Math 2370 – Fall 2008  
Practice Problems VI

Problem 1: Show that if $S: \mathbb{R}^2 \to \mathbb{R}^2$ is a projection, then $S = 0$, or $S = I$, or there is a basis $B$ such that the matrix representation of $S$ with respect to $B$ is $S_B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Problem 2: Find integers $a, b, c, d$ such that the following two matrices are similar over $\mathbb{Q}$:

\[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & a \\
1 & 0 & 0 & b \\
0 & 1 & 0 & c \\
0 & 0 & 1 & d
\end{bmatrix}
\]

Problem 3: Let $A$ be a block diagonal matrix with the decomposition

\[
A = \begin{bmatrix}
A_1 & 0 & 0 & 0 \\
0 & A_2 & 0 & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & 0 & A_n
\end{bmatrix}
\]

where $A_i$ are square. Show that $\det A = \det A_1 \det A_2 \cdots \det A_n$.

Problem 4: Let $E$ be an $n \times n$ matrix with the block decomposition $E = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where $A$ and $D$ are square.

(a) Show that if $A$ is invertible then $E$ can be decomposed as $E = \begin{bmatrix} I & 0 \\ W & I \end{bmatrix} \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$.

(b) Using the decomposition show that $\det E = \det A \det(D - CA^{-1}B)$.

Problem 5: Using the properties of the determinant function $D(a_1, \ldots, a_n)$ show that $D(a_1, \ldots, a_n) = 0$ if the $j$-th component of each of the vectors $a_1, \ldots, a_n$ is zero.

Problem 6: Find a matrix $A$ of integer entries such that $A \neq I$ and $A^3 = I$.

(Hint: A permutation can be represented by a matrix multiplication.)