Show all your work, justify your results.

Problem 1: (20 points)
(a) Show that if $\Lambda$ is a complex diagonal matrix, then there is a complex diagonal unitary matrix $U$ such that $\Lambda = U\Lambda$.
(b) Show that a complex matrix $A$ is normal if and only if there is a unitary matrix $V$ such that $A^* = VA$.

Problem 2: (20 points) Let $A$ be a complex self-adjoint matrix.
(a) Show that $A^{2k} \geq 0$ for all positive integers $k$, and $e^A > 0$.
(b) Show that $A \geq 0$ with $\text{rank}(A) = r$ if and only if $A = \sum_{i=1}^{r} v_i v_i^*$ where $v_i$ are orthogonal nonzero column vectors.

Problem 3: (15 points) Let $\rho(A)$ be the spectral radius of a matrix $A$. Show that $\rho(A^k) = \rho(A)^k$ for all positive integers $k$.

Problem 4: (25 points) Let $\| \cdot \|_2$ be the matrix norm induced by the Euclidean vector norm, and $\| \cdot \|_F$ be the Frobenius norm, i.e., $\| A \|_2 = (\sum_{i,j} |a_{ij}|^2)^{1/2}$.
(a) Show that $\| A \|_2$ is unitarily invariant, i.e., $\| UAV \|_2 = \| A \|_2$ for all unitary matrices $U$ and $V$.
(b) Show that $\| A \|_2 = (\sum_i \sigma_i^2)^{1/2}$, where $\sigma_i$ are the singular values of $A$.
(c) Show that $\| A \|_F \leq \| A \|_2 \leq \sqrt{n} \| A \|_2$ and the bounds are sharp.

Problem 5: (20 points)
(a) Show that an irreducible matrix cannot have a 0 row or a column.
(b) What is the maximum number of zeros one can find in an irreducible $n \times n$ matrix? Prove your result.
(c) Suppose $A$ is a real square matrix whose Gershgorin circles are all mutually disjoint. Show that all eigenvalues of $A$ are real.