Problem: Let $X$, $Y$ and $Z$ be subspaces of a linear vector space (not necessarily finite-dimensional). Show that
\[ (X \cap Y) + (X \cap Z) = X \cap (Y + (X \cap Z)) \]

Solution: In order to show that \( (X \cap Y) + (X \cap Z) = X \cap (Y + (X \cap Z)) \) we need to show that (i) \( (X \cap Y) + (X \cap Z) \subseteq X \cap (Y + (X \cap Z)) \) and (ii) \( (X \cap Y) + (X \cap Z) \supseteq X \cap (Y + (X \cap Z)) \).

Proof of (i): By definition, for any \( x \in (X \cap Y) + (X \cap Z) \) there exist \( y \in X \cap Y \) and \( z \in X \cap Z \) such that \( x = y + z \). As both \( y \) and \( z \) are in \( X \), we have \( x \in X \). Furthermore, as \( y \in Y \), we also have \( x \in (X \cap Y) \). Thus \( x \in X \cap (Y + (X \cap Z)) \).

Proof of (ii): If \( x \in X \cap (Y + (X \cap Z)) \) then \( x \in X \). Also, we have \( x = y + z \) with \( y \in Y \) and \( z \in X \cap Z \). As both \( y \) and \( z \) are in \( X \), so is \( y \). Thus, \( y \in X \cap Y \) and hence \( x \in (X \cap Y) + (X \cap Z) \).