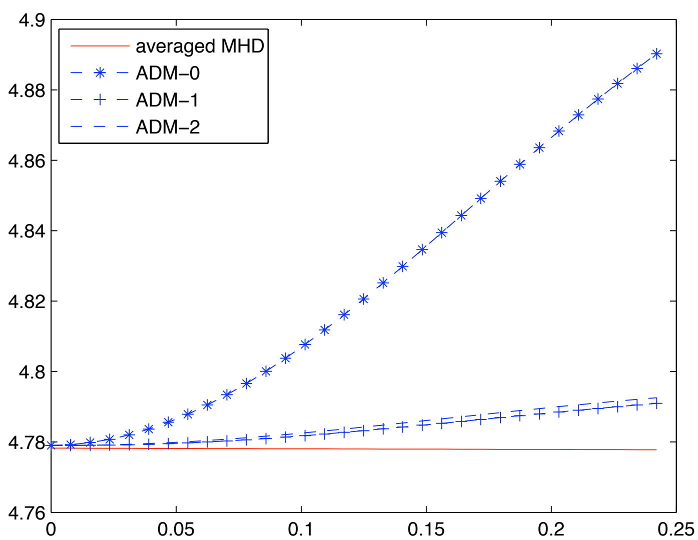

FIG. 6.1. *ADM Enstrophy vs. averaged MHD*

Figure 6.2 shows that the graph of the models energy is hardly distinguishable from that of the averaged MHD.

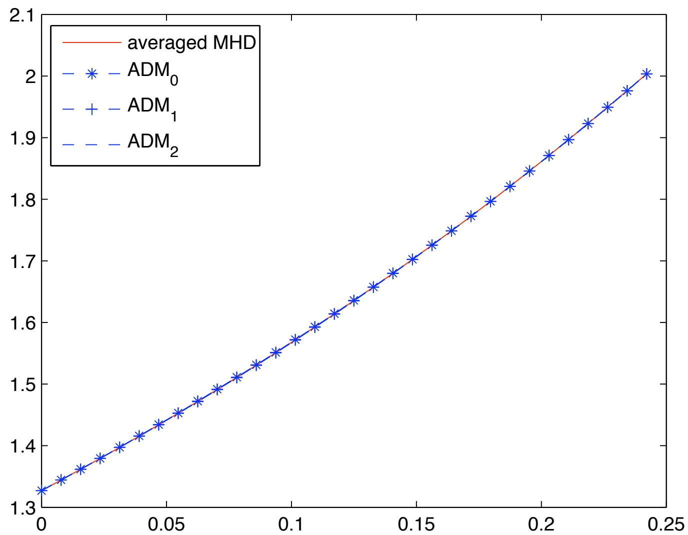

FIG. 6.2. *ADM Energy vs. averaged MHD*

Zooming in at the final time $t = 0.25$ we verify that the ADM energy approximates the averaged MHD energy better as the model's order increases, see Figure 6.3.

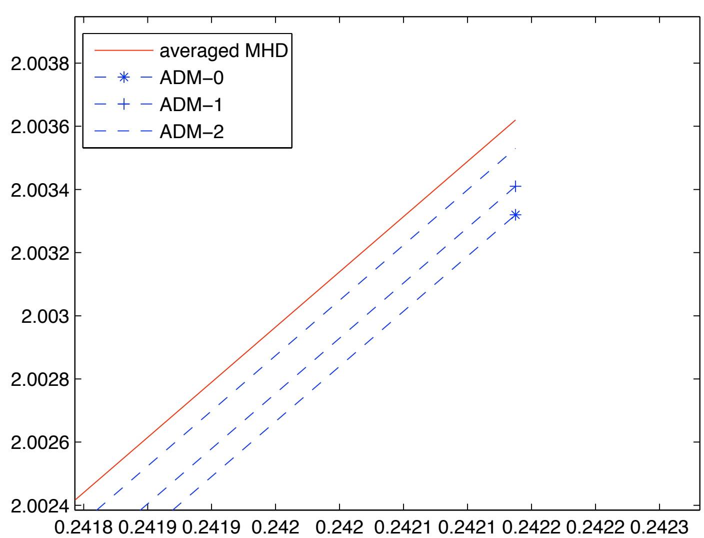

FIG. 6.3. *ADM Energy vs. averaged MHD: zoom in*

Finally, we introduce another test problem - the two-dimensional wave propagation with the nonlinear magnetic field increasing in time. Consider the MHD flow in $\Omega = (0,1) \times (0,1)$, with the Reynolds number and magnetic Reynolds number $Re = 10^4, Re_m = 10^4$, the final time $T = 1/8$.

We construct the solution as

$$u = \begin{pmatrix} 0.75 + 0.25\cos(2\pi(x-t))\sin(2\pi(y-t))e^{-8\pi^2 t\nu} \\ 0.75 - 0.25\sin(2\pi(x-t))\cos(2\pi(y-t))e^{-8\pi^2 t\nu} \end{pmatrix},$$
$$p = -\frac{1}{64}(\cos(4\pi(x-t)) + \cos(4\pi(y-t)))e^{-16\pi^2 t\nu},$$
$$B = \begin{pmatrix} y^3 e^t \\ x^3 e^t \end{pmatrix}.$$

and compute the right hand sides accordingly.

We compare the model solution to the average of the known true solution of the problem. The following tables verify the claimed convergence rates in the $L^2(0,T;L^2(\Omega))$ and $L^2(0,T;H^1(\Omega))$ norms.

TABLE 6.7
Zeroth ADM vs. averaged MHD, $Re = 10^4$, $Re_m = 10^4$.

h	$\Vert w-\bar{u}\Vert_{L^2(L^2)}$	*rate*	$\Vert W-\bar{B}\Vert_{L^2(L^2)}$	*rate*	$\Vert w-\bar{u}\Vert_{L^2(H^1)}$	*rate*
1/4	0.0128497		0.0114325		0.120428	
1/8	0.00860029	0.58	0.0051792	1.14	0.0866042	0.48
1/16	0.00390914	1.14	0.00187599	1.47	0.0490398	0.82
1/32	0.0012649	1.63	0.00055774	1.75	0.018325	1.42
1/64	0.000346	1.87	0.00014841	1.91	0.005388	1.77

TABLE 6.8
First ADM vs. averaged MHD, $Re = 10^4$, $Re_m = 10^4$.

h	$\Vert w-\bar{u}\Vert_{L^2(L^2)}$	*rate*	$\Vert W-\bar{B}\Vert_{L^2(L^2)}$	*rate*	$\Vert w-\bar{u}\Vert_{L^2(H^1)}$	*rate*
1/4	0.0115473		0.0112871		0.120186	
1/8	0.0075245	0.62	0.0054775	1.04	0.066946	0.84
1/16	0.00219	1.78	0.0016639	1.72	0.0218235	1.62
1/32	0.00046929	2.22	0.00033461	2.31	0.00515495	2.08
1/64	0.000106067	2.15	0.000074052	2.18	0.0012762	2.01

TABLE 6.9
Second ADM vs. averaged MHD, $Re = 10^4$, $Re_m = 10^4$.

h	$\Vert w-\bar{u}\Vert_{L^2(L^2)}$	*rate*	$\Vert W-\bar{B}\Vert_{L^2(L^2)}$	*rate*	$\Vert w-\bar{u}\Vert_{L^2(H^1)}$	*rate*
1/4	0.0116245		0.012034		0.118043	
1/8	0.00745374	0.64	0.00571368	1.07	0.068952	0.78
1/16	0.00228231	1.71	0.00154496	1.89	0.0214066	1.69
1/32	0.000461252	2.31	0.000293358	2.4	0.00519608	2.04
1/64	0.000101224	2.19	0.0000613646	2.26	0.00132037	1.98

7. Conclusions. A Large Eddy Simulation approach to the MagnetoHydroDynamic Turbulence was considered. The Approximate Deconvolution Models were introduced for the incompressible MHD equations, and this family of models was analyzed. We proved the existence and uniqueness of solutions, and their convergence in the weak sense to a solution of the MHD equations, as the filtering widths are decreased to zero. We proved the accuracy of the model both theoretically (by establishing an á priori bound on the model's consistency error) and numerically (the results of the computational tests are listed in the previous section).

Also, all models in the family of the ADMs are proven to possess the physical properties of the MHD - the energy and helicity of the models are conserved, and the models are also proven to preserve the Alfvén waves, a unique feature of the MHD equations. The physical fidelity of the models was also verified computationally. The test results demonstrate that both the solution and the energy of the averaged MHD equations are approximated better, as one increases the models' order N (from zeroth ADM to the first ADM, and from the first to the second ADM). This gives a freedom of choosing the model's order N, based on the desired accuracy of approximation and the available computational power. Finally, the tests demonstrate that in the situations when the direct numerical simulation is no longer available (flows with high Reynolds and magnetic Reynolds numbers), the solution can still be obtained by the ADM approach. Note also, that in order to acquire the higher accuracy of approximating the MHD solution with the ADM solution, one needs to increase both the order of the model in use, and the degrees of piecewise polynomials in the corresponding finite element spaces.

REFERENCES

[1] N.A. Adams and S. Stolz, *Deconvolution methods for subgrid-scale approximation in large-eddy simulation*, Modern Simulation Strategies for Turbulent Flow, R.T. Edwards, 2001.

[2] H. Alfvén, *Existence of electromagnetic-hydrodynamic waves*, Nature **150** (1942), 405.

[3] C. Amrouche and V. Girault, *Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension*, Czechoslovak Math. J. **44(119)** (1994), no. 1, 109–140.

[4] V. Barbu, T. Havârneanu, C. Popa, and S.S. Sritharan, *Exact controllability for the magnetohydrodynamic equations*, Comm. Pure Appl. Math. **56(6)** (2003), 732–783.

[5] T. Berger, J. Kim, C. Lee, and J. Lim, *Turbulent boundary layer control utilizing Lorentz force*, Phys. Fluids **12** (2000), 631649.

[6] L.C. Berselli, T. Iliescu, and W.J. Layton, *Mathematics of large eddy simulation of turbulent flows*, Scientific Computation, Springer-Verlag, Berlin, 2006.

[7] T.G. Cowling, *Magnetohydrodynamics*, Interscience Tracts on Physics and Astronomy, New York, 1957.

[8] A. Das and R.D. Moser, *Filtering boundary conditions for LES and embedded boundary simulations*, DNS/LES progress and challenges (C. Liu, L. Sakeland, and T. Beutner, eds.), Greyden Press, Columbus, 2001, pp. 389–396.

[9] ———, *Optimal large-eddy simulation of forced Burgers equation*, Phys. Fluids **14** (2002), no. 12, 4344–4351.

[10] P. A. Davidson, *An introduction to magnetohydrodynamics*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2001.

[11] A. Dunca and Y. Epshteyn, *On the Stolz-Adams deconvolution model for the large eddy simulation of turbulent flows*, SIAM J. Math. Anal. **37** (2006), no. 6, 1890–1902.

[12] A. Dunca, V. John, W. Layton, and N. Sahin, *Numerical analysis of large eddy simulation*, DNS/LES progress and challenges (C. Liu, L. Sakeland, and T. Beutner, eds.), Greyden Press, Columbus, 2001, pp. 359–364.

[13] A. Dunca, V. John, and W. J. Layton, *The commutation error of the space averaged Navier-Stokes equations on a bounded domain*, Contributions to current challenges in mathematical fluid mechanics, Adv. Math. Fluid Mech., Birkhäuser, Basel, 2004, pp. 53–78.

[14] A.V. Fursikov, *Optimal control of distributed systems. Theory and applications*, Translations of Mathematical Monographs, vol. 187, American Mathematical Society, Providence, RI, 2000, Translated from the 1999 Russian original by Tamara Rozhkovskaya.

[15] A. Gailitis and O. Lielausis, *On a possibility to reduce the hydrodynamical resistance of a plate*

in an electrolyte, (in Russian), Applied Magnetohydrodynamics. Reports of the Physics Institute **12** (1961), 143–146.

[16] R. Griesse and K. Kunisch, *Optimal control for a stationary MHD system in velocity-current formulation*, SIAM J. Control Optim. **45** (2006), no. 5, 1822–1845.

[17] P. Grisvard, *Boundary value problems in non-smooth domains*, Univ. of Maryland, Dept. of Math. Lecture Notes (1980), no. 19.

[18] M.D. Gunzburger, A.J. Meir, and J. Peterson, *On the existence, uniqueness, and finite element approximation of solutions of the equations of stationary, incompressible magnetohydrodynamics*, Math. Comp. **56** (1991), no. 194, 523–563.

[19] M.D. Gunzburger, J. Peterson, and C. Trenchea, *The velocity and magnetic field tracking problem for MHD flows with distributed controls*, Int. J. Pure Appl. Math. **42** (2008), no. 2, 289–296

[20] M.D. Gunzburger and C. Trenchea, *Analysis and discretization of an optimal control problem for the time-periodic MHD equations*, J. Math. Anal. Appl. **308** (2005), no. 2, 440–466.

[21] C. Henoch and J. Stace, *Experimental investigation of a salt water turbulent boundary layer modified by an applied streamwise magnetohydrodynamic body force*, Physics of Fluids, **7** (1995), 1371–1383.

[22] ———, *Experimental investigations of a salt water turbulent boundary layer modified by an applied streamwise magnetohydrodynamic body force*, Phys. Fluids **7** (1995), 1371–1381.

[23] V. John, *Large eddy simulation of turbulent incompressible flows*, Lecture Notes in Computational Science and Engineering, vol. 34, Springer-Verlag, Berlin, 2004, Analytical and numerical results for a class of LES models.

[24] A. Labovschii, W. Layton, M. Manica, M. Neda, L. Rebholz, I. Stanculescu, and C. Trenchea, *Mathematical architecture of approximate deconvolution models of turbulence*, Quality and Reliability of Large-Eddy Simulations, ERCOFTAC, Springer, 2007.

[25] A. Labovschii and C. Trenchea, *Large eddy simulation for MHD flows*, submitted.

[26] L. Landau and E. Lifschitz, *Électrodynamique des milieux continus*, Physique théorique, vol. tome VIII, MIR, Moscow, 1969.

[27] W. Layton, M. Sussman, and C. Trenchea, *Bounds on energy, magnetic helicity and cross helicity dissipation rates of approximate deconvolution models of turbulence for MHD flows*, Numer. Funct. Anal. Optim. (2010), accepted.

[28] W. Layton and C. Trenchea, *The Das-Moser commutator closure for filtering through a boundary is well posed*, Mathematical and Computer Modelling (2010), accepted.

[29] J. Leray, *Sur le mouvement d'un fluide visqueux emplissant l'espace*, Acta Math. **63** (1934), no. 1, 193–248.

[30] J.L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Études mathématiques, Dunod Gauthiers-Villars, 1969.

[31] J.C.S. Meng, P. J. Hendricks, and J.D. Hrubes, *Superconducting electromagnetic thrusters*, Sea Technology **33** (1992), 29–39.

[32] J.C.S. Meng, C.W. Henoch, and J.D. Hrubes, *Seawater electromagnetohydrodynamics: A new frontier*, Magnetohydrodynamics **30** (1994), 401–418.

[33] J.-L. Meyer, F. Durand, R. Ricou, and C. Vives, *Steady flow of liquid aluminum in a rectangular-vertical ingot mold, thermally or electromagnetically activated*, Metall. Trans. B **15B** (1984), 471–478.

[34] T.M. Özgökmen, T. Iliescu, and P.F. Fischer, *Large eddy simulation of stratified mixing in a three-dimensional lock-exchange system*, Ocean Modelling **26** (2009), 134–155.

[35] P.J. Prescott and F.P. Incropera, *Magnetically damped convection during solidification of a binary metal alloy*, J. Heat Transfer (Trans. ASME) **115** (1993), 302–310.

[36] P. Sagaut, *Large eddy simulation for incompressible flows*, third ed., Scientific Computation, Springer-Verlag, Berlin, 2006, An introduction, Translated from the 1998 French original, With forewords by Marcel Lesieur and Massimo Germano, With a foreword by Charles Meneveau.

[37] J. Sakane, B.Q. Li, and J.W. Evans, *Mathematical modeling of meniscus profile and melt flow in electromagnetic casters*, Metall. Trans. B **19B** (1988), 397–408.

[38] M. Sermange and R. Temam, *Some mathematical questions related to the MHD equations*, Comm. Pure Appl. Math. **36** (1983), 635–664.

[39] J.A. Shercliff, *A Textbook of Magnetohydrodynamics*, Pergamon, Oxford, 1965.

[40] N.S. Singh and P.R. Bandyopandhyay, *Linear feedback control on boundary layer using electromagnetic microtiles*, Trans. ASME J. Fluid Engrg. **119** (1997), 852–858.

[41] L. Tartar, *Topics in nonlinear analysis*, Publications Mathématiques d'Orsay, vol. 13, Université de Paris-Sud, Départment de Mathématique, Orsay, 1978.

[42] R. Temam, *Navier-Stokes equations and nonlinear functional analysis*, CBMS-NSF Regional

Conference Series in Applied Mathematics, Society for Industrial and Applied Mathematics, Philadelphia, 1995.

[43] A. Tsinober, *MHD flow drag reduction*, Viscous Drag Reduction in Boundary Layers, (D.M. Busnell and J.N. Hefner, eds.), Progress in Astronautics and Aeronautics, vol. 123, American Institute of Astronautics and Aeronautics, Reston, VA, 1990, pp. 327–349.

[44] A. B. Tsinober and A. G. Shtern, *Possibility of increasing the flow stability in a boundary layer by means of crossed electric and magnetic fields*, Magnetohydrodynamics **3** (1967), no. 2, 152–154.

[45] Ch. Vivès and R. Ricou, *Fluid flow phenomena in a single phase coreless induction furnace*, Metall. Trans. B **16B** (1985), 227–235.