Problem 1: Which of the following matrices is unitarily similar to a diagonal matrix and why (or why not)?

\[
\begin{pmatrix}
1 & i \\
-i & 0
\end{pmatrix},
\begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 1 \\
-1 & -1
\end{pmatrix},
\begin{pmatrix}
1 & 1 \\
0 & 2
\end{pmatrix}
\]

Problem 2: Given the space of \( P_n \) complex polynomials of degree less than \( n \) in the variable \( t \), with inner product

\[
\varphi(f, g) = \int_0^1 f(t) \overline{g(t)} \, dt
\]

(a) Is the multiplication operator \( T \) which acts as \( Tf(t) = tf(t) \) a self-adjoint map from \( P_n \) to itself?

(b) Is the differentiation operator \( D \) a self-adjoint map from \( P_n \) to itself?

Problem 3: Let \( O \) be \( 3 \times 3 \) orthogonal matrix with determinant 1. Show that it represents a rotation about some line in \( \mathbb{R}^3 \). Find this line and the angle of rotation.

Problem 4: Let \( A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \). Find an orthogonal matrix \( Q \) such that \( Q^T AQ \) is diagonal.

Problem 5: Let \( V \) be a vector space endowed with two scalar products \( (.,.)_1 \) and \( (.,.)_2 \). Suppose that \( (v,v)_1 = (v,v)_2 \) for every vector \( v \) in \( V \). Then \( (v,w)_1 = (v,w)_2 \) for every pair of vectors \( v, w \) in \( V \).

Problem 6: Let \( X \) be a finitely-dimensional Euclidean space and \( T \) is a linear map on \( X \). Show that the range of \( T^* \) is the orthogonal complement of the nullspace of \( T \).

Problem 7: Let \( V \) be the real Euclidean space consisting of real-valued continuous functions on the interval \( -2 \leq t \leq 2 \) with the scalar product

\[
(f, g) = \int_{-2}^{2} f(t) g(t) \, dt
\]

Let \( W \) be the subspace of odd functions. Find the orthogonal complement of \( W \).

Problem 8: Let \( X \) be a finitely-dimensional real Euclidean space and let \( B \) be a linear map such that \( (Bx,x) \geq 0 \) for all \( x \) in \( V \).

(a) Show that \( (Bx,x) = 0 \) implies \( (Bx,y) + (x,By) = 0 \) for all \( y \) in \( V \).

(b) Deduce from (a) that the nullspace of \( B \) equals the nullspace of \( B^* \) and hence that the nullspace and range of \( B \) are orthogonal.
**Problem 9:** Let $V$ be a finite-dimensional real Euclidean space. A linear map $T$ is said to be a reflection with respect to a plane $S_u$, defined as the span of vectors orthogonal to a given vector $u$ in $V$, if $T(u) = -u$ and $T(w) = w$ for all $w$ in $S_u$.

(a) Show that $T$ is given by

$$T(v) = v - 2 \frac{(v,u)}{(u,u)} u$$

(b) Show that $T$ is an isometry.

**Problem 10:** Let $V$ be the space of all $n \times n$ matrices over the reals. For $A, B$ from $V$ define $(A, B) = \text{tr}(B^T A)$.

(a) Show that $(\cdot, \cdot)$ is a scalar product on $V$.

(b) Let $E_{ij}$ be a matrix in $V$ whose $i$-th row and $j$-th column entry is 1 and all other entries are 0. Show that the matrices $E_{ij}$, $i, j = 1, 2, \ldots, n$ form an orthonormal basis for $V$.

(c) For $A$ in $V$ let $f(A) = \sum_{i,j=1}^n (i + j)a_{ij}$ where $a_{ij}$, $i, j = 1, 2, \ldots, n$ are the elements of matrix $A$. Find a matrix $B$ such that $f(A) = (A, B)$ for all $A$ in $V$ and show that $B$ is unique.