The ARS Open Coloring Axiom with a Large Continuum

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Summary

We will be discussing the following theorem (joint with I. Neeman) from my thesis:

**Theorem (G., Neeman)**

The Abraham-Rubin-Shelah Open Coloring Axiom is consistent with \(2^{\aleph_0} = \aleph_3\).

An outline of the talk:

- In Section 1: background.
- In Section 2: the technique of preassigning colors in ARS.
- In Section 3: outline the main ideas of our result.
- In Section 4: sketch the construction of preassignments in both the original ARS setting and in our context.
Notation and Basic Definitions

Definition

- For a set $A$, $[A]^2$ denotes all two-element subsets of $A$.
- A (2-) Coloring of $A$ is a function $\chi : [A]^2 \rightarrow \{0, 1\}$.
- A subset $H$ of $A$ is 0-homogeneous (resp. 1-homogeneous) if $\chi$ takes the constant value 0 (resp. 1) on $[H]^2$. $H \subseteq A$ is said to be homogeneous if it is either 0 or 1 homogeneous.

The general question here: which colorings have large homogeneous set? A fundamental result:

Theorem (Ramsey)

Any 2-coloring $\chi$ of $\omega$ has an infinite homogeneous set.
The most naive generalization of Ramsey’s theorem to $\omega_1$ is false. However, by placing topological restrictions on the colorings, we obtain consistent generalizations of Ramsey’s theorem to $\omega_1$. These are called Open Coloring Axioms, the first of which appeared in [1] (or perhaps in Abraham-Shelah).

**Definition**

A coloring $\chi : [\omega_1]^2 \rightarrow \{0, 1\}$ is **open** if with respect to some 2nd countable, Hausdorff topology on $\omega_1$, $\chi^{-1}(\{c\})$ is open in the product, for each $c \in \{0, 1\}$.

The **Abraham-Rubin-Shelah Open Coloring Axiom**, abbreviated OCA$_{ARS}$, states that for any open coloring $\chi$ on $\omega_1$, there exist a sequence $\langle H_n : n \in \omega \rangle$ of $\chi$-homogeneous subsets of $\omega_1$ so that $\omega_1 = \bigcup_n H_n$. 
In [1], the authors showed the following theorem:

**Theorem (ARS)**

\[ \text{OCA}_{ARS} \text{ is consistent}. \]

We will discuss their result more later. For now, let’s look at an example. We will often switch between a coloring on \( \omega_1 \) and a coloring on a set of size \( \aleph_1 \), usually a subset of \( \mathbb{R} \) or \( \mathbb{R}^2 \).
Example

Let $A \subseteq \mathbb{R}$ have size $\aleph_1$ and let $f : A \longrightarrow \mathbb{R}$ be injective. We define a coloring $\chi$ on $\text{Graph}(f)$ by setting

$$\chi((a, f(a)), (b, f(b))) = 0$$

if $f \upharpoonright \{a, b\}$ is order-preserving, and otherwise we set $\chi$ to be 1.

- Because $f$ is injective, the value of $\chi$ on $((a, f(a)), (b, f(b)))$ can be witnessed by a pair of open sets in the plane, which yields that $\chi$ is open.
- Note that 0-homogeneous (resp. 1-) sets correspond to increasing (resp. decreasing) subfunctions of $f$. 
OCA\textsubscript{ARS} therefore implies that any such $f$ can be written as a countable union of monotonic subfunctions. (In fact, Todorčević proved that under MA, OCA\textsubscript{ARS} is a consequence of this.)

As a result, we see that OCA\textsubscript{ARS} has some effect on the size of $2^{\aleph_0}$:

**Fact**

OCA\textsubscript{ARS} implies that $2^{\aleph_0} \geq \aleph_2$.

By a result of Dushnik and Miller ([2]), the CH implies that there exists an injective $f : \mathbb{R} \rightarrow \mathbb{R}$ with no uncountable monotonic subfunction. In particular, $f$ is not a union of countably-many monotonic subfunctions.
The conjunction of OCA\textsubscript{ARS} with Todorčević’s Open Coloring Axiom (also known as the Open Graph Axiom) implies that $2^{\aleph_0} = \aleph_2$, by a result of Justin Moore ([3]).

In this vein, I. Farah showed (unpublished) that Todorčević’s OCA restricted to graphs of size $\aleph_1$ is consistent with $2^{\aleph_0} > \aleph_2$.

However, the question of whether OCA\textsubscript{ARS} on its own is powerful enough to decide the value of $2^{\aleph_0}$ remained open. To explain why, we need to dive deeper into the original result from [1].
To obtain their model, ARS iterate (with finite support) to add the requisite homogeneous sets for open colorings. The main technical difficulty is to obtain c.c.c. posets which do this.

**Key Point**

The natural first guess won’t work. If $\chi$ is open, the poset consisting exactly of finite approximations to the decomposition of $\omega_1$ into countably-many $\chi$-homogeneous sets will likely not be c.c.c.

Something else is needed. This “extra something” is a **Preassignment of Colors**. We give a very general definition on the next slide.
Adding Homogeneous sets by c.c.c. Forcing

Definition

Let $\chi$ be an open coloring on $\omega_1$ and $f : \omega_1 \rightarrow \{0, 1\}$ an arbitrary function. We define $\mathbb{Q}(\chi, f)$ to be the poset where conditions are finite partial functions $p$ with $\text{dom}(p) \subset \omega$ so that for each $n \in \text{dom}(p)$,

- $p(n) \subseteq \omega_1$ is finite;
- $f$ is constant on $p(n)$, say with value $c_n \in \{0, 1\}$;
- $p(n)$ is $\chi$-homogeneous with color $c_n$.

Any such function $f$ will be a preassignment of colors. The role of $f$ is to decide, in the ground model, whether each $\alpha < \omega_1$ will be generically placed in some 0-homogeneous set or in some 1-homogeneous set.
Adding Homogeneous sets by c.c.c. Forcing

A preassignment $f$ allows us to write $Q(\chi, f)$ as a product of two posets, one of which decomposes $f^{-1}(\{0\})$ into countably-many $0$-homogeneous sets and the other of which decomposes $f^{-1}(\{1\})$ into countably-many $1$-homogeneous sets. Most $f$ will lead to $Q(\chi, f)$ collapsing $\omega_1$. However, ARS (and previously, Abraham and Shelah) showed that effective preassignments can be built over models of the CH; this is the key technical result in their theorem.

**Proposition (ARS)**

Suppose that $V \models \text{CH}$ and $\chi \in V$ is an open coloring on $\omega_1$. Then there exists a preassignment $f \in V$ so that $Q(\chi, f)$ is c.c.c.
Roughly, the construction of a preassignment is done locally, inside the intervals of some club on $\omega_1$ coming from a continuous, $\in$-increasing sequence of elementary submodels decomposing some countably-closed model $M$ of size $\aleph_1$.

Inside the $\alpha$th interval, the preassignment is built so as to diagonalize out of all closed sets in all finite products of the space which are member of the $\alpha$th model, ensuring that no such closed set is the topological closure of an antichain of conditions (this is useful since $\chi$ is open).

Since our spaces are 2nd countable and the CH holds, there are only $\omega_1$-many such closed sets, and therefore all such closed sets live in $M$.

Therefore, each such closed set is captured by a tail of the models on the sequence.
The CH Obstacle

Thus to obtain their model, ARS build an $\omega_2$-length iteration, over a model of the CH, of posets adding homogeneous sets. At each proper initial segment CH holds, so the next (name for a) preassignment can be constructed. However, it appears that the CH is necessary to build effective preassignments; we call this the CH-obstacle. Since an iteration of $\aleph_1$-sized posets all of whose initial segments satisfy the CH can only lead to a model in which $2^{\aleph_0}$ is at most $\aleph_2$, we are left with the following question:

**Question**

Can we build a model of $\text{OCA}_{ARS}$ in which $2^{\aleph_0}$ is greater than $\aleph_2$?

Yes! We now turn to outlining the main features of our solution.
The general theme of our solution is the following:

- We only construct (names for) preassignments with respect to a small (size $\aleph_2$) alphabet $A$ of c.c.c., size $\aleph_1$, and hence CH-preserving, iterations. (So we only construct $\aleph_2$-many preassignments.)

- But we combine these short “alphabetical posets” to construct much longer ones which result in models wherein $2^{\aleph_0} > \aleph_2$.

One can check by nice-name and cardinality arguments that executing this theme requires constructing names for preassignments with substantial amounts of symmetry which we now explain.
Symmetric Names

Recall that ARS showed that, in a model of the CH, if $\mathbb{P}$ is c.c.c. of size $\aleph_1$, then for any $\mathbb{P}$-name $\dot{\chi}$ for an open coloring, there exists a $\mathbb{P}$-name $\dot{f}$ for a preassignment so that $\mathbb{P} \models Q(\dot{\chi}, \dot{f})$ is c.c.c. We need the following result (in fact, *much more*):

**Fact (G., Neeman)**

Suppose that the CH holds, that $\mathbb{P}$ is productively c.c.c. and that $\dot{\chi}$ is a $\mathbb{P}$-name for an open coloring. Then there is a single $\mathbb{P}$-name $\dot{f}$ so that if $G_L, G_R$ are mutually generic filters for $\mathbb{P}$, then

$$Q(\dot{\chi}[G_L], \dot{f}[G_L]) \times Q(\dot{\chi}[G_R], \dot{f}[G_R])$$

is c.c.c. in $V[G_L \times G_R]$. 
Constructing such symmetric names takes us beyond the techniques of ARS; determining exactly what “symmetry” means leads to the notion of a *Partition Product*. We will sketch the definition.

Roughly, a partition product is a type of restricted memory iteration (these use only partial generic information at each stage) with isomorphism and coherent overlap conditions on the memories.

- We will first give a concrete example;
- then we will move to the more general definition.
Example

Suppose that $P = \text{Add}(\omega, \omega_1)$ is the poset to add an $\omega_1$-sequence $\langle \dot{r}_\nu : \nu < \omega_1 \rangle$ of Cohen reals, and let $\dot{Q}$ be the $P$-name for the poset to add a new real almost disjoint from the $\dot{r}_\nu$.

In addition to combinations like $\prod_{j \in J}(P \ast \dot{Q})$ (mutually generic) and $P \ast \prod_{i \in I} \dot{Q}$ (complete agreement), we can form variations where we interpret $\dot{Q}$ by generics sharing a partial amount of information. For example, say $\nu < \omega_1$, and consider

$$(P \upharpoonright \nu) \ast (P \upharpoonright [\nu, \omega_1])^2,$$

adding two sequences of Cohen reals $\vec{r} := \langle \dot{r}_i : i < \omega_1 \rangle$ and $\vec{s} := \langle \dot{s}_i : i < \omega_1 \rangle$ so that $\vec{r}$ and $\vec{s}$ agree below $\nu$ and are mutually generic past $\nu$. In the extension, we then get a “blended” product

$$\dot{Q}[\vec{r}] \times \dot{Q}[\vec{s}].$$
Example

Note: the poset \((P \upharpoonright \nu) \ast (P \upharpoonright [\nu, \omega_1])^2 \ast (\hat{Q}(\hat{r}) \times \hat{Q}(\hat{s}))\) can be “rearranged.” For instance, it is isomorphic to

\[
U := P \ast \hat{Q}[\hat{r}] \ast (P \upharpoonright [\nu, \omega_1]) \ast \hat{Q}[\hat{s}].
\]

As long (roughly) as the reals relevant for interpreting \(\hat{Q}\) appear at an earlier “stage” than when \(\hat{Q}\) does, we can rearrange as we please.

**Honesty in advertising:** \(U\) is the sort of thing that occurs “in nature” and is a specific instance of what we mean by a partition product. The origin of the name: the rearranged version with all of the reals appearing first has many “products” of interpretations of the \(\hat{Q}\)’s.
RM iterations and Partition Products

Definition

\( \mathcal{R} \) is a finite support restricted memory “iteration” on \( X \) of iterand names \( \langle \dot{U}_\xi : \xi \in X \rangle \), with memory function \( \xi \mapsto b_{\mathcal{R}}(\xi) \) (\( \xi \in X \)) if:

1. for each \( \xi \in X \), \( b_{\mathcal{R}}(\xi) \subseteq X \cap \xi \) and \( b_{\mathcal{R}}(\xi) \) is closed under \( b_{\mathcal{R}} \);
2. \( \dot{U}_\xi \) is an \( \mathcal{R} \upharpoonright b_{\mathcal{R}}(\xi) \)-name;
   - \textbf{Note:} \( \mathcal{R} \upharpoonright b_{\mathcal{R}}(\xi) \) is a RMI of \( \langle \dot{U}_\zeta : \zeta \in b_{\mathcal{R}}(\xi) \rangle \) by (1);
3. conditions in \( \mathcal{R} \) are finite partial functions \( p \) on \( X \). For each \( \xi \in \text{dom}(p) \), \( p(\xi) \) is a canonical \( \mathcal{R} \upharpoonright b_{\mathcal{R}}(\xi) \)-name for an element of \( \dot{U}_\xi \);
4. \( q \preceq p \) in \( \mathcal{R} \) iff \( \text{dom}(q) \supseteq \text{dom}(p) \) and for all \( \xi \in \text{dom}(q) \), \( q \upharpoonright b_{\mathcal{R}}(\xi) \models_{\mathcal{R}} b_{\mathcal{R}}(\xi) q(\xi) \preceq p(\xi) \).
In general, given $R$ and $b$ as above, if $Y \subseteq X$ is memory-closed (i.e., $\xi \in Y$ implies $b_R(\xi) \subseteq Y$) then $R \upharpoonright Y$ is also an RMI and a regular suborder of $R$.

Note that $R$ is only a dense subset of an iteration.

We now isolate the situations where we can “rearrange” the coordinates in an RMI.

**Definition**

Let $b$ be a memory function on $X$ and $\sigma : X \longrightarrow X^*$ a bijection. $\sigma$ is an acceptable rearrangement if for all $\zeta, \xi \in X$, $\zeta \in b(\xi)$ implies $\sigma(\zeta) < \sigma(\xi)$. 
Rearrangements give Isomorphisms

Given an RMI $R$ of $\langle \dot{U}_\xi : \xi \in X \rangle$ with $b$ as a memory function, and given an acceptable rearrangement $\sigma$ of $b$, $\sigma$ then induces an isomorphism from $R$ to an RMI $R^*$ on $X^*$. The memory function for $R^*$, denoted by $\sigma(b)$, is defined as

$$\sigma(b)(\sigma(\xi)) = \sigma[b(\xi)];$$

$\sigma(b)$ is a memory function by definition of $\sigma$ acceptable.
As mentioned previously, we build our iterations by arranging the members of a small alphabet of short iterations (the ones for which we build names for preassignments). We now work towards explicating this idea.

**Definition**

A *restricted memory alphabet* is a pair of sequences

\[ P = \langle P(\delta) : \delta < \gamma \rangle \] (of RMIs) and

\[ Q = \langle Q(\delta) : \delta < \gamma \rangle, \] where

\[ \gamma \leq \omega_2 \] and where each \( Q(\delta) \) is a \( P(\delta) \)-name for a poset.
Definition

An RMI $\mathcal{R}$ with iterands $\langle \hat{U}_\xi : \xi \in X \rangle$ and memory function $b_{\mathcal{R}}$ is called a simplified partition product (based upon a restricted memory alphabet $\mathcal{P}, \mathcal{Q}$) if there exist functions $\text{index}_{\mathcal{R}}$ and $\pi^\mathcal{R}_{\xi}$ for each $\xi \in X$ so that for each $\xi \in X$,

1. $\text{index}^\mathcal{R}_R(\xi) \leq \text{lh}(P)$; say $\text{index}^\mathcal{R}_R(\xi) = \mu$;
2. $\pi^\mathcal{R}_{\xi} : \text{dom}(P(\mu)) \rightarrow b_{\mathcal{R}}(\xi)$ is an acceptable rearrangement of $P(\mu)$;
3. $\mathcal{R} \upharpoonright b_{\mathcal{R}}(\xi)$ is exactly equal to the $\pi^\mathcal{R}_{\xi}$-rearrangement of $P(\mu)$;
4. $\hat{U}_\xi = \pi^\mathcal{R}_{\xi}(\hat{Q}(\mu))$ (the translate of this name).
(Simplified) Partition Products

So we see that in a simplified partition product $\mathcal{R}$, the restriction of $\mathcal{R}$ to a given memory $b_\mathcal{R}(\xi)$ is a “copy” of one of the alphabetical posets on $\mathcal{P}$; $\text{index}_\mathcal{R}(\xi)$ is responsible for selecting (an index for) this poset. Then at stage $\xi$ itself, we force with the “translate” of $\dot{Q}(\text{index}_\mathcal{R}(\xi))$ by the rearrangement. This what we meant by isomorphism conditions earlier.

To form *Partition Products*, we then impose constraints on how the memory sets overlap and how the bijections interact in the case of overlap.
Essentially: given $\xi, \zeta$ in $X$, the overlap $b_R(\zeta) \cap b_R(\xi)$ pulls back via $\pi^R_\xi$ to a “simple” subset of $\text{dom}(\mathbb{P}_{\text{index}_R}(\xi))$.

1. In the case that $\zeta$ and $\xi$ have the same index, this should just be an initial segment of $\text{dom}(\mathbb{P}_{\text{index}_R}(\xi))$.

2. If $\zeta$ has index below $\xi$, then this pull-back should sit inside $\text{dom}(\mathbb{P}_{\text{index}_R}(\xi))$ in a definable way.

The **Point**: this constrains the number of regular suborders of a sufficiently simple, yet still rich, type (ones in which uncountable antichains could occur). We can then ensure that an appropriate $\aleph_1$-sized, countably closed model contains isomorphic copies of each such simple type, which is necessary to catch our tail when we build preassignments (more below).
An Overview of Our Construction

We now provide an overview of the construction of our partition products; we describe in the final section how to construct names for preassignments.

1. By recursion we define sequences $\mathbb{P} = \langle \mathbb{P}(\alpha) : \alpha \leq \omega_2 \rangle$ and $\dot{\mathbb{Q}} = \langle \dot{\mathbb{Q}}(\alpha) : \alpha < \omega_2 \rangle$.

2. Each $\mathbb{P}_\alpha$ will be a partition product over the alphabet $\mathbb{P} \upharpoonright \alpha$ and $\dot{\mathbb{Q}} \upharpoonright \alpha$ and $\dot{\mathbb{Q}}(\alpha)$ will be a $\mathbb{P}(\alpha)$-name.

3. The final partition product, $\mathbb{P}_{\omega_2}$, has size $\aleph_3$ (don’t let that subscript $\omega_2$ fool you!)

4. We do the construction in $L$ using condensation, so that each $\mathbb{P}(\alpha)$ is defined uniformly from $\alpha$ and $\mathbb{P} \upharpoonright \alpha$ and $\dot{\mathbb{Q}} \upharpoonright \alpha$. 
A Successor Stage: $\mathbb{P}(\alpha)$

We maintain as a recursion hypothesis that for each $\alpha$, every partition product built over $\mathbb{P} \upharpoonright \alpha$ and $\dot{Q} \upharpoonright \alpha$ is c.c.c. (So essentially no problems at limits).

At stage $\alpha$, we define $\mathbb{P}(\alpha) \upharpoonright \gamma$ by recursion. At each stage we look at the $\gamma$-th “good” level of $L$ which sees that $\alpha$-th “local $\omega_2$,” call it $\kappa_\alpha$. As long as we’re below the level where $\kappa_\alpha$ is collapsed (if $\alpha < \omega_2$), we take hulls of $\omega_1$ to determine the memory and index at stage $\gamma$.

Once $\kappa_\alpha$ is seen to have size $\aleph_1$, we halt. Then we have to determine $\dot{Q}(\alpha)$...
A Successor Stage: $\dot{Q}_\alpha$

To determine $\dot{Q}(\alpha)$, we select a $\mathbb{P}(\alpha)$-name $\dot{\chi}_\alpha$ for an ARS coloring.

We then need to build a single $\mathbb{P}(\alpha)$-name $\dot{f}_\alpha$ for a preassignment so that

(*): Any partition product based upon $\mathbb{P} \upharpoonright (\alpha + 1)$ and $\dot{Q} \upharpoonright (\alpha + 1)$ is c.c.c. (this is what we mean, more precisely, by “symmetric” names)

Using the coherence conditions on PPs, we can take a countably-closed model $M$ of size $\aleph_1$ which contains the construction and then argue that if (*) were to fail, then there would be some counterexample which lives in $M$.

Our task is then to build $\dot{f}_\alpha$ in such a way that no partition product in $M$ can lead to a counterexample.
Constructing Preassignments: the Set-Up

We are now going to describe how to construct preassignments, first in the original ARS setting, and then in our more general setting. Assume the CH for what follows.

Notation

Using the CH, fix $M \prec H(\omega_3)$ with $|M| = \aleph_1$ and $\omega M \subseteq M$. Write

$$M = \bigcup_{\gamma < \omega_1} M_\gamma$$

where $\langle M_\gamma : \gamma < \omega_1 \rangle$ is a continuous, $\in$-increasing sequence of ctb esms of $M$. Finally, let $\delta_\gamma := M_\gamma \cap \omega_1 \in \omega_1$, and fix an enumeration $\langle \nu_{\gamma,n} : n < \omega \rangle$ of $[\delta_\gamma, \delta_{\gamma+1})$. We will later have that various parameters are in $M_0$.  

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Let’s first construct preassignments which are effective in the ARS setting. Fix an open coloring $\chi$ with $\chi \in M_0$. We will build the preassignment $f$ locally, defining it on one interval $[\delta_\gamma, \delta_{\gamma+1})$ at a time. The property which $f$ needs to satisfy is the following:

**Proposition (ARS/AS)**

There exists a function $f : \omega_1 \rightarrow \{0, 1\}$ so that for any $\gamma < \omega_1$, $n < \omega$, and any $C \subseteq \omega_1^n$ which is in $M_\gamma$, the following implication holds:

- if $\langle \nu_\gamma, 0, \ldots, \nu_\gamma, n-1 \rangle \in C$, then there exists a tuple $\langle \mu_0, \ldots, \mu_{n-1} \rangle$ in $M_\gamma \cap C$ so that for each $k < n$,

$$\chi(\mu_k, \nu_\gamma, k) = f(\nu_\gamma, k).$$
Preassignments in ARS: Reflection to Preassigned Color

Proof

Fix some $\gamma < \omega_1$. $f$ is constructed recursively, assuming that $f \upharpoonright \{\nu_{\gamma,k} : k < n\}$ is constructed satisfying the above proposition for the finite tuple $\langle \nu_{\gamma,0}, \ldots, \nu_{\gamma,n-1} \rangle$. We will just show how to preassign either 0 or 1 to the first ordinal ordinal $\nu_{\gamma,0} \in [\delta_{\gamma}, \delta_{\gamma+1})$.

Suppose for a contradiction that neither 0 nor 1 can be preassigned to $\nu := \nu_{\gamma,0}$.

- Since 0 cannot be preassigned, there exists a $C_0 \subseteq \omega_1$ with $C_0 \in M_\gamma$ so that $\nu \in C_0$, but for any $\mu \in M_\gamma \cap C_0$, $\chi(\mu, \nu) = 1$.

- Since 1 cannot be preassigned, there exists a $C_1 \subseteq \omega_1$ with $C_1 \in M_\gamma$ so that $\nu \in C_1$, but for any $\mu \in M_\gamma \cap C_1$, $\chi(\mu, \nu) = 0$. 

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Proof.

Now $C_0 \cap C_1$ is also in $M_\gamma$. And $\nu \in C_0 \cap C_1$. Apply the elementarity of $M_\gamma$ to find some $\mu \in M_\gamma$ so that $\mu \in C_0 \cap C_1$. But $\mu \neq \nu$ (since $\nu \notin M_\gamma$), so $\chi(\mu, \nu)$ is defined. Then $\chi(\mu, \nu)$ is neither 0 nor 1, a contradiction.

Preassigning to the next ordinals is similar, and it involves taking projections.
Now we want to consider how to generalize this to construct a symmetric preassignment. We will consider only the simplest case which is relevantly different. Fix a productively c.c.c. poset $\mathbb{P}$ of size $\aleph_1$ and a $\mathbb{P}$-name $\dot{\chi}$ for an open coloring on $\omega_1$.

**Notation**

If $\dot{\tau}$ is a $\mathbb{P}$-name and we force with $\mathbb{P} \times \mathbb{P}$, we use $\dot{\tau}_L$ (resp. $\dot{\tau}_R$) to denote the $\mathbb{P}^2$-name for the interpretation of $\mathbb{P}$ under the Left (resp. Right) generic.
Our goal is to construct a single $\mathbb{P}$-name $\dot{f}$ for a preassignment so that

$$\mathbb{P} \times \mathbb{P} \models Q(\dot{\chi}[\dot{G}_L], \dot{f}[\dot{G}_L]) \times Q(\dot{\chi}[\dot{G}_R], \dot{f}[\dot{G}_R])$$

is c.c.c.

We now assume that $\mathbb{P}, \dot{\chi}$ are in $M_0$. As in the ARS case, we construct $\dot{f}$ locally, one interval at a time, satisfying a condition similar to the previous one.
Reflection for Multiple “Branches”

Proposition

There exists a $\mathbb{P}$-name $\dot{f}$ so that for any $\gamma < \omega_1$, any $n$, and any $\mathbb{P}^2$-name $\dot{C}$ for a closed subset of $\omega_1^n$ which is in $M_\gamma$, the following implication holds:

- $\models_{\mathbb{P} \times \mathbb{P}}$ if $\langle \nu_\gamma, 0, \ldots, \nu_\gamma, n-1 \rangle \in \dot{C}$, then there exists a tuple $\langle \mu_0, \ldots, \mu_{n-1} \rangle$ of ordinals below $\delta_\gamma$ in $\dot{C}$ so that for each $k < n$,

$$\dot{\chi}_L(\mu_k, \nu_\gamma, k) = \dot{f}_L(\nu_\gamma, k) \land \dot{\chi}_R(\mu_k, \nu_\gamma, k) = \dot{f}_R(\nu_\gamma, k).$$

So each pair $\langle \mu_k, \nu_\gamma, k \rangle$ must get the appropriate color under each “branch”, i.e., under both interpretations of the coloring $\dot{\chi}$. 
Here we view each value $\hat{f}(\nu)$ as a \textit{canonical color name}, i.e., a function from a maximal antichain $\hat{a}_\nu$ of $\mathbb{P}$ into $\{0, 1\}$ with the natural interpretation.

For the purposes of illustration, we will just show how to preassign to $\nu_{\gamma,0}$ below a single condition (i.e., the first stage in the construction of the first canonical color name $\hat{f}(\nu_{\gamma,0})$).
Preassigning Below a Single Condition

**Definition**

Fix $p \in \mathbb{P}$. If $q \leq p$ and $c \in \{0, 1\}$, then we say that $q \mapsto c$ is **good** if the following holds: For any $\mathbb{P}^2$-name $\dot{C}$ in $M_\gamma$ for a subset of $\omega_1$, $\langle q, q \rangle$ forces that

- if $\nu \in \dot{C}$, then there exist $\mu$ in $M_\gamma \cap \dot{C}$ so that
  
  $$\dot{\chi}_L(\mu, \nu) = c \text{ and } \dot{\chi}_R(\mu, \nu) = c.$$  

We now need to prove the following:

**Lemma**

There exists $q \leq p$ and $c \in \{0, 1\}$ so that $q \mapsto c$ is good.
Preassigning Below a Single Condition

Proof

If there exists some \( q \leq p \) so that \( q \mapsto 0 \) is good, we are done. So suppose that this is false. We then show that \( p \mapsto 1 \) is good.

Suppose otherwise. Then \( \langle p, p \rangle \) does not force the desired statement. Thus there exist \( q_L, q_R \leq p \) and \( \dot{C} \in M_\gamma \) so that \( \langle q_L, q_R \rangle \) forces the following:

- \( \nu \in \dot{C} \), but
- for any \( \mu \in M_\gamma \cap \dot{C} \), either \( \dot{\chi}_L(\mu, \nu) = 0 \) or \( \dot{\chi}_R(\mu, \nu) = 0 \).

By assumption, for \( Z \in \{L, R\} \), \( q_Z \mapsto 0 \) is not good. Thus we can fix some \( q_{Z,L}, q_{Z,R} \leq q_Z \) and \( \dot{C}_Z \in M_\gamma \) so that \( \langle q_{Z,L}, q_{Z,R} \rangle \) forces

- \( \nu \in \dot{C}_Z \), but
- for any \( \mu \in M_\gamma \cap \dot{C}_Z \), either \( \dot{\chi}_L(\mu, \nu) = 1 \) or \( \dot{\chi}_R(\mu, \nu) = 1 \).
Proof

Now we force in $\mathbb{P}^4$ below the condition $\langle q_L, L, q_L, R, q_R, L, q_R, R \rangle$ to obtain generics $G_{X,Y}$ for each $X, Y \in \{L, R\}$, with $q_{X,Y} \in G_{X,Y}$. Let $H$ be the product of these four generic filters. Some observations:

- for each $X, Y \in \{L, R\}$, $\nu \in \hat{\mathcal{C}}[G_{L,X} \times G_{R,Y}]$, because $\langle q_{L,X}, q_{R,Y} \rangle \leq \langle q_L, q_R \rangle$.
- $\nu$ is in $\hat{\mathcal{C}}_Z[G_{Z,L} \times G_{Z,R}]$ for each $Z \in \{L, R\}$, by choice of $\langle q_{Z,L}, q_{Z,R} \rangle$.
- Since $\mathbb{P}^4$ is c.c.c., we have $\nu \notin M_\gamma[H]$, and so by elementarity we may find some $\mu < \delta_\gamma$ in the same sets.

By choice of $\langle q_{Z,L}, q_{Z,R} \rangle$, we know that either $\check{\chi}[G_{Z,L}](\mu, \nu) = 1$ or $\check{\chi}[G_{Z,R}](\mu, \nu) = 1$. Let $d(Z) \in \{L, R\}$ select which of these is true.
Preassigning Below a Single Condition

Proof.

Note that we then have that \( \dot{\chi}[G_Z,d(Z)](\mu,\nu) = 1 \) for each \( Z \in \{L, R\} \). Now \( \langle q_L,d(L), q_R,d(R) \rangle \) extends \( \langle q_L, q_R \rangle \), and by choice of \( \langle q_L, q_R \rangle \), since \( \nu \in \dot{C}[G_L,d(L) \times G_R,d(R)] \), we must have that either

\[
\dot{\chi}[G_L,d(L)](\mu,\nu) = 0 \text{ or } \dot{\chi}[G_R,d(R)](\mu,\nu) = 0,
\]

a contradiction!

We will close with a few vague remarks about how the proof works in full generality by abstracting a general proof schema.
In the above proof we wanted to argue that \( p \mapsto 1 \) is good. To see that no extension of \( p \) forces the negation of the desired conclusion, we embedded \( P \) into a much larger poset, namely \( P^4 \), by adding various counterexamples to preassigning 0.

As a result, we saw that along at least one branch of \( P^4 \), we achieved the desired colors.

In general, if we’ve constructed the \( \beta \)th partition product \( P_\beta \) and want to construct \( f_\beta \) for \( \chi_\beta \), we continue to follow this schema.

However, the construction becomes much more complicated since various copies of \( P_\beta \) inside the larger partition product needn’t be disjoint.

The proof then works by induction on the height of the agreement between images of the \( P_\beta \) inside larger partition products.
Thanks for listening!!


