Math 2371 – Spring 2009

Quiz #3

Problem 1:
(a) Show that in a complex Euclidean space, if \( A \) is such that \( (x, Ax) > 0 \) for all \( x \neq 0 \), then \( A \) is self-adjoint.
(b) Show that the statement is not true in a real Euclidean space.

Proof: (a) Suppose that a map \( A \) is positive, i.e., \( (x, Ax) > 0 \) for all \( x \neq 0 \). Then \( (x, Ax) \) is real for all \( x \neq 0 \), which implies that \( A \) is self-adjoint as follows:

\[
(x + y, A(x + y)) = (x, Ax) + (x, Ay) + (Ax, y) + (y, Ay) \quad \text{implies} \quad \text{Im}(x, Ay) = \text{Im}(Ax, y)
\]

\[
(x + iy, A(x + iy)) = (x, Ax) - i(x, Ay) + i(Ax, y) - (y, Ay) \quad \text{implies} \quad \text{Re}(x, Ay) = \text{Re}(Ax, y)
\]

(b) Consider, for example, the matrix \( A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \) and \( x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \). Then

\[
(x, Ax) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + x_2^2 \geq 0 \quad \text{for all} \quad x \neq 0 , \quad \text{but} \quad A \quad \text{is not symmetric.}