Problem 1: (a) Use the adjoint formula to compute the inverse of the following matrix:
\[
\begin{bmatrix}
\cos \theta & 0 & -\sin \theta \\
0 & 1 & 0 \\
\sin \theta & 0 & \cos \theta
\end{bmatrix}
\]
(b) Use Cramer’s rule to solve the following system over the field of rational numbers:
\[
\begin{align*}
3x - 2y &= 7 \\
3y - 2z &= 6 \\
3z - 2x &= -1
\end{align*}
\]

Problem 2: Show that if the \( n \times n \) matrix \((I - AB)\) is invertible then \((I - BA)\) is invertible.

Problem 3: Let \( V \) be a finite dimensional linear space and let \( T \in L(V,V) \). Given a subspace \( W \) of \( V \), set \( T^{-1}(W) = \{x \in V : Tx \in W\} \).
(a) Show that \( T^{-1}(W) \) is a subspace of \( V \).
(b) Show that \( \dim T^{-1}(W) \leq \dim N_T + \dim W \), where \( N_T \) is the nullspace of \( T \).
(c) If \( S \in L(V,V) \), show that \( \text{rank } ST \geq \text{rank } T + \text{rank } S - \dim V \).

Problem 4: An \( n \times n \) matrix \( A \) is skew-symmetric if \( A^T = -A \). Let \( A \) be a skew-symmetric \( n \times n \) matrix with real entries and with \( n \) odd.
(a) Show that \( \det A = 0 \).
(b) Show that all the eigenvalues of \( A \) are pure imaginary.

Problem 5: Find all eigenvectors and eigenvalues of the backward shift operator \( T \in L(\mathbb{C}^\infty, \mathbb{C}^\infty) \) defined by \( T(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots) \).

Problem 6: Suppose that \( S, T \in L(V,V) \) where \( V \) is finite dimensional
(a) Show that if \( \dim R_T = k \), then \( T \) has at most \( k + 1 \) distinct eigenvalues.
(b) Show that \( ST \) and \( TS \) have the same eigenvalues.
(c) Show that if every vector in \( V \) is an eigenvector of \( T \) then \( T = aI \).

Problem 7: Show that \( n \times n \) complex matrix \( A \) is never similar to \( A + I \).