NAME: ______________________________________

Show all your work, justify your results.
Number and sign every page.

Problem 1 (15 points): Suppose that \( x \) and \( y \) are vectors in a linear space \( V \) and \( M \subset V \) is a subspace. Let \( K \) be the space spanned by \( M \) and \( x \), and let \( H \) be the space spanned by \( M \) and \( y \). Show that if \( x \in H \) and \( x \notin M \) then \( y \in K \).

Problem 2 (15 points): Let \( X \) be a finite-dimensional linear space and \{\( x_1, \ldots, x_n \)\} be a basis of \( X \). Let \( Y \) be a vector space over the same field and \{\( y_1, \ldots, y_n \)\} be any vectors in \( Y \).

(a) Show that there is a unique linear map \( T : X \to Y \) such that \( Tx_j = y_j \), \( j = 1, \ldots, n \).

(b) Show that if \{\( y_1, \ldots, y_n \)\} are linearly independent then \( T \) is invertible.

Problem 3 (15 points): Let \( A, B \) be complex square matrices. Show that if the minimal polynomial of \( A \) is equal to the characteristic polynomial of \( B \), and the minimal polynomial of \( B \) is equal to the characteristic polynomial of \( A \), then \( A \) and \( B \) are similar.

Problem 4 (15 points): Find the characteristic polynomial, minimal polynomial and Jordan canonical form for the matrix
\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & -1 & 0 \\
0 & 2 & -1
\end{bmatrix}
\]

Problem 5 (20 points): (True or false.) Give proofs of true statements and matrix counterexamples for false statements.

(a) If all eigenvalues of \( A \) are 0 then \( A = 0 \).

(b) Every invertible matrix is diagonalizable.

(c) If \( N \) is nilpotent with \( N^3 = 0 \) then \( I + N \) has a square root (i.e., there is a matrix \( M \) such that \( M^2 = I + N \).)

(d) If \( A \) commutes with \( B \) and \( B \) commutes with \( C \) then \( C \) commutes with \( A \).

(e) If \( A \) is anti self-adjoint then \( e^A \) is unitary.

Problem 6 (20 points): Show that \( T \) is a self-adjoint map on a complex Euclidean space \( X \) if and only if \( (Tx, x) \) is real for all \( x \) from \( X \).