**Problem 1:** Which of the following matrices is unitarily similar to a diagonal matrix and why (or why not)?

\[
\begin{pmatrix}
1 & i \\
-i & 0
\end{pmatrix},
\begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 1 \\
-1 & -1
\end{pmatrix},
\begin{pmatrix}
1 & 1 \\
0 & 2
\end{pmatrix}
\]

**Problem 2:** Given the space of \( P_n \) complex polynomials of degree less than \( n \) in the variable \( t \), with inner product

\[
\varphi(f, g) = \int_0^1 f(t)g(t)dt
\]

(a) Is the multiplication operator \( T \) which acts as \( Tf(t) = tf(t) \) a self-adjoint map from \( P_n \) to itself?

(b) Is the differentiation operator \( D \) a self-adjoint map from \( P_n \) to itself?

**Problem 3:** Let \( A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \). Find an orthogonal matrix \( Q \) such that \( Q^T AQ \) is diagonal.

**Problem 4:** Let \( V \) be a vector space endowed with two scalar products \((.,.)_1 \) and \((.,.)_2 \). Suppose that \((v,v)_1 = (v,v)_2 \) for every vector \( v \) in \( V \). Then \((v,w)_1 = (v,w)_2 \) for every pair of vectors \( v, w \) in \( V \).

**Problem 5:** Let \( X \) be a finitely-dimensional Euclidean space and \( T \) is a linear map on \( X \). Show that the range of \( T^* \) is the orthogonal complement of the nullspace of \( T \).

**Problem 6:** Let \( V \) be the real Euclidean space consisting of real-valued continuous functions on the interval \(-2 \leq t \leq 2\) with the scalar product

\[
(f, g) = \int_{-2}^{2} f(t)g(t)dt
\]

Let \( W \) be the subspace of odd functions. Find the orthogonal complement of \( W \).

**Problem 7:** Let \( V \) be a finite-dimensional real Euclidean space. A linear map \( T \) is said to be a reflection with respect to a plane \( S_u \), defined as the span of vectors orthogonal to a given vector \( u \) in \( V \), if \( T(u) = -u \) and \( T(w) = w \) for all \( w \) in \( S_u \).

(a) Show that \( T \) is given by

\[
T(v) = v - 2\frac{(v,u)}{(u,u)}u
\]

(b) Show that \( T \) is an isometry.

**Problem 8:** Let \( V \) be the space of all \( n \times n \) matrices over the reals. For \( A, B \) from \( V \) define \((A,B) = \text{tr}(B^TA)\).

(a) Show that \((.,.)\) is a scalar product on \( V \)

(b) Let \( E_{ij} \) be a matrix in \( V \) whose \( i \)-th row and \( j \)-th column entry is 1 and all other entries are 0. Show that the matrices \( E_{ij} \) \( i, j = 1, 2, ..., n \) form an orthonormal basis for \( V \).

(c) For \( A \) in \( V \) let \( f(A) = \sum_{i,j=1}^{n} (i + j)a_{ij} \) where \( a_{ij} \) \( i, j = 1, 2, ..., n \) are the elements of matrix \( A \). Find a matrix \( B \) such that \( f(A) = (A,B) \) for all \( A \) in \( V \) and show that \( B \) is unique.