Math 2371, Midterm Exam practice, Spring 2009
Instructor: D. Swigon
SOLUTIONS

Problem 1: Let *A* be an anti-selfadjoint map on a finite-dimensional complex Euclidean space. Show that
(a) $A-I$ is invertible.
(b) If $U=(A+I)(A-I)^{-1}$ then *U* is unitary and $U-I$ is invertible.

Proof: (a) If *A* is anti-selfadjoint then its eigenvalues are pure imaginary. Thus 1 is not an eigenvalue and hence $A-I$ is invertible.
(b) *A* has a spectral decomposition $A=\sum_k ia_k P_k$ where a_k are nonzero real numbers and the orthogonal projectors P_k obey $P_k^2=P_k$, $P_k P_j=0$ if $i\neq j$, $P_k^*=P_k$, and $\sum_k P_k=I$. The matrix *U* then obeys: $U=\sum_k \frac{ia_k+1}{ia_k-1}P_k=\sum_k \frac{2ia_k+1-a_k^2}{a_k^2+1}P_k$. As a_k are nonzero, the eigenvalues of *U* are distinct from 1 and $U-I$ is invertible. Furthermore, it follows that

$$UU^*=\sum_k \frac{2ia_k+1-a_k^2}{a_k^2+1}P_k\sum_j \frac{-2ia_j+1-a_j^2}{a_j^2+1}P_j^*$$

$$=\sum_{j,k}\frac{2ia_k+1-a_k^2}{a_k^2+1}\,\frac{-2ia_j+1-a_j^2}{a_j^2+1}P_kP_j$$

$$=\sum_k \frac{(2ia_k+1-a_k^2)(-2ia_k+1-a_k^2)}{(a_k^2+1)^2}P_k=\sum_k P_k I$$

and hence *U* is unitary.
(Another proof using the fact that $(A+I),(A-I),(A+I)^{-1},(A-I)^{-1}$ all commute can also be constructed.)

Problem 2: Prove or disprove:
(a) If a complex square matrix *A* is an isometry and involution ($A^2=I$) then it is selfadjoint.
(b) Complex square matrices AA^* and A^*A are always unitarily similar.

Proof: (a) Statement is true. The matrix *A* obeys $AA^*=I=A^2$. Furthermore, *A*, being isometry, is invertible. Thus $A^*=A$.
(b) Statement is true. Let $A=RU$ be the polar decomposition of *A*. Then
$AA^*=RUU^*R^*=RR^*=R^2=R^*R=UU^*R^*RUU^*=UA^*AU^*$
and hence AA^* and A^*A are unitarily similar.

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Problem 3: Show that if *A* is positive linear map on a finite-dimensional complex Euclidean space and *AB* is selfadjoint then $|(ABx,x)| \le \|B\|(Ax,x)$.

Proof: Consider the function $R(x) = (ABx,x)/(Ax,x)$. Since AB is selfadjoint and *A* is positive, the function is real-valued and attains its minimum b_1 and maximum b_n on the sphere $\|x\| = 1$. Thus $|(ABx,x)| \le \max\{|b_1|,|b_n|\}(Ax,x)$. Furthermore, the minimizer x_1 and maximizer x_n obey $ABx_1 = b_1 Ax_1$, $ABx_n = b_n Ax_n$. and, since *A* is positive, *A* is invertible and we have $Bx_1 = b_1 x_1$, $Bx_n = b_n x_n$. In other words, b_1 and b_n are eigenvalues of *B*. Clearly, $b_1{}^2 = \dfrac{(Bx_1,Bx_1)}{(x_1,x_1)} \le \max\limits_{x\neq 0}\dfrac{(Bx,Bx)}{(x,x)} = \sigma_1{}^2 = \|B\|^2$ and the same holds for b_n.

Problem 4: (a) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ be matrices in $\mathbb{C}^{2\times 2}$. Find the general form of a 2x2 complex matrix *C* such that $A \le C$ (i.e., $(Ax,x) \le (Cx,x)$ for all *x* in $\mathbb{C}^2$) and $B \le C$. Is there at least one such matrix *C* for which $C \le I$?
(b) How does the answer differ if the matrices *A*, *B*, *C* are taken to be from $\mathbb{R}^{2\times 2}$ and inequalities are required to hold for all *x* in $\mathbb{R}^2$?

Proof: Let $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$. If $C - A$ is to be nonnegative then *C* must obey $c_{11} \ge 1$, $c_{22} \ge 0$, $c_{12} = \overline{c_{21}}$ and $(c_{11}-1)c_{22} \ge |c_{12}|^2$. If $C - B$ is to be nonnegative then *C* must obey $c_{11} \ge 0$, $c_{22} \ge 1$, $c_{12} = \overline{c_{21}}$ and $c_{11}(c_{22}-1) \ge |c_{12}|^2$. Thus, to obey both inequalities requires that

$$C = \begin{bmatrix} \alpha & \gamma + i\delta \\ \gamma - i\delta & \beta \end{bmatrix} \quad \text{where } \alpha,\beta,\gamma,\delta \text{ are real, } \alpha,\beta \ge 1,\ \gamma^2 + \delta^2 \le \min\{(\alpha-1)\beta, \alpha(\beta-1)\}$$

If $I - C$ is to be nonnegative then we need $\alpha = \beta = 1$, $\gamma = \delta = 0$ and hence there is one such a matrix, $C = I$.

Problem 5: (20 points) Let $A(t)$ be differentiable invertible square matrix valued function. Compute $\frac{d^2}{dt^2}\det A(t)$ in terms of $\dot{A}(t)$ and $\ddot{A}(t)$.

Proof: Using the rules for differentiating det $A(t)$ and tr $A(t)$ we obtain

$$\begin{aligned}
\frac{d^2}{dt^2}\det A &= \frac{d}{dt}\det A\,\mathrm{tr}\left(A^{-1}\dot{A}\right) \\
&= \det A\left[\mathrm{tr}\left(A^{-1}\dot{A}\right)\right]^2 + \det A\ \mathrm{tr}\left(\frac{d}{dt}A^{-1}\dot{A}\right) \\
&= \det A\left\{\left[\mathrm{tr}\left(A^{-1}\dot{A}\right)\right]^2 + \mathrm{tr}\left(-A^{-1}\dot{A}A^{-1}\dot{A}\right) + \mathrm{tr}\left(A^{-1}\ddot{A}\right)\right\} \\
&= \det A\left\{\left[\mathrm{tr}\left(A^{-1}\dot{A}\right)\right]^2 - \mathrm{tr}\left(A^{-1}\dot{A}\right)^2 + \mathrm{tr}\left(A^{-1}\ddot{A}\right)\right\}
\end{aligned}$$