Problem 3: Let $P, S, T$ be $n \times n$ complex matrices.

(a) Show that if $P$ is a projection (i.e., $P^2 = P$) then $\text{tr}P \geq 0$
(b) Prove if true or give a counterexample if false: $\text{tr}(ST) = (\text{tr}S)(\text{tr}T)$
(c) Show that if $\text{tr}(ST) = 0$ for all $S$ then $T = 0$.
(d) Give an example of a real $2 \times 2$ matrix $T$ such that $\text{tr}(T^2) < 0$.

Solution:

(a) We have $\text{tr}P = \sum \lambda_i$, where $\lambda_i$ are the eigenvalues of $P$. Using the definition we have $\lambda_i x_i = Px_i = P^2 x_i = P \lambda_i x_i = \lambda_i^2 x_i$ and hence any eigenvalue of $P$ obeys $\lambda_i = \lambda_i^2$. The only solutions are $\lambda_i = 0$ or $\lambda_i = 1$, which proves the claim.

(b) False. Consider, for example, the case $S = I$. The relation $\text{tr}T = \text{tr}(IT) = (\text{tr}I)(\text{tr}T) = n\text{tr}T$ is false unless $\text{tr}T = 0$. Thus, pick any matrix $T$ with nonzero trace.

(c) We have $\text{tr}(ST) = \sum \sum S_{ij} T_{ji} = 0$ for any $S$. Pick $S$ such that $S_{ij} = 1$ for some $i$ and $j$ and $S_{ij} = 0$ otherwise. The equation implies $T_{ij} = 0$. As this applies to any choice of $i$ and $j$, we have $T = 0$.

(d) We have $\text{tr}(T^2) = \sum \sum T_{ij}^2 T_{ji} = T_{11}^2 + 2T_{12}T_{21} + T_{22}^2 < 0$. This will be true if $2T_{12}T_{21} < -T_{11}^2 - T_{22}^2$. Pick, for example, $T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. 