1. Find the fundamental matrix solution $\Psi(t)$ for the system

$$x_1' = 2x_2$$
$$x_2' = -x_1 + 3x_2$$

Hint: Fractions and square roots are not necessary in finding the solution, and the final answer has no fractions or square roots.

2. (a) Find a set of two linearly independent real solutions to the system

$$\mathbf{x}' = \begin{bmatrix} 2 & 4 \\ -1 & 2 \end{bmatrix} \mathbf{x}.$$

b. Find a solution to the system of part (a) which satisfies $x_1(0) = 1,\ x_2(0) = 2$.

3. (a) For what values of k does a solution exist to the following system:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= -1 \\ 4x_1 + 5x_2 + 6x_3 &= 2 \\ 7x_1 + 8x_2 + 9x_3 &= k. \end{aligned}$$

(b). Choose one of the values of k for which there is a solution. For that value of k, find the general solution to the system.

4. (a). Determine if the following set of vectors is linearly independent.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

b. Find a basis for the subspace of R^4 which consists of all linear combinations of the three vectors in (a).

5. The equation

$$u'' + \frac{1}{t}u' - \frac{1}{t^2}u = 0$$

has solutions

$$u_1(t) = t, \quad u_2(t) = \frac{1}{t}.$$

(a). Show that these form a basic set of solutions for the equation on the interval $0 < t < \infty$.

(b). Find the 2×2 companion system of differential equations for this second order equation, and find a matrix solution Φ such that $\Phi(1)^{-1}$ exists. (You do not have to find the fundamental solution $\Psi(t)$.)

(c) Find a particular solution to

$$u'' + \frac{1}{t}u' - \frac{1}{t^2}u = t^2$$

by using either variation of parameters for the companion system or variation of parameters for the original equation.

7. A mass of 1 kg is suspended from a spring with spring constant 5 N/m. The spring is stretched a distance of 0.1 m initially and released. The damping coefficient is $C = 2$ kg/s.

(a) Determine the displacement $y(t)$ of the spring from its rest position as a function of t.

(b) Does the spring oscillate around the rest position? Explain your answer.

8. Find the general real solution of the following equations.

(a) $y'' + 2y' + 2y = 0$

(b) $y'' + 2y' + 2y = e^t \sin t,$

(c) $y' + t\,y = e^{-\frac{1}{2}t^2}$

9. (a). Give the companion system for the third order equation

$$y''' - y = 0.$$

b. Find the eigenvalues of the associated matrix.

c. Does the third order equation have any solutions such that $y(t)$ oscillates? Explain your answer.