Math 2371 – Spring 2009
Practice Problems IV

Problem 1: Let \( A \) be a linear map on a finite-dimensional complex Euclidean space. Show that if \( A + A^* > 0 \) then
(a) \( A \) is invertible and \( A^{-1} + (A^{-1})^* > 0 \).
(b) Eigenvalues of \( A \) have positive real parts.

Problem 2: (a) Find 2x2 matrices \( A \) and \( B \) such that \( 0 < A < B \) such that \( AB + BA \) is not positive.
(b) Find 2x2 matrices \( A \) and \( B \) such that \( 0 < A < B \) such that \( A^2 \) is not less than \( B^2 \).

Problem 3: Show that if \( A \) is positive selfadjoint, then so is \( A, A^T, A^* \), and \( A^{-1} \).

Problem 4: Let \( A \) be a positive selfadjoint \( n \times n \) matrix.
(a) Show that \( a_{ii}a_{jj} > |a_{ij}|^2 \) for all \( i, j = 1, 2, \ldots, n \).
(b) What can you conclude about \( A \) if one of its diagonal entries is zero?
(c) Show that the largest entry of \( A \) is on the diagonal.

Problem 5: Show that the following special matrices are positive (if you need hints, see HJ, p.401, Problems 16, 17, 18)
(a) \( A = [a_{ij}] \) where \( a_{ij} = 1/(i + j - 1) \)
(b) \( A = [a_{ij}] \) where \( a_{ij} = 1/(i + j) \)
(c) \( A = [a_{ij}] \) where \( a_{ij} = \min\{i, j\} \)

Problem 6: Let \( A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} \). Compute \( \sqrt{A} \). Find an orthogonal matrix \( Q \) such that \( QAQ^T \) is diagonal.

Problem 7: Suppose \( A \) is nonnegative selfadjoint \( n \times n \) matrix and rank \( A = r < n \). Show that \( A \) has \( r \times r \) positive principal submatrix.

Problem 8: If \( A \) and \( B \) are positive, show that the block matrix \( \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \) is positive.

Problem 9: Let \( A \) be \( n \times n \) real symmetric positive matrix and \( b \) a real vector. Define \( \phi(x) = (x, Ax) - 2(b, x) \). Show that real vector \( z \) obeys \( Az = b \) if and only if \( \phi(z) \leq \phi(x) \) for all \( x \).