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Nonlinear dynamics and stability in a multi-group asset flow model

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Abstract. The multi-group asset flow model for asset price dynamics incorporates distinct motivations, e.g., trend and fundamentals (value) and assessments of value by different groups of investors. The stability and bifurcation properties are established for the curve of equilibria. We prove that if all trader groups focus on fundamentals, then all equilibria are stable. For systems in which there is one fundamental and one momentum (trend) group, we establish conditions for stability. In particular, an equilibrium that is stable becomes unstable as the time scale on which momentum investors focus diminishes. The computations examine the excursions, which we define as the maximum deviation in price of the trajectory from its initial price located near the curve of equilibria.

Key words. asset price dynamics, asset flow, momentum, trend, fundamental value, stability of price dynamics

AMS subject classifications. 91G80, 34C60, 34D20, 37N40

1. Introduction.

During the past half-century, asset price dynamics have been modeled within the framework of classical finance which has the efficient market hypothesis as its foundation. As all informed participants have the same public information, the theory stipulates that there will be widespread agreement on the valuation of the asset [1]. Any imbalances in orders will be quickly exploited by these investors who, for all practical purposes have infinite "arbitrage capital" compared to the uninformed investors. Hence, any behavioral biases or cognitive errors on the part of a group of investors would not alter the price much beyond adding some noise. The valuation of an asset is subject to change due to a number of factors, economic, political, natural (e.g. weather), etc. To the trader or investor, these changes in valuation can be regarded as random processes. Consequently, a typical model of asset prices involves an equation such as

$$\frac{dP}{P} = \sigma dX + \mu dt \tag{1.1}$$

where P is the asset price at time t, while X is a normal random variable (mean 0 and variance 1), σ^2 is the variance of returns, and μ is the drift (average return), so that μdt is the expected return on the investment in time dt.

Within this classical model, the role of different groups with distinct assessments of value or different strategies is marginalized. A measure of stability of a market is expressed in an averaged form into the variance, σ^2. From the perspective of many practitioners, this classical view is a default theory that is useful only until one can obtain deeper models and link them to markets quantitatively. There is some limited evidence that an asset that was volatile last week will be volatile next week. There are also forecasts (though often varying widely) on

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the expected return, μ. Thus, if one has no additional information or insight, a model such as (1.1) is acceptable in terms of pricing options, for example, or calculating the value at risk.

There are many questions, however, that are left unanswered by this approach. Some of these have been at the forefront of problems confronting the finance community. For example, a number of "flash crashes" have occurred recently including the Dow's 600 point drop in five minutes on May 6, 2010. From the perspective of classical finance, one would only be able to say that we have seen a very unusual event. Stepping away from the purely mathematical model, one can see that such a drop indicates the absence of limit buy orders until the price has fallen dramatically. Since there was no significant news prior to the abrupt drop, it suggests that a significant fraction of traders have motivations or strategies beyond long-term valuation. This also suggests the absence of a large pool of cash that is ready to take advantage of errors made by less informed investors, contrary to the assumptions of classical finance.

The example above illustrates the need for a theory that (i) considers different groups of investors with distinct assessments of valuation, rather than trivializing all but one homogeneous group; (ii) incorporates the finiteness of assets; (iii) considers different motivations for trades, e.g., trend-based or momentum trading (defined below). Since the equation (1.1) is built on a foundation of infinite arbitrage, it would be difficult to add terms to render this finite. Hence, a different approach is needed.

Within this model the information available on the value of the asset is expressed through $P_a^{(i)}(t)$, where i denotes the Group. Thus, different groups may have distinct assessments of the value at the same time, t. Moreover, the valuation will be subject to randomness, so that one can regard it as a stochastic term. In terms of the classical theory, the only source of change would be through this valuation. Furthermore, since all groups would have the same public information, they would agree on the valuation, $P_a(t) = P_a^{(i)}(t)$. Our main objective is to examine price dynamics arising from sources other than valuation; hence we focus on constant $P_a^{(i)}(t)$.

Randomness is generally present in many aspects of a market beyond valuation; it is also manifested in asset levels (e.g., an influx of cash for a particular group), and changes in motivations due to random events that may inspire or inhibit risk taking, for example. The presence of noise from these factors means that an equilibrium that is unstable will quickly depart from this equilibrium point and eventually settle into a stable equilibrium. The issue of understanding randomness within the context of the asset flow equations (described in Section 2) is discussed further in the Conclusion (Section 5).

The asset flow equations discussed in the next section are based on a modeling approach that incorporates these ideas naturally. Each group of investors is endowed with a set of assets (shares in a single stock or index, plus cash), its assessment of valuation, and motivations such as buying due to undervaluation or rising prices (momentum). Other motivations are also easily included in these equations, though we restrict ourselves to those two in this paper. The different groups participate in a single market where price changes are governed by supply and demand for the stock.

The system of equations we study is based on the approach of Caginalp and collaborators since 1990 (see [5] for references) and involves coupled ordinary differential equations and algebraic equations. We consider the "closed" system in which the number of shares and cash of the entire system is fixed. For the multi-group model, this has been developed in [5] and [10]. In [5] the authors proved the existence of a continuum of equilibria and characterized the stability. While equilibrium consists of a single point in classical finance, the asset flow equations permit a curve of equilibria that depends on the parameters in the system such as the cash and share endowment of each group, the parameters characterizing their motivations, etc. In the distinguished limit as (a) all groups have the same assessment of value, (b) the assets available for arbitrage approach infinity, (c) the motivations of the traders are based on value alone (not on price history, for example), and (d) explicit randomness is introduced into the price, one expects that solutions of the asset flow equations will converge to those of (1.1). A formal argument that this limit is attained has been made in [5]. A rigorous proof that solutions of the asset flow equations converge, in an appropriate sense, to those of (1.1) remains a research problem.

In this paper we consider the study of the stability and bifurcation properties of the curve of equilibria. We find that in a system comprised of G purely fundamental traders there is a G-1 dimensional manifold of equilibrium states and all equilibria are stable with respect to any perturbation away from the equilibrium manifold (Theorem 3.5). Although the equilibrium price is in a narrow range defined by the extremes of equilibrium prices of individual groups (if trading alone) (Theorem 3.4), the equilibrium cash and shares of each group can range anywhere between zero and the maximum amount. As a result, there is an inherent natural indeterminacy of the equilibrium price which may result in price fluctuations and drift. In the simplest case of fixed trading preferences the equilibrium price, cash, and shares are determined by the initial condition (Theorem 3.2). In addition, we find that the presence of a group with trend-based trading preferences leads to destabilization of equilibria in a way that is characterized by Theorem 3.7.

One of our goals is an aspect of stability that is crucial in practice, but is often neglected in theoretical studies. Upon perturbation, a point that is an unstable equilibrium moves away from the initial point. Stability studies often focus on the initial rate of change. However, from the point of view of finance, the important questions are (i) what is the stable equilibrium at which the solutions (in particular the price) settles, and (ii) what is the maximum deviation of the price from the initial value on the path to the stable value. We call this the "excursion," and note that it is a crucial concept since many financial events are triggered by a large deviation in price, rather than the ultimate settling point.

From the perspective of finance, some important issues involve the dependence of the maximum excursion and the new (stable) equilibrium on the parameters in the system, and how a stable point can become unstable as parameters of the system vary. In the example of the flash crash discussed above, a relatively steady S&P 500 index value appeared to become unstable without any apparent cause. One mechanism for this may be a gradual changing of parameters (e.g., a decrease in the time scale of interest for momentum traders [6]) that moves the stable equilibrium value to an unstable one. If the resulting excursion is small, the transition in the equilibrium points would not attract much attention. However, if this excursion is large relative to the historical variance, then the transition can become very significant in terms of markets. For example, if the market is at a stable equilibrium, and the relative fraction of traders using short term momentum strategies increases, in the absence of any changes in valuation, then the system can move from a stable to an unstable equilibrium.

The magnitude of the excursion would then depend on a number of factors such as the liquidity value (which we define as the ratio of total cash in the system to the total asset value) relative to the fundamental value. We present a systematic and rigorous analysis of the stability of these asset flow equations, and introduce the concept of excursions, which we define in terms of maximum price deviation from the initial price (near the curve of equilibria) as the trajectory moves from an unstable point to a stable one, i.e. it equals $\max_t(|P(t) - P(0)|)$.

This paper is organized as follows. In Section 2 we present the mathematical model. Section 3 features theoretical results such as conditions for equilibrium and stability. Section 4 presents numerical results including excursions from unstable to stable equilibria. While the vast majority of analysis in dynamical systems focuses on the initial onset of stability, the ultimate magnitude of the deviation from the curve of equilibria is of great importance in many applications. In the conclusion (Section 5) we discuss the implications of our results in terms of finance, and directions for future research.

2. Model.

As noted above, we consider the model developed and refined by Caginalp and collaborators (see [5] and the references therein). This model is based upon the flow of cash and shares between investor groups. These groups make investment decisions based upon nonclassical motivations, such as the recent trend in price[1], as well as the typical rationale of the trading price's deviation from the fundamental (intrinsic) value. For G investor groups this model[2] has the form

$$\frac{dP}{dt} = F - P \tag{2.1}$$

$$\frac{dN^{(i)}}{dt} = \frac{k^{(i)}M^{(i)}}{F} - \left(1 - k^{(i)}\right) N^{(i)} \tag{2.2}$$

$$\frac{dM^{(i)}}{dt} = -k^{(i)}M^{(i)} + \left(1 - k^{(i)}\right) N^{(i)}F \tag{2.3}$$

with $i = 1, 2, ..., G$ where P is the price of the asset, $M^{(i)}$ is the amount of cash investor group i has, $N^{(i)}$ is the number of shares group i has, and F is defined as

$$F = \frac{\sum_{i=1}^{G} k^{(i)}M^{(i)}}{\sum_{i=1}^{G}\left(1 - k^{(i)}\right) N^{(i)}}. \tag{2.4}$$

[1] [3], [4], and [8] provide empirical support for the inclusion of this factor.

[2] The derivation of the continuous model (2.1)-(2.3) from its discrete counterpart is given in [5]. In [5] a timescale parameter, τ, is defined as the relaxation time it takes for a non-equilibrium situation to return to equilibrium. Thus, this parameter describes the time scale on which traders react to changes in the system. It is assumed that τ equals the discrete time interval between trading periods. This choice is reflective of an efficient market in which the market returns to equilibrium in one time period. In this paper we assume τ is set to one via a rescaling of time. As noted in the Conclusion (Section 5), one could utilize optimization methods (see [14]) to estimate the time scale for readjustment to equilibrium and hence determine the time interval represented by the trajectories. Note that the parameter values utilized in Section 4 are consistent with the results of [14]. Further, this time scale and the $c_j^{(i)}$ (defined later in this section) parameter values may vary considerably from one market to another. As the purpose of this paper is to study stability properties of this model in a general framework, we leave the establishment of a stronger connection between this model and real world markets as a research problem.

Dividing both sides of equation 2.1 by P yields

$$\frac{1}{P}\frac{dP}{dt} = F/P - 1. \tag{2.5}$$

The right hand side of equation (2.5) may be interpreted as excess demand (see [5]). Thus, in this model the relative change in price is proportional to excess demand, a common microeconomic principle (see [16] and [22]).

The time-dependent function $k^{(i)}$ denotes the transition rate from investor group i holding cash to holding shares [7]. Alternately, it may be thought of as the proportion of cash investor group i submits for purchase of the asset, while $\tilde{k}^{(i)}$ corresponds to the proportion of shares the investor group sells. We set $\tilde{k}^{(i)} = 1 - k^{(i)}$ in this article and assume $0 < k^{(i)} < 1$.

The system (2.1)-(2.4) has two conserved quantities, namely the total amount of cash $\sum_{i=1}^{G} M^{(i)} = M_0$ and the total number of shares $\sum_{i=1}^{G} N^{(i)} = N_0$. Throughout this paper we assume there is no borrowing of cash or shorting of shares. Hence, we have the following assumption: $M^{(i)}$ and $N^{(i)}$ are non-negative and less than M_0 and N_0, respectively. A key quantity, discussed in [2] and [5], is the ratio $L = M_0/N_0$, termed the liquidity value[3]. We rescale the system, representing the rescaled variables in boldface type; i.e., $\mathbf{N}^{(i)} = N^{(i)}/N_0$, $\mathbf{M}^{(i)} = M^{(i)}/M_0$, $\mathbf{P} = P/L$, and $\mathbf{F} = F/L$, yielding

$$\frac{d\mathbf{P}}{dt} = \mathbf{F} - \mathbf{P} \tag{2.6}$$

$$\frac{d\mathbf{N}^{(i)}}{dt} = \frac{k^{(i)}\mathbf{M}^{(i)}}{\mathbf{F}} - \left(1 - k^{(i)}\right)\mathbf{N}^{(i)} \tag{2.7}$$

$$\frac{d\mathbf{M}^{(i)}}{dt} = -k^{(i)}\mathbf{M}^{(i)} + \left(1 - k^{(i)}\right)\mathbf{N}^{(i)}\mathbf{F} \tag{2.8}$$

where

$$\mathbf{F} = \frac{\sum_{i=1}^{G} k^{(i)}\mathbf{M}^{(i)}}{\sum_{i=1}^{G}\left(1 - k^{(i)}\right)\mathbf{N}^{(i)}}. \tag{2.9}$$

Given this rescaling and the above assumption regarding the absence of borrowing and shorting, we assume throughout the remainder of this paper that $\mathbf{M}^{(i)} \in [0,1]$ and $\mathbf{N}^{(i)} \in [0,1]$. Note that, in view of the conservation of the total number of shares and cash and the positivity of $k^{(i)}$, one has $\mathbf{F} > 0$. Also, note that the equations (2.7) and (2.8) together yield

$$\mathbf{F}\frac{d\mathbf{N}^{(i)}}{dt} + \frac{d\mathbf{M}^{(i)}}{dt} = 0. \tag{2.10}$$

[3] In [2] the authors showed that the equilibrium price for a single investor group model with both fundamental and trend-based trading preferences (see below for definitions) must lie between the fundamental value of the asset and this liquidity value. Analogously, in [5] it was shown that in the case of two groups, one with fundamental trading preference and the other with trend-based preference, the system (2.1)-(2.4), (2.14), (2.16), and (2.17) has a range of stable and unstable equilibria with price value between the fundamental (intrinsic) value of the asset and the liquidity value. Note the authors in [5] assumed $tanh\,(x) \simeq x$ and did not rescale by L.

The basic exchange laws (2.6)-(2.9) are complemented by a set of equations for $k^{(i)}$. The $k^{(i)}$ are defined by the sentiment functions $\zeta_j^{(i)}(t)$, which reflect traders' motivation and can be functions of many different factors, such as the discount from valuation (i.e., the classical motivation), the recent price trend, the relation between the current price and the trader's purchase price or recent high price, etc. In order to obtain a closed system of equations, $\zeta_j^{(i)}(t)$ should be functions of the variables $\mathbf{N}^{(i)}$, $\mathbf{M}^{(i)}$, $\mathbf{P}$, $\mathbf{F}$, $\zeta_j^{(i)}(t)$ and their derivatives. Throughout this paper we will focus on the two basic motivations:

1. *Trend-based traders*:

$$\zeta_1^{(i)}(t) = q_1^{(i)} c_1^{(i)} \int_{-\infty}^{t} e^{-c_1^{(i)}(t-\tau)} \frac{1}{\mathbf{P}(\tau)} \frac{d\mathbf{P}(\tau)}{d\tau} d\tau \tag{2.11}$$

2. *Fundamental* (value-based) *traders*:

$$\zeta_2^{(i)}(t) = q_2^{(i)} c_2^{(i)} \int_{-\infty}^{t} e^{-c_2^{(i)}(t-\tau)} \frac{\mathbf{P}_a^{(i)}(\tau) - \mathbf{P}(\tau)}{\mathbf{P}_a^{(i)}(\tau)} d\tau, \tag{2.12}$$

where $q_j^{(i)}$ represents the magnitude of the effect, i.e. how strongly investor group i is affected by motivation j. The negative exponential multiplying the time elapsed in equation (2.11) reflects the notion that individuals weight recent events more strongly than past events ([15]). In addition, this factor also helps to smooth historical data so that large spikes do not alter these values abruptly. The $c_1^{(i)}$ parameter corresponds to the inverse of the time scale of interest, i.e. if group 1 is interested in the recent trend in price over the past 10 days, then $c_1^{(1)}$ is 1/10.

As described in [2], the fundamental trading group makes decisions to buy/sell based upon the relative difference between the current asset price and its assessment of the asset's fundamental value, i.e. $(\mathbf{P}_a^{(i)}(t) - \mathbf{P}(t))/\mathbf{P}_a^{(i)}(t)$ where $\mathbf{P}_a^{(i)} = P_a^{(i)}/L$ is investor group i's estimate of the fundamental value. As different investors within the fundamental group may take longer to react to changes in this quantity than others, the longer the deviation between the price and fundamental value persists, the greater the proportion of traders who act on it. The negative exponential multiplying the time elapsed in equation (2.12) models this delay. The value $1/c_2^{(i)}$ is the time scale. Thus, a large $c_2^{(i)}$ indicates that group i reacts quickly to changes in the relative deviation between price and fundamental value.[4]

These sentiment functions were first introduced in [7]. As one investor group may be influenced by multiple motivations, the $\zeta_j^{(i)}$ are summed over all motivations, i.e.

$$\zeta^{(i)} = \sum_{j=1}^{K} \zeta_j^{(i)}. \tag{2.13}$$

[4]Throughout the remainder of this paper we assume $c_j^{(i)} > 0$ and $q_j^{(i)} > 0$.

In this paper we will restrict our focus to homogeneous investor groups, i.e. each group is affected by only one sentiment. Thus, the subscript identifying the specific sentiment will only be included as necessary. The relationship between the sentiment functions and the trading proportions is given via a monotone increasing function that maps $(-\infty, \infty)$ onto $(0, 1)$ and obeys $h(0) = 1/2$, such as, for example:

$$k^{(i)} = h\left(\zeta^{(i)}\right) = \frac{1}{2}\left[1 + \tanh\left(\zeta^{(i)}\right)\right]. \tag{2.14}$$

The form of the equations (2.11) and (2.12) allows us to write $\frac{d\zeta_j^{(i)}}{dt}$ as functions:

$$\frac{d\zeta_j^{(i)}}{dt} = \psi^{(i)}\left(\mathbf{F}, \mathbf{P}, \zeta_j^{(i)}\right). \tag{2.15}$$

In particular, the differentiation of equations (2.11) and (2.12) with respect to the variable t yields

$$\frac{d\zeta_1^{(i)}}{dt} = c_1^{(i)}\left[q_1^{(i)}\frac{1}{\mathbf{P}}\frac{d\mathbf{P}}{dt} - \zeta_1^{(i)}\right] = c_1^{(i)}\left[q_1^{(i)}\frac{\mathbf{F} - \mathbf{P}}{\mathbf{P}} - \zeta_1^{(i)}\right]. \tag{2.16}$$

and

$$\frac{d\zeta_2^{(i)}}{dt} = c_2^{(i)}\left[q_2^{(i)}\frac{\mathbf{P}_a^{(i)} - \mathbf{P}}{\mathbf{P}_a^{(i)}} - \zeta_2^{(i)}\right] \tag{2.17}$$

which, together with equations (2.6)-(2.9) and (2.14) form a closed system of equations for $(\mathbf{P}, \mathbf{N}^{(i)}, \mathbf{M}^{(i)}, \zeta_j^{(i)})(t)$.

The wealth, $w^{(i)} := M^{(i)} + N^{(i)}P$, of each investor group (rescaled by the total amount of cash, M_0) is defined as

$$\mathbf{W}^{(i)} := \mathbf{M}^{(i)} + \mathbf{N}^{(i)}\mathbf{P}. \tag{2.18}$$

In view of the conservation relations for shares and price, i.e.

$$\sum_{i=1}^{G}\mathbf{N}^{(i)} = 1, \qquad \sum_{i=1}^{G}\mathbf{M}^{(i)} = 1, \tag{2.19}$$

we have

$$\sum_{i=1}^{G}\mathbf{W}^{(i)} = 1 + \mathbf{P}. \tag{2.20}$$

3. Analytical Results.

3.1. Fixed Trading Preferences. For illustrative purposes let us first examine the dynamics of a simplified system in which $k^{(i)}$ are assumed to be constant. This is an approximation to the situation, for example, when the price trend is near zero, the valuation is not changing, and the trading volume is such that the cash position relative to the wealth in the stock is not changing significantly.

The dynamics of the system with fixed $k^{(i)}$ is simplifed by the fact that $\mathbf{F}$ is constant along any trajectory:

Lemma 3.1. *If* $k^{(i)} = const$*, then the dynamics of the system* (2.6) - (2.8) *obeys* $d\mathbf{F}(t)/dt = 0$

Proof. By definition

$$\mathbf{F} = \frac{\sum_{j=1}^{G} k^{(j)}\mathbf{M}^{(j)}}{\sum_{j=1}^{G} \tilde{k}^{(j)}\mathbf{N}^{(j)}}.$$

Differentiating with respect to time yields

$$\begin{aligned}\frac{d\mathbf{F}}{dt} &= \frac{\sum_{j=1}^{G} \tilde{k}^{(j)}\mathbf{N}^{(j)} \sum_{j=1}^{G} k^{(j)}\dot{\mathbf{M}}^{(j)} - \sum_{j=1}^{G} k^{(j)}\mathbf{M}^{(j)} \sum_{j=1}^{G} \tilde{k}^{(j)}\dot{\mathbf{N}}^{(j)}}{\left[\sum_{j=1}^{G} \tilde{k}^{(j)}\mathbf{N}^{(j)}\right]^2} \\ &= \frac{\sum_{j=1}^{G} k^{(j)}\left(\dot{\mathbf{M}}^{(j)} + \dot{\mathbf{N}}^{(j)}\mathbf{F}\right) - \mathbf{F}\sum_{j=1}^{G} \dot{\mathbf{N}}^{(j)}}{1 - \sum_{j=1}^{G} k^{(j)}\mathbf{N}^{(j)}} \\ &= 0.\end{aligned}$$

Note that the last equality follows from equation (2.10) and the conservation laws (2.19). ■

It follows from Lemma 3.1, (2.6), and the definition of $\mathbf{F}$ that $\mathbf{F} = \mathbf{P}_{eq}$ where the equilibrium price $\mathbf{P}_{eq}$ is determined by the initial conditions and trading preferences as

$$\mathbf{P}_{eq} = \frac{\sum_{j=1}^{G} k^{(j)}\mathbf{M}^{(j)}(0)}{\sum_{j=1}^{G} \left(1 - k^{(j)}\right)\mathbf{N}^{(j)}(0)}. \tag{3.1}$$

Note that $\mathbf{P}_{eq}$ is independent of the initial price $\mathbf{P}(0)$.

Another consequence of Lemma 3.1 is that the system (2.6) -(2.8) splits into $G+1$ subsystems of equations, the first being a single equation for the price: $\dot{\mathbf{P}} = \mathbf{P}_{eq} - \mathbf{P}$ with the solution

$$\mathbf{P}(t) = \mathbf{P}_{eq} + \left(\mathbf{P}(0) - \mathbf{P}_{eq}\right)e^{-t}. \tag{3.2}$$

Each of the remaining subsystems is a system of two equations (2.7)-(2.8) with fixed i and with $\mathbf{F} = \mathbf{P}_{eq}$. The equation (2.10) implies that $\dot{\mathbf{M}}^{(i)} + \dot{\mathbf{N}}^{(i)}\mathbf{P}_{eq} = 0$ for each i which can be integrated to obtain a condition on the trajectory $\left(\mathbf{M}^{(i)}(t), \mathbf{N}^{(i)}(t)\right)$

$$\mathbf{M}^{(i)}(t) + \mathbf{N}^{(i)}(t)\,\mathbf{P}_{eq} = \mathbf{M}^{(i)}(0) + \mathbf{N}^{(i)}(0)\,\mathbf{P}_{eq}. \tag{3.3}$$

The equilibrium values $(\mathbf{M}_{eq}^{(i)}, \mathbf{N}_{eq}^{(i)})$ corresponding to a given initial condition can be found by solving for each i a system of two linear equations, namely the equilibrium version of (3.3), and

$$k^{(i)}\mathbf{M}_{eq}^{(i)} - \left(1 - k^{(i)}\right) \mathbf{N}_{eq}^{(i)} \mathbf{P}_{eq} = 0, \tag{3.4}$$

which follows from (2.8). The solutions are

$$\mathbf{N}_{eq}^{(i)} = k^{(i)} \left(\frac{\mathbf{M}^{(i)}(0)}{\mathbf{P}_{eq}} + \mathbf{N}^{(i)}(0) \right) \tag{3.5}$$

and

$$\mathbf{M}_{eq}^{(i)} = \left(1 - k^{(i)}\right) \left(\mathbf{M}^{(i)}(0) + \mathbf{N}^{(i)}(0)\, \mathbf{P}_{eq}\right). \tag{3.6}$$

Substitution of (3.3) into (2.7) and (2.8) then implies that

$$\frac{d\mathbf{N}^{(i)}}{dt} = \mathbf{N}_{eq}^{(i)} - \mathbf{N}^{(i)} \tag{3.7}$$

$$\frac{d\mathbf{M}^{(i)}}{dt} = \mathbf{M}_{eq}^{(i)} - \mathbf{M}^{(i)} \tag{3.8}$$

i.e.,

$$\mathbf{N}^{(i)}(t) = \mathbf{N}_{eq}^{(i)} + \left(\mathbf{N}^{(i)}(0) - \mathbf{N}_{eq}^{(i)}\right) e^{-t} \tag{3.9}$$

$$\mathbf{M}^{(i)}(t) = \mathbf{M}_{eq}^{(i)} + \left(\mathbf{M}^{(i)}(0) - \mathbf{M}_{eq}^{(i)}\right) e^{-t} \tag{3.10}$$

which completes the solution of the system. Note that the formulas (3.5)-(3.6) imply that the equilibrium values obey (3.1) and the constraints (2.19), as long as the initial conditions obey (2.19) as well.

We have proven the following result.

Theorem 3.2. *For a system composed of traders governed by (2.6)-(2.8) with fixed $k^{(i)}$, the set $\mathcal{E}$ of equilibrium points is the set of all points $(\mathbf{P}_{eq}, \mathbf{M}_{eq}^{(i)}, \mathbf{N}_{eq}^{(i)}) \in (0, \infty) \times [0, 1]^{2G}$ that obey*

$$\sum_{i=1}^{G} \mathbf{N}_{eq}^{(i)} = 1, \qquad \sum_{i=1}^{G} \mathbf{M}_{eq}^{(i)} = 1,$$

$$k^{(i)}\mathbf{M}_{eq}^{(i)} - \left(1 - k^{(i)}\right) \mathbf{N}_{eq}^{(i)} \mathbf{P}_{eq} = 0, \qquad i = 1, ..., G.$$

Each equilibrium in $\mathcal{E}$ is asymptotically Lyapunov stable and attracting all trajectories starting in the set of initial conditions consisting of arbitrary $\mathbf{P}(0)$ and of $(\mathbf{M}^{(i)}(0), \mathbf{N}^{(i)}(0))$ compatible with the relations

$$\mathbf{P}_{eq} = \frac{\sum_{j=1}^{G} k^{(j)}\mathbf{M}^{(j)}(0)}{\sum_{j=1}^{G} \left(1 - k^{(j)}\right) \mathbf{N}^{(j)}(0)},$$

$$\mathbf{N}_{eq}^{(i)} = k^{(i)} \left(\frac{\mathbf{M}^{(i)}(0)}{\mathbf{P}_{eq}} + \mathbf{N}^{(i)}(0) \right), \qquad i = 1, ..., G,$$

and

$$\mathbf{M}_{eq}^{(i)} = \left(1 - k^{(i)}\right) \left(\mathbf{M}^{(i)}(0) + \mathbf{N}^{(i)}(0)\, \mathbf{P}_{eq}\right), \qquad i = 1, ..., G.$$

Remark 1 The set $\mathcal{S}$ of all initial conditions (i.e., trajectories) compatible with a given equilibrium $(\mathbf{P}_{eq}, \mathbf{M}_{eq}^{(i)}, \mathbf{N}_{eq}^{(i)})$ is a $G+1$ dimensional hyperplane in $(0,\infty) \times [0,1]^{2G}$ defined by the equations (3.5) and (3.6).

Remark 2 There are important practical situations in which the hypothesis, $k^{(i)} = \zeta_1^{(i)} + \zeta_2^{(i)} = const$, is satisfied.

(1) Consider a single investor group, Group i. Under suitable conditions on $c_1^{(i)}$, $q_1^{(i)}$, $c_2^{(i)}$, $q_2^{(i)}$ we can have $P(t)$ and $P_a^{(i)}(t)$ both varying with time, but remaining equidistant from each other. For example, suppose they are linearly increasing with $P(t) > P_a^{(i)}(t)$ so that $\zeta_1^{(i)} > 0$ and $\zeta_2^{(i)} < 0$ (see Figure 3.1). As these are constant, so is their sum, $k^{(i)}$. In practical terms this means that an overvalued situation in which prices are rising steadily with valuation is stable. This is often observed in markets, in that an overvalued stock whose fundamentals are steadily improving together with price can continue until the situation changes, e.g., a downturn in fundamentals.

(2) A more surprising situation is that in which the price, $P(t)$, is increasing and concave downward, while $P_a^{(i)}(t)$ is decreasing and concave upward (see Figure 3.1). At first glance, this seems to be a dangerously unstable situation since the price is deviating more and more from the valuation. However, $\zeta_1^{(i)}$ is positive and increasing in time, while $P_a^{(i)}(t)$ is decreasing in time (whether initially positive or negative). This means that with suitable parameters characterizing the investor populations, e.g., comparable time scales and magnitudes for trend and valuation, the two components in sentiment can neutralize one another, and the conditions of the theorem would be satisfied, leading to stability.

3.2. Variable Trading Preferences.

We now consider the more general situation in which the trading preferences are variable such as in the motivations in (2.11) and (2.12). Recall that $k^{(i)} = h\left(\zeta^{(i)}\right)$ where h is a monotone increasing function, and suppose that

$$\dot{\zeta}^{(i)} = \psi^{(i)} \left(\mathbf{F}, \mathbf{P}, \zeta^{(i)}\right). \tag{3.11}$$

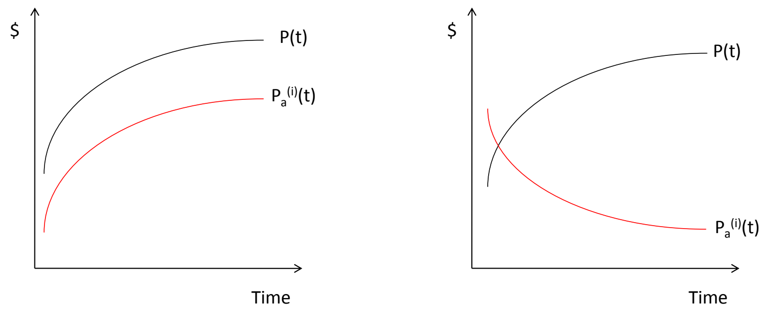


Figure 3.1. *Practical scenarios for Theorem 3.2. The figure on the left shows $P(t) > P_a(t)$, both monotonically increasing and equidistant from each other during the time period. In the figure on the right $P(t)$ is concave downward and increasing, while $P_a(t)$ is decreasing and concave upward. In both situations a suitable choice of parameter values could lead to one component of the sentiment function canceling the other thereby satisfying the hypothesis of Theorem 3.2.*

An equilibrium $(\mathbf{P}_{eq}, \mathbf{N}_{eq}^{(i)}, \mathbf{M}_{eq}^{(i)}, \zeta_{eq}^{(i)})$ of the system (2.6)-(2.9), (3.11) is a solution of the following system of equations with $i = 1, ..., G$:

$$\left[1 - h\left(\zeta_{eq}^{(i)}\right)\right] \mathbf{N}_{eq}^{(i)} \mathbf{P}_{eq} = h\left(\zeta_{eq}^{(i)}\right) \mathbf{M}_{eq}^{(i)} \tag{3.12}$$

$$\psi^{(i)}\left(\mathbf{P}_{eq}, \mathbf{P}_{eq}, \zeta_{eq}^{(i)}\right) = 0, \tag{3.13}$$

$$\sum_{j=1}^{G} \mathbf{M}_{eq}^{(j)} = 1, \qquad \sum_{j=1}^{G} \mathbf{N}_{eq}^{(j)} = 1. \tag{3.14}$$

Suppose we have two distinct investor groups, one comprised of fundamental traders with sentiment functions (2.12), labeled by indices I^f, the other trend-based traderswith sentiment functions (2.11) (labeled with indices I^t). In this case, (3.13) reduces to $\zeta_{eq}^{(i)} = \frac{q^{(i)}\left(\mathbf{P}_a^{(i)} - \mathbf{P}_{eq}\right)}{\mathbf{P}_a^{(i)}}$ for $i \in I^f$ and $\zeta_{eq}^{(i)} = 0$ for $i \in I^t$. Moreover, the set of equilibria can be parametrized by the values of $\mathbf{M}_{eq}^{(j)}$ as follows:

Theorem 3.3. *In a system comprised of fundamental traders (with indices I^f) and trend-based traders (with indices I^t), the set $(\mathbf{P}_{eq}, \mathbf{N}_{eq}^{(i)}, \mathbf{M}_{eq}^{(i)}, \zeta_{eq}^{(i)})$ of equilibrium solutions is in one-to-one correspondence with the set*

$$\mathcal{S} = \left\{ (\mathbf{M}_{eq}^{(1)}, ..., \mathbf{M}_{eq}^{(G)}) \in [0,1]^G \,\middle|\, \sum_{i=1}^{G} \mathbf{M}_{eq}^{(i)} = 1 \right\}.$$

Proof. Suppose $(\mathbf{M}_{eq}^{(1)}, ..., \mathbf{M}_{eq}^{(G)}) \in \mathcal{S}$. By expressing $\mathbf{N}_{eq}^{(j)}$ from equations (3.12) and substituting into the conservation formula for total number of shares one obtains the condition

$$Q\left(\mathbf{P}_{eq}\right) = \sum_{i=1}^{G} \mathbf{N}_{eq}^{(i)} = 1, \tag{3.15}$$

where

$$Q(\mathbf{P}) = \sum_{i\in I^f} \frac{h\left(\frac{q^{(i)}\left(\mathbf{P}_a^{(i)}-\mathbf{P}\right)}{\mathbf{P}_a^{(i)}}\right)\mathbf{M}_{eq}^{(i)}}{\left[1-h\left(\frac{q^{(i)}\left(\mathbf{P}_a^{(i)}-\mathbf{P}\right)}{\mathbf{P}_a^{(i)}}\right)\right]\mathbf{P}} + \sum_{i\in I^t} \frac{\mathbf{M}_{eq}^{(i)}}{\mathbf{P}}. \tag{3.16}$$

Since $h\left(\frac{q^{(i)}\left(\mathbf{P}_a^{(i)}-\mathbf{P}\right)}{\mathbf{P}_a^{(i)}}\right)$ is a monotone decreasing function of $\mathbf{P}$, $Q(\mathbf{P})$ is monotone decreasing in $\mathbf{P}$ for all $\mathbf{P}$. Furthermore, $Q(\mathbf{P}) \to 0$ as $\mathbf{P} \to \infty$ and $Q(\mathbf{P}) \to \infty$ as $\mathbf{P} \to 0$. It follows that the equation $Q(\mathbf{P}) = 1$ has a unique solution $\mathbf{P}_{eq} > 0$. The values $\mathbf{N}_{eq}^{(j)}$ can be computed from (3.12), i.e.,

$$\mathbf{N}_{eq}^{(i)} = \frac{h(\zeta_{eq}^{(i)})\mathbf{M}_{eq}^{(i)}}{\left[1-h\left(\zeta_{eq}^{(i)}\right)\right]\mathbf{P}_{eq}}, \qquad i \in I^f \tag{3.17}$$

$$\mathbf{N}_{eq}^{(i)} = \mathbf{M}_{eq}^{(i)}/\mathbf{P}_{eq}, \qquad i \in I^t \tag{3.18}$$

It follows that $\mathbf{N}_{eq}^{(i)}$ are non-negative for all i and hence, in view of (3.15), $0 \le \mathbf{N}_{eq}^{(i)} \le 1$. Thus any point in $\mathcal{S}$ defines a unique equilibrium of the system. Conversely, for any equilibrium one has $(\mathbf{M}_{eq}^{(1)}, ..., \mathbf{M}_{eq}^{(G)}) \in \mathcal{S}$. ■

A result analogous to Theorem 3.3 states that equilibria can also be parametrized by the values of $\mathbf{N}_{eq}^{(i)}$. The corresponding equation to solve for $\mathbf{P}_{eq}$ is then

$$\begin{aligned} 1 = \tilde{Q}(\mathbf{P}) &= \sum_{i=1}^{G} \mathbf{M}_{eq}^{(i)} \\ &= \sum_{i\in I^f} \frac{\left[1-h\left(\frac{q^{(i)}\left(\mathbf{P}_a^{(i)}-\mathbf{P}\right)}{\mathbf{P}_a^{(i)}}\right)\right]\mathbf{P}}{h\left(\frac{q^{(i)}\left(\mathbf{P}_a^{(i)}-\mathbf{P}\right)}{\mathbf{P}_a^{(i)}}\right)} \mathbf{N}_{eq}^{(i)} + \sum_{i\in I^t} \mathbf{N}_{eq}^{(i)}\mathbf{P}. \end{aligned}$$

In some situations it may be of importance to determine the range of equilibrium prices that can be attained as one traverses the set of equilibria of a given system. Let $\mathbf{P}^{(i)}$ be the equilibrium price of investor group i corresponding to the singular situation in which all other investor groups are absent from the system, i.e., $\mathbf{M}_{eq}^{(j)} = \mathbf{N}_{eq}^{(j)} = 0$ for all $j \neq i$. In the case of a system comprised of two distinct investor groups,

$$h\left(\frac{q^{(i)}\left(\mathbf{P}_a^{(i)} - \mathbf{P}^{(i)}\right)}{\mathbf{P}_a^{(i)}}\right) = \frac{\mathbf{P}^{(i)}}{1+\mathbf{P}^{(i)}}, \qquad i \in I^f \tag{3.19}$$

$$\mathbf{P}^{(i)} = 1, \qquad i \in I^t \tag{3.20}$$

One has the following result.

Theorem 3.4. *In a system comprised of fundamental traders and trend-based traders, the range of equilibrium prices in the set of equilibria is*

$$\min_i \mathbf{P}^{(i)} \leq \mathbf{P}_{eq} \leq \max_i \mathbf{P}^{(i)}.$$

Proof. In view of Theorem 3.3 any equilibrium of the system can be parameterized by the values of $(\mathbf{M}_{eq}^{(1)}, ..., \mathbf{M}_{eq}^{(G)}) \in \mathcal{S}$. Note that the function $Q(\mathbf{P})$ in (3.16) is a convex combination of monotone decreasing functions $g^{(i)}(\mathbf{P})$ where

$$g^{(i)}(\mathbf{P}) = \frac{h\left(\frac{q^{(i)}\left(\mathbf{P}_a^{(i)} - \mathbf{P}\right)}{\mathbf{P}_a^{(i)}}\right)}{\left[1 - h\left(\frac{q^{(i)}\left(\mathbf{P}_a^{(i)} - \mathbf{P}\right)}{\mathbf{P}_a^{(i)}}\right)\right]\mathbf{P}}, \qquad i \in I^f \tag{3.21}$$

$$g^{(i)}(\mathbf{P}) = 1/\mathbf{P}, \qquad i \in I^t \tag{3.22}$$

and hence

$$\min_i g^{(i)}(\mathbf{P}) \leq Q(\mathbf{P}) \leq \max_i g^{(i)}(\mathbf{P}) \tag{3.23}$$

Note that $g^{(i)}(\mathbf{P}^{(i)}) = 1$. Suppose that, contrary to statement of the theorem, $\mathbf{P}_{eq} < \min_i \mathbf{P}^{(i)}$. By monotonicity of $g^{(j)}$, $g^{(j)}(\mathbf{P}_{eq}) > \max_i g^{(j)}(\mathbf{P}^{(i)}) \geq 1$. In view of (3.23), $Q(\mathbf{P}_{eq}) \geq \min_j g^{(j)}(\mathbf{P}_{eq}) > 1$ which is a contradiction with (3.15).

Similarly, if we suppose that $\mathbf{P}_{eq} > \max_i \mathbf{P}^{(i)}$, we conclude that $g^{(j)}(\mathbf{P}_{eq}) < \min_i g^{(j)}(\mathbf{P}^{(i)}) \leq 1$ which, in view of (3.23) implies $Q(\mathbf{P}_{eq}) \leq \min_j g^{(j)}(\mathbf{P}_{eq}) < 1$ which is also a contradiction with (3.15). The bounds on $\mathbf{P}_{eq}$ are sharp because they can be attained in the singular cases outlined above. ■

Next, we consider the stability of the equilibrium $(\mathbf{P}_{eq}, \mathbf{N}_{eq}^{(i)}, \mathbf{M}_{eq}^{(i)}, \zeta_{eq}^{(i)})$. Let us denote the following quantities:

$$\begin{aligned}
\alpha &= \sum_{j=1}^{G} k^{(j)} \mathbf{M}_{eq}^{(j)} \\
\beta &= \sum_{j=1}^{G} \left(1 - k^{(j)}\right) \mathbf{N}_{eq}^{(j)} \\
v^{(i)} &= \frac{\left(1 - k^{(i)}\right) \mathbf{N}_{eq}^{(i)}}{\beta} \\
k^{(i)} &= h(\zeta_{eq}^{(i)}) \\
\theta^{(i)} &= h'\left(\zeta_{eq}^{(i)}\right) \\
\mathbf{W}^{(i)} &= \mathbf{M}_{eq}^{(i)} + \mathbf{N}_{eq}^{(i)} \mathbf{P}_{eq}^{(i)}
\end{aligned}$$

where α is the equilibrium demand and β is the equilibrium supply, i.e., $\mathbf{P}_{eq} = \alpha/\beta$. Finally, let δ_{ij} be the standard delta function.

The linearized dynamics of the system (2.6) - (2.8), (3.11) near equilibrium are determined by a Jacobian that has the following block diagonal form (notice the order of the variables)

$$A = \begin{bmatrix} A^{(1,1)} & \dots & A^{(1,j)} & \dots & A^{(1,G+1)} \\ \vdots & & \vdots & & \vdots \\ A^{(j,1)} & \dots & A^{(j,j)} & \dots & A^{(j,G+1)} \\ \vdots & & \vdots & & \vdots \\ A^{(G+1,1)} & \dots & A^{(G+1,j)} & \dots & -1 \end{bmatrix} \tag{3.24}$$

where the block matrices are defined for $j = 1, ..., G$ as

$$A^{(i,j)} = \frac{\partial\left(\dot{\mathbf{M}}^{(i)}, \dot{\mathbf{N}}^{(i)}, \dot{\zeta}^{(i)}\right)}{\partial\left(\mathbf{M}^{(j)}, \mathbf{N}^{(j)}, \zeta^{(j)}\right)}, \quad A^{(G+1,j)} = \frac{\partial \dot{\mathbf{P}}}{\partial\left(\mathbf{M}^{(j)}, \mathbf{N}^{(j)}, \zeta^{(j)}\right)},$$

$$A^{(j,G+1)} = \frac{\partial\left(\dot{\mathbf{M}}^{(j)}, \dot{\mathbf{N}}^{(j)}, \dot{\zeta}^{(j)}\right)}{\partial \mathbf{P}}.$$

The matrix A can be transformed into a similar matrix $\tilde{A} = QAQ^{-1}$, which has the same block-diagonal structure but is more sparse and more amenable to eigenvalue analysis (see Appendix A):

$$\tilde{A}^{(j,j)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\left(1 - v^{(j)}\right) & \beta\left(1 - v^{(j)}\right)\mathbf{W}^{(j)}\theta^{(j)} \\ 0 & -\frac{\partial\psi^{(j)}}{\partial\mathbf{F}}\frac{1}{\beta^2} & \frac{\partial\psi^{(j)}}{\partial\mathbf{F}}\frac{\mathbf{W}^{(j)}\theta^{(j)}}{\beta} + \frac{\partial\psi^{(j)}}{\partial\zeta^{(j)}} \end{bmatrix}, \qquad 1 \le j \le G$$

$$\tilde{A}^{(i,j)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & v^{(i)} & -\beta v^{(i)}\mathbf{W}^{(j)}\theta^{(j)} \\ 0 & -\frac{\partial\psi^{(i)}}{\partial\mathbf{F}}\frac{1}{\beta^2} & \frac{\partial\psi^{(i)}}{\partial\mathbf{F}}\frac{\mathbf{W}^{(j)}\theta^{(j)}}{\beta} \end{bmatrix}, \qquad 1 \le i,j \le G, \quad i \ne j$$

$$\tilde{A}^{G+1,j} = \begin{bmatrix} 0 & -1/\beta^2 & \mathbf{W}^{(j)}\theta^{(j)}/\beta \end{bmatrix}, \qquad 1 \le j \le G$$

and

$$\tilde{A}^{j,G+1} = \begin{bmatrix} 0 \\ 0 \\ \frac{\partial\psi^{(i)}}{\partial\mathbf{P}} \end{bmatrix}. \qquad 1 \le j \le G$$

The trading strategies discussed above imply that $\frac{\partial\psi^{(i)}}{\partial\mathbf{F}} > 0$ for trend-based traders and $\frac{\partial\psi^{(i)}}{\partial\mathbf{F}} = 0$ for fundamental traders. In addition, $\frac{\partial\psi^{(i)}}{\partial\mathbf{P}} < 0$, $\frac{\partial\psi^{(i)}}{\partial\zeta^{(i)}} < 0$, and $0 < \theta^{(i)} < 1/2$ for both groups.

We shall analyze the stability of two important cases: (i) all traders in the group have fundamental trading preferences and (ii) the group has two traders - one with fundamental and the other one with trend-based preferences. In both cases Theorem 3.3 implies that the equilibrium of the system is unaffected by $c^{(i)}$ and therefore we focus our investigation on the dependence of the stability on $c^{(i)}$.

3.2.1. Case (i): All fundamental traders. Suppose that all traders follow a fundamental strategy with distinct constants and estimates of the fundamental price, i.e.,

$$\psi^{(i)}\left(\mathbf{F},\mathbf{P},\zeta^{(i)}\right)=c^{(i)}\left(q^{(i)}\frac{\mathbf{P}_a^{(i)}-\mathbf{P}}{\mathbf{P}_a^{(i)}}-\zeta_1^{(i)}\right).$$

In this case the characteristic polynomial of $\tilde{A}$ (and thus A) is given by

$$p_A(\lambda)=\lambda^G\det(\lambda I-B)$$

where the zero eigenvalue of multiplicity G is a consequence of the constraint (2.10) relating any pair of variables $(\mathbf{M}^{(i)},\mathbf{N}^{(i)})$. The matrix B (after a permutation of rows and columns) is a block diagonal matrix

$$B=\begin{bmatrix} B^{(1,1)} & B^{(1,2)} & 0_{G\times 1}\\ 0_{G\times G} & B^{(2,2)} & B^{(2,3)}\\ B^{(3,1)} & B^{(3,2)} & -1\end{bmatrix}$$

with

$$B^{(1,1)}=\begin{bmatrix} -\left(1-v^{(1)}\right) & \cdots & v^{(1)} & \cdots & v^{(1)}\\ \vdots & & \vdots & & \vdots\\ v^{(j)} & \cdots & -\left(1-v^{(j)}\right) & \cdots & v^{(j)}\\ \vdots & & \vdots & & \vdots\\ v^{(G)} & \cdots & v^{(G)} & \cdots & -\left(1-v^{(G)}\right)\end{bmatrix},$$

$$B^{(1,2)}=\begin{bmatrix} \left(1-v^{(1)}\right)\beta\mathbf{W}^{(1)}\theta^{(1)} & \cdots & -v^{(1)}\beta\mathbf{W}^{(j)}\theta^{(j)} & \cdots & -v^{(1)}\beta\mathbf{W}^{(G)}\theta^{(G)}\\ \vdots & & \vdots & & \vdots\\ -v^{(j)}\beta\mathbf{W}^{(1)}\theta^{(1)} & \cdots & \left(1-v^{(j)}\right)\beta\mathbf{W}^{(j)}\theta^{(j)} & \cdots & -v^{(j)}\beta\mathbf{W}^{(G)}\theta^{(G)}\\ \vdots & & \vdots & & \vdots\\ -v^{(G)}\beta\mathbf{W}^{(1)}\theta^{(1)} & \cdots & -v^{(G)}\beta\mathbf{W}^{(j)}\theta^{(j)} & \cdots & \left(1-v^{(G)}\right)\beta\mathbf{W}^{(G)}\theta^{(G)}\end{bmatrix},$$

$$B^{(2,2)}=\begin{bmatrix} -c^{(1)} & \cdots & 0 & \cdots & 0\\ \vdots & & \vdots & & \vdots\\ 0 & \cdots & -c^{(j)} & \cdots & 0\\ \vdots & & \vdots & & \vdots\\ 0 & \cdots & 0 & \cdots & -c^{(G)}\end{bmatrix},$$

$$B^{(2,3)}=\begin{bmatrix} -q^{(1)}c^{(1)}/\mathbf{P}_a^{(1)} & \cdots & -q^{(j)}c^{(j)}/\mathbf{P}_a^{(j)} & \cdots & -q^{(G)}c^{(G)}/\mathbf{P}_a^{(G)}\end{bmatrix}^T,$$

$$B^{(3,1)} = \left[-1/\beta^2 \quad \cdots \quad -1/\beta^2 \right],$$

and

$$B^{(3,2)} = \left[\mathbf{W}^{(1)}\theta^{(1)}/\beta \quad \cdots \quad \mathbf{W}^{(j)}\theta^{(j)}/\beta \quad \cdots \quad \mathbf{W}^{(G)}\theta^{(G)}/\beta \right].$$

In the special case of two fundamental traders we have

$$B = \begin{bmatrix} -\left(1-v^{(1)}\right) & v^{(1)} & \left(1-v^{(1)}\right)\beta\mathbf{W}^{(1)}\theta^{(1)} & -v^{(1)}\beta\mathbf{W}^{(2)}\theta^{(2)} & 0 \\ v^{(2)} & -\left(1-v^{(2)}\right) & -v^{(2)}\beta\mathbf{W}^{(1)}\theta^{(1)} & \left(1-v^{(2)}\right)\beta\mathbf{W}^{(2)}\theta^{(2)} & 0 \\ 0 & 0 & -c^{(1)} & 0 & -q^{(1)}c^{(1)}/\mathbf{P}_a^{(1)} \\ 0 & 0 & 0 & -c^{(2)} & -q^{(2)}c^{(2)}/\mathbf{P}_a^{(2)} \\ -1/\beta^2 & -1/\beta^2 & \mathbf{W}^{(1)}\theta^{(1)}/\beta & \mathbf{W}^{(2)}\theta^{(2)}/\beta & -1 \end{bmatrix} \tag{3.25}$$

Let C be the $2G \times 2G$ principal submatrix of B defined as

$$C = \begin{bmatrix} B^{(1,1)} & B^{(1,2)} \\ 0_{G\times G} & B^{(2,2)} \end{bmatrix}$$

The matrix C is diagonalizable and has eigenvalues $\left\{0,\ -1,\ -c^{(j)}\right\}$, $j = 1, 2, ..., G$, where the -1 eigenvalue has multiplicity $G-1$. Diagonalization of C for two traders is accomplished via the left eigenvectors

$$V = \begin{bmatrix} 1 & 1 & 0 & 0 \\ v^{(2)} & -v^{(1)} & \frac{v^{(2)}\beta\mathbf{W}^{(1)}\theta^{(1)}}{c^{(1)}-1} & -\frac{v^{(1)}\beta\mathbf{W}^{(2)}\theta^{(2)}}{c^{(2)}-1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

By applying analogous diagonalization to the full matrix B we obtain the matrix M which has a mammillary form:

$$M = \begin{bmatrix} M^{(1,1)} & 0_{G\times G} & M^{(1,3)} \\ 0_{G\times G} & M^{(2,2)} & M^{(2,3)} \\ M^{(3,1)} & M^{(3,2)} & -1 \end{bmatrix}$$

where

$$M^{(1,1)} = \begin{bmatrix} 0 & 0_{1\times(G-1)} \\ 0_{(G-1)\times 1} & -I_{(G-1)\times(G-1)} \end{bmatrix},$$

$$M^{(2,2)} = \begin{bmatrix} -c^{(1)} & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & -c^{(j)} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & -c^{(G)} \end{bmatrix},$$

$$M^{(3,1)} = \left[-1/\beta^2 \quad 0_{1\times(G-1)} \right]^T,$$

and

$$M^{(3,2)} = \left[\mathbf{W}^{(1)}\theta^{(1)}/\beta \quad \cdots \quad \mathbf{W}^{(j)}\theta^{(j)}/\beta \quad \cdots \quad \mathbf{W}^{(G)}\theta^{(G)}/\beta\right].$$

(The submatrix $M^{(1,3)}$ is not important for stability considerations.) In the case of two investors

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -v^{(2)}\frac{q^{(1)}c^{(1)}\mathbf{W}^{(1)}\theta^{(1)}\beta}{\mathbf{P}_a^{(1)}\left(c^{(1)}-1\right)} + v^{(1)}\frac{q^{(2)}c^{(2)}\mathbf{W}^{(2)}\theta^{(2)}\beta}{\mathbf{P}_a^{(2)}\left(c^{(2)}-1\right)} \\ 0 & 0 & -c^{(1)} & 0 & -q^{(1)}c^{(1)}/\mathbf{P}_a^{(1)} \\ 0 & 0 & 0 & -c^{(2)} & -q^{(2)}c^{(2)}/\mathbf{P}_a^{(2)} \\ -1/\beta^2 & 0 & \mathbf{W}^{(1)}\theta^{(1)}/\beta & \mathbf{W}^{(2)}\theta^{(2)}/\beta & -1 \end{bmatrix}$$

The matrix M has the characteristic polynomial

$$p_M(\lambda) = (\lambda+1)^{G-1}\lambda \det\left(\lambda I - \tilde{M}\right)$$

where

$$\tilde{M} = \begin{bmatrix} M^{(3,3)} & M^{(3,4)} \\ M^{(4,3)} & -1 \end{bmatrix}.$$

In the case of two investors,

$$\tilde{M} = \begin{bmatrix} -c^{(1)} & 0 & -q^{(1)}c^{(1)}/\mathbf{P}_a^{(1)} \\ 0 & -c^{(2)} & -q^{(2)}c^{(2)}/\mathbf{P}_a^{(2)} \\ \mathbf{W}^{(1)}\theta^{(1)}/\beta & \mathbf{W}^{(2)}\theta^{(2)}/\beta & -1 \end{bmatrix}. \tag{3.26}$$

The single zero eigenvalue of M results from the conservation of the total amount of cash in the system.

The sign structure of the matrix $\tilde{M}$ is

$$\operatorname{sgn}(\tilde{M}) = \begin{bmatrix} -I_{G\times G} & -1_{G\times 1} \\ 1_{1\times G} & -1 \end{bmatrix}. \tag{3.27}$$

and hence $\tilde{M}$ satisfies the necessary conditions for sign stability of Quirk & Ruppert [20] and May [18], namely, (i) $\tilde{M}_{jj} \le 0$ for all j, (ii) $\tilde{M}_{jj} < 0$ for at least one j, (iii) $\tilde{M}_{ij}\tilde{M}_{ji} \le 0$ for all $i \ne j$, (iv) in the graph associated with $\tilde{M}$ there is no closed path of length 3 or more, and (v) $\det(\tilde{M}) \ne 0$. As a result, one can conclude that $\tilde{M}$ has no eigenvalues with positive real part. Furthermore, the matrix $\tilde{M}$ satisfies Jeffries sufficient condition (color test) [17] since every node of the graph associated with $\tilde{M}$ is self-regulating, and hence all eigenvalues of $\tilde{M}$ have negative real parts.

Alternatively, for two fundamental investor groups one can use the Routh-Hurwitz criterion (see [13]) to show that the equilibrium is stable for all positive $c^{(1)}$ and $c^{(2)}$. Indeed, set

$$
\begin{aligned}
A_2 &= c^{(1)} + c^{(2)} + 1, \\
A_1 &= c^{(1)}c^{(2)} + c^{(1)}\left(1 + \frac{q^{(1)}\mathbf{W}^{(1)}\theta^{(1)}}{\mathbf{P}_a^{(1)}\beta}\right) + c^{(2)}\left(1 + \frac{q^{(2)}\mathbf{W}^{(2)}\theta^{(2)}}{\mathbf{P}_a^{(2)}\beta}\right), \\
A_0 &= c^{(1)}c^{(2)}\left(1 + \frac{q^{(1)}\mathbf{W}^{(1)}\theta^{(1)}}{\mathbf{P}_a^{(1)}\beta} + \frac{q^{(2)}\mathbf{W}^{(2)}\theta^{(2)}}{\mathbf{P}_a^{(2)}\beta}\right)
\end{aligned}
$$

where

$$p_{\tilde{M}}(\lambda) = \lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$$

is the characteristic polynomial for $\tilde{M}$. The Routh-Hurwitz criterion implies that the eigenvalues of $\tilde{M}$ have negative real parts if and only if $A_2 > 0$, $A_1 > 0$, $A_0 > 0$, and $A_1A_2 > A_0$. The criteria $A_j > 0$, $j = 0, 1, 2$ are clearly satisfied for all positive $c^{(1)}$ and $c^{(2)}$. After substitution, the final Routh-Hurwitz criterion is equivalent to:

$$
\begin{aligned}
c^{(1)}c^{(2)}\left(c^{(1)} + c^{(2)} + 2\right) + c^{(1)}\left(c^{(1)} + 1\right)\left(1 + \frac{q^{(1)}\mathbf{W}^{(1)}\theta^{(1)}}{\mathbf{P}_a^{(1)}\beta}\right) \\
+ c^{(2)}\left(c^{(2)} + 1\right)\left(1 + \frac{q^{(2)}\mathbf{W}^{(2)}\theta^{(2)}}{\mathbf{P}_a^{(2)}\beta}\right) > 0.
\end{aligned}
$$

which also holds for all positive $c^{(1)}$ and $c^{(2)}$.

We have proved the following result.

Theorem 3.5. *In a system composed of traders with pure fundamental trading preferences every equilibrium is asymptotically stable for all positive $c^{(i)}$, $i = 1, \ldots G$, within the class of trajectories compatible with that equilibrium.*

3.2.2. Case (ii): One fundamental trading group and one trend-based trading group.

Consider a system consisting of two trading groups. Group 1 follows a pure trend-based strategy, while group 2 follows a pure fundamental strategy, i.e.

$$\psi^{(1)}\left(\mathbf{F}, \mathbf{P}, \zeta^{(1)}\right) = c^{(1)}\left(q^{(1)}\frac{\mathbf{F} - \mathbf{P}}{\mathbf{P}} - \zeta^{(1)}\right)$$

and

$$\psi^{(2)}\left(\mathbf{F}, \mathbf{P}, \zeta^{(2)}\right) = c^{(2)}\left(q^{(2)}\frac{\mathbf{P}_a^{(2)} - \mathbf{P}}{\mathbf{P}_a^{(2)}} - \zeta^{(2)}\right).$$

In this scenario we have the following equilibrium solution[5]

$$\zeta_{eq}^{(1)} = 0 \tag{3.28}$$

$$\zeta_{eq}^{(2)} = q^{(2)} \frac{\mathbf{P}_a - \mathbf{P}_{eq}}{\mathbf{P}_a} \tag{3.29}$$

$$k_{eq}^{(1)} = 1/2 \tag{3.30}$$

$$k_{eq}^{(2)} = h\left(q^{(2)} \frac{\mathbf{P}_a - \mathbf{P}_{eq}}{\mathbf{P}_a}\right) \tag{3.31}$$

$$\begin{aligned} Q\left(\mathbf{P}_{eq}\right) &= \frac{\mathbf{M}_{eq}^{(1)}}{\mathbf{P}_{eq}} + \frac{h\left(q^{(2)} \frac{\mathbf{P}_a - \mathbf{P}_{eq}}{\mathbf{P}_a}\right)}{\left[1 - h\left(q^{(2)} \frac{\mathbf{P}_a - \mathbf{P}_{eq}}{\mathbf{P}_a}\right)\right] \mathbf{P}_{eq}} \left(1 - \mathbf{M}_{eq}^{(1)}\right) \\ &= 1. \end{aligned} \tag{3.32}$$

Due to the assumption of conservation of cash in this system, we set $\mathbf{M}_{eq}^{(2)} = \left(1 - \mathbf{M}_{eq}^{(1)}\right)$ in equation (3.32).

We can parameterize this equilibrium solution by the equilibrium price, $\mathbf{P}_{eq}$, to obtain

$$\mathbf{M}_{eq}^{(1)} = \frac{\mathbf{P}_{eq} - \left(1 + \mathbf{P}_{eq}\right) h\left(q^{(2)} \frac{\mathbf{P}_a - \mathbf{P}_{eq}}{\mathbf{P}_a}\right)}{1 - 2h\left(q^{(2)} \frac{\mathbf{P}_a - \mathbf{P}_{eq}}{\mathbf{P}_a}\right)} \tag{3.33}$$

$$\mathbf{N}_{eq}^{(1)} = \frac{\mathbf{P}_{eq} - \left(1 + \mathbf{P}_{eq}\right) h\left(q^{(2)} \frac{\mathbf{P}_a - \mathbf{P}_{eq}}{\mathbf{P}_a}\right)}{\mathbf{P}_{eq}\left[1 - 2h\left(q^{(2)} \frac{\mathbf{P}_a - \mathbf{P}_{eq}}{\mathbf{P}_a}\right)\right]}. \tag{3.34}$$

The range of values of $\mathbf{P}_{eq}$ for which solutions (3.33) and (3.34) stay within $(0, 1)$ depends on the function h and the constants $q^{(2)}, \mathbf{P}_a$. Note that equilibrium is again independent of the $c^{(i)}$ and therefore it is reasonable to investigate stability of the equilibrium as a function of these parameters.

The transformed Jacobian $\tilde{A}$ in this case has the characteristic polynomial

$$p_{\tilde{A}}(\lambda) = \lambda^2 \det\left(\lambda I - B\right)$$

where after appropriate permutations we have

$$B = \begin{bmatrix} -\left(1 - v^{(1)}\right) & v^{(1)} & \left(1 - v^{(1)}\right)\beta \mathbf{W}^{(1)}\theta^{(1)} & -v^{(1)}\beta \mathbf{W}^{(2)}\theta^{(2)} & 0 \\ v^{(2)} & -\left(1 - v^{(2)}\right) & -v^{(2)}\beta \mathbf{W}^{(1)}\theta^{(1)} & \left(1 - v^{(2)}\right)\beta \mathbf{W}^{(2)}\theta^{(2)} & 0 \\ -\frac{c^{(1)}q^{(1)}}{\alpha\beta} & -\frac{c^{(1)}q^{(1)}}{\alpha\beta} & -c^{(1)} + \frac{c^{(1)}q^{(1)}\mathbf{W}^{(1)}\theta^{(1)}}{\alpha} & \frac{c^{(1)}q^{(1)}\mathbf{W}^{(2)}\theta^{(2)}}{\alpha} & -q^{(1)}c^{(1)}/\mathbf{P}_{eq} \\ 0 & 0 & 0 & -c^{(2)} & -q^{(2)}c^{(2)}/\mathbf{P}_a \\ -1/\beta^2 & -1/\beta^2 & \mathbf{W}^{(1)}\theta^{(1)}/\beta & \mathbf{W}^{(2)}\theta^{(2)}/\beta & -1 \end{bmatrix}.$$

[5] As only one group is focused on the valuation (and to simplify notation) we set: $\mathbf{P}_a := \mathbf{P}_a^{(2)}$.

(Note that the matrix B is identical to that in the case (i) with the exception of the third line.) With an additional similarity transformation using

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -c^{(1)}q^{(1)}/\mathbf{P}_{eq} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

we obtain $\tilde{B} = RBR^{-1}$ which resembles matrix B in (3.25):

$$\tilde{B} = \begin{bmatrix} -\left(1-v^{(1)}\right) & v^{(1)} & \left(1-v^{(1)}\right)\beta\mathbf{W}^{(1)}\theta^{(1)} & -v^{(1)}\beta\mathbf{W}^{(2)}\theta^{(2)} & \frac{c^{(1)}q^{(1)}\left(1-v^{(1)}\right)\beta^2\mathbf{W}^{(1)}\theta^{(1)}}{\alpha} \\ v^{(2)} & -\left(1+v^{(2)}\right) & -v^{(2)}\beta\mathbf{W}^{(1)}\theta^{(1)} & \left(1-v^{(2)}\right)\beta\mathbf{W}^{(2)}\theta^{(2)} & -\frac{c^{(1)}q^{(1)}v^{(2)}\beta^2\mathbf{W}^{(1)}\theta^{(1)}}{\alpha} \\ 0 & 0 & -c^{(1)} & 0 & -\left[c^{(1)}\right]^2 q^{(1)}/\mathbf{P}_{eq} \\ 0 & 0 & 0 & -c^{(2)} & -c^{(2)}q^{(2)}/\mathbf{P}_a \\ -1/\beta^2 & -1/\beta^2 & \mathbf{W}^{(1)}\theta^{(1)}/\beta & \mathbf{W}^{(2)}\theta^{(2)}/\beta & -1+\frac{c^{(1)}q^{(1)}\mathbf{W}^{(1)}\theta^{(1)}}{\alpha} \end{bmatrix}.$$

Specifically, the $2G \times 2G$ principal submatrix of $\tilde{B}$ formed by the rows and columns from 1 to $2G$, is identical to the matrix C in the case of all fundamental traders, i.e., it has eigenvalues $\left\{0, -1, -c^{(1)}, -c^{(2)}\right\}$. Diagonalization of $\tilde{B}$ yields the matrix

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -\frac{c^{(1)}q^{(1)}v^{(2)}\beta\mathbf{W}^{(1)}\theta^{(1)}}{\left(c^{(1)}-1\right)\mathbf{P}_{eq}} + \frac{c^{(2)}q^{(2)}v^{(1)}\beta\mathbf{W}^{(2)}\theta^{(2)}}{\left(c^{(2)}-1\right)\mathbf{P}_a} \\ 0 & 0 & -c^{(1)} & 0 & -\left[c^{(1)}\right]^2 q^{(1)}/\mathbf{P}_{eq} \\ 0 & 0 & 0 & -c^{(2)} & -c^{(2)}q^{(2)}/\mathbf{P}_a \\ -1/\beta^2 & 0 & \mathbf{W}^{(1)}\theta^{(1)}/\beta & \mathbf{W}^{(2)}\theta^{(2)}/\beta & -1+\frac{c^{(1)}q^{(1)}\mathbf{W}^{(1)}\theta^{(1)}}{\alpha} \end{bmatrix}.$$

As before, the characteristic polynomial of M obeys

$$p_M(\lambda) = (\lambda+1)\,\lambda \det\left(\lambda I - \tilde{M}\right) \tag{3.35}$$

where

$$\tilde{M} = \begin{bmatrix} -c^{(1)} & 0 & -\left[c^{(1)}\right]^2 q^{(1)}/\mathbf{P}_{eq} \\ 0 & -c^{(2)} & -c^{(2)}q^{(2)}/\mathbf{P}_a \\ \mathbf{W}^{(1)}\theta^{(1)}/\beta & \mathbf{W}^{(2)}\theta^{(2)}/\beta & -1+\frac{c^{(1)}q^{(1)}\mathbf{W}^{(1)}\theta^{(1)}}{\alpha} \end{bmatrix}. \tag{3.36}$$

Again, the single zero eigenvalue of M results from the conservation law for total amount of cash. The differences between the stability properties for the case with two fundamental trading groups and the current case with one trend-based and one fundamental trading group boil down to differences between the matrix $\tilde{M}$ here and the one in (3.26).

The first observation about stability of equilibria that we can make is that any equilibrium of the system is stable for sufficiently small $c^{(1)}$, the inverse time scale of the trend-based trading group:

Lemma 3.6. *An equilibrium of a system composed of one trading group with trend-based preference and one with fundamental trading preference is stable for all* $c^{(2)}$ *within the class of trajectories compatible with that equilibrium if, for the trend-based group,*

$$c^{(1)} < \frac{\alpha}{q^{(1)}\mathbf{W}^{(1)}\theta^{(1)}}.$$

Proof. If the hypothesis holds then the matrix $\tilde{M}$ in (3.36) has the sign signature (3.27) and hence it is sign stable. ■

A sharper result can be obtained by analyzing the characteristic polynomial of $\tilde{M}$

$$p_{\tilde{M}}(\lambda) = \lambda^3 + A_2\lambda^2 + A_1\lambda + A_0.$$

where

$$\begin{aligned} A_2 &= 1 + c^{(2)} + c^{(1)}\mathbf{U} \\ A_1 &= c^{(1)} + c^{(2)}\mathbf{V} + c^{(1)}c^{(2)}\mathbf{U} \\ A_0 &= c^{(1)}c^{(2)}\mathbf{V}. \end{aligned}$$

where we introduced the equilibrium-dependent quantities

$$\mathbf{U} = 1 - \frac{q^{(1)}\mathbf{W}^{(1)}\theta^{(1)}}{\alpha} \tag{3.37}$$

$$\mathbf{V} = 1 + \frac{q^{(2)}\mathbf{W}^{(2)}\theta^{(2)}}{\beta\mathbf{P}_a} \tag{3.38}$$

Note that $\mathbf{U} < 1$ and $\mathbf{V} > 1$.

Routh-Hurwitz theory implies that all roots of $\tilde{M}$ have negative real parts if and only if $A_2 > 0$, $A_1 > 0$, $A_0 > 0$, and $A_1A_2 > A_0$. The fourth inequality can be rewritten as

$$\begin{aligned} &\left(c^{(1)} + c^{(2)}\mathbf{U}\right)c^{(1)}c^{(2)}\mathbf{U} + \left[c^{(2)}\right]^2\mathbf{U} + \left[c^{(1)}\right]^2\mathbf{V} \\ &\qquad + c^{(1)}c^{(2)}\left(1 + \mathbf{U}\mathbf{V} + \mathbf{U} - \mathbf{V}\right) + c^{(1)}\mathbf{V} + c^{(2)} > 0. \end{aligned}$$

Note the following:

1. All four inequalities are satisfied for sufficiently small $c^{(1)}$ and $c^{(2)}$.
2. The first two inequalities are satisfied for all positive $c^{(1)}$ and $c^{(2)}$ if and only if $\mathbf{U} > 0$.
3. All four inequalities hold for all positive $c^{(1)}$ and $c^{(2)}$ if $1 < \mathbf{V} < \frac{1+\mathbf{U}}{1-\mathbf{U}}$.
4. The fourth inequality fails if $\mathbf{U} \sim 0$, $\mathbf{V} > \frac{1+\mathbf{U}}{1-\mathbf{U}}$, $c^{(1)} > 1/(\mathbf{V}-1)$, and $c^{(2)} > \frac{\left(1+c^{(1)}\right)c^{(1)}\mathbf{V}}{c^{(1)}(\mathbf{V}-1)-1}$.

To summarize, we have the following theorem.

Theorem 3.7. *A system composed of one investor group with trend-based preference and one with fundamental trading preference is stable (within the class of compatible trajectories) for all* $c^{(1)}$ *and* $c^{(2)}$ *at every equilibrium for which the following condition holds*

$$2\left(1-\frac{q^{(1)}\mathbf{W}^{(1)}\theta^{(1)}}{\alpha}\right)-\frac{q^{(1)}\mathbf{W}^{(1)}\theta^{(1)}}{\alpha}\frac{q^{(2)}\mathbf{W}^{(2)}\theta^{(2)}}{\beta\mathbf{P}_a}>0.$$

If the condition fails and if $q^{(1)}\mathbf{W}^{(1)}\theta^{(1)}>\alpha-\varepsilon$ *at an equilibrium with* $\varepsilon>0$ *sufficiently small, then there exist* $c^{(1)}$ *and* $c^{(2)}$ *for which that equilibrium is unstable.*

In [5] the authors studied a variation of this two group model in which the $y=\tanh(x)$ function was approximated by $y\simeq x$ for $-1<x<1$.

4. Numerical Results. In this section we study numerically the dynamics of the conserved two-group system (2.6)-(2.8); (2.16) and (2.17), where Group 1 is focused solely on the recent trend in price and Group 2 is focused solely on the price's relative deviation from its fundamental value[6]. In [5] the criteria for stability of the approximated system, i.e. $\tanh(x)\simeq x$, were determined numerically. The computations in that paper showed the existence of a stability threshold price, $\mathbf{P}_{eq}^{tr}$, where the stability of the equilibrium changed as the price moved through this value with other parameters held fixed. It was confirmed for several parameter regimes (in [12]) that the linearized system had two strictly negative eigenvalues, a zero eigenvalue, and a pair of purely imaginary eigenvalues at this price, $\mathbf{P}_{eq}^{tr}$. In this section we focus on the full (non-approximate) system and consider the following aspects of the dynamics:

1. The simulations indicate that trajectories starting near unstable equilibrium points terminate near stable equilibria. We observe that as the trajectory starting points approach $\mathbf{P}_{eq}^{tr}$ from the unstable side, the ending points move toward $\mathbf{P}_{eq}^{tr}$ from the stable side. Thus, a plot of the ending price versus the beginning price mirrors the line $y=x$ when the starting price is near a stable equilibrium and appears to be a monotone decreasing function of the starting price ($\mathbf{P}(0)$) once $\mathbf{P}(0)>\mathbf{P}_{eq}^{tr}$, i.e. once $\mathbf{P}(0)$ is near an unstable equilibrium.

2. Just as important as the trajectory's ending price is the excursion which the trajectory experiences as it attains this point. We define the excursion as the maximum deviation in price from the trajectory's initial price near the curve of equilibria. While the starting and ending prices may be close to one another, it is evident that the prices along the trajectory may deviate significantly from these values. We track the maximum and minimum prices attained along the trajectory and plot them on a log scale to facilitate comparisons. This is particularly relevant given recent events, i.e. the housing bubble/bust episode and subsequent market turbulence during 2008-2009 and the European debt crisis of 2011. These maximum and minimum prices, which can vary drastically from the initial price, are indications that this model may be utilized to study non-classical phenomena like bubbles.

[6]As in the preceding section(s), we utilize the simplified notation: $q_1:=q_1^{(1)}$; $q_2:=q_2^{(2)}$; $c_1:=c_1^{(1)}$; $c_2:=c_2^{(2)}$; and $\mathbf{P}_a^{(2)}=\mathbf{P}_a$.

It is also interesting to note what happens to each trading group's assets, hence its wealth, as the price evolves over time. Figures 4.8 and 4.13 contain plots of the trend-based trading group's shares and cash levels versus time[7] as well as logarithmic plots of Group 1's wealth and the price versus time[8]. As all equilibria are stable in Case 1, these figures are omitted.

Finally, we consider the eigenvalues of the linearized system to better understand the trajectories' dynamics. As noted above (and in [5]), the linearized system admits a 0 and a -1 eigenvalue. The remaining three eigenvalues are discussed.

This analysis is achieved by assigning practical[9] values to certain parameters. In this section we will restrict our attention to three representative cases:

(1) Small magnitudes for the motivations and one unit time scales for both groups ($q_1 = q_2 = 1$, $c_1 = c_2 = 1$);

(2) Small motivation magnitude for the momentum group; large motivation magnitude for the fundamental group; and shorter time scales for both groups ($q_1 = 1$, $q_2 = 5$, $c_1 = c_2 = 5$); and

(3) Large magnitudes for the motivations and shorter time scales for both groups ($q_1 = q_2 = 5$, $c_1 = c_2 = 5$). We restrict each case to the scenario $\mathbf{P}_a = 0.8 < \mathbf{P}_{eq} < 1$.

In the computations displayed in Figures 4.1, 4.4, and 4.9 the curve of equilibria is plotted as the union of green and red points, where the green points correspond to stable and the red unstable equilibria.

Case 1. $q_1 = q_2 = 1$, $c_1 = c_2 = 1$

In the computations displayed in Figure 4.1 all equilibrium points are stable. This figure also contains five sample trajectories[10] which begin at the points labeled "1B", "2B", ..., "5B" and end at the points labeled "1E", "2E", ..., "5E."

Figure 4.2 plots 100 average (trajectory) ending prices versus average beginning prices. To produce this figure the curve of equilibrium points was discretized into 100 points. Then, five starting points were randomly selected about each equilibrium point. The Matlab ode23s function was utilized to solve the system. The five starting (alternately ending) prices are then averaged for each equilibrium point and plotted. As all equilibrium points are stable, the trajectory ending points lie on the line $y = x$, which is represented by the green line in Figure 4.2.

Figure 4.3 verifies that all permissible equilibrium points, i.e. those satisfying the assumptions (*i*) $0 < \mathbf{M}^{(i)} < 1$, (*ii*) $0 < \mathbf{N}^{(i)} < 1$, and (*iii*) $0 < \mathbf{P}_a < 1$, are stable as the real parts

[7] As assets are conserved and the system has been rescaled, in order to compute the value of the fundamental trading group's cash, for example, simply subtract the trend-based trading group's cash level from one.

[8] Logarthmic plots are utilized for the price and wealth figures to facilitate comparison of the different trajectories.

[9] Previous studies, [9] and [14], obtained practical values for the q_i and c_i based upon optimization (using closed-end fund data) and statistical (using experimental data) methods. The parameter values utilized in this section are consistent with these previous findings.

[10] The trajectories in Figures 4.1, 4.4, and 4.9 are color-coded as follows: 1-blue, 2-red, 3-cyan, 4-magenta, and 5-black.

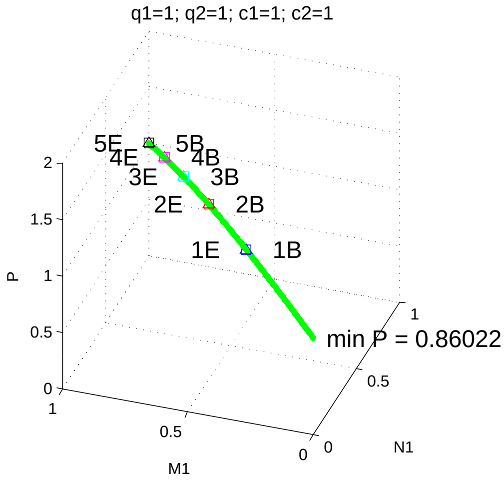


Figure 4.1. *Examples of five trajectories for Case 1. The curve represents the permissible range (i.e. $0.86022 \leq \mathbf{P}_{eq} < 1$) of equilibrium prices, while the color green indicates these equilibria are stable. The five marked points correspond to trajectories that begin (and therefore terminate) near these stable equilibria.*

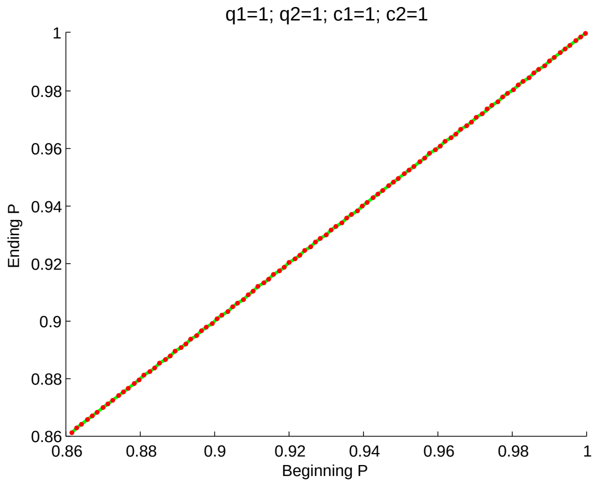


Figure 4.2. *Average equilibrium price versus the starting price over 100 trajectories. Note that since all trajectories start near stable equilibria the ending points lie along the line $y = x$ (denoted in green).*

(blue curves) of the pair of complex eigenvalues are strictly negative. The remaining three eigenvalues are 0, -1, and -1. Thus, in this case four of the eigenvalues have negative real parts and the last eigenvalue is a 0 due to the curve of equilibria.

While there exists a range of possible equilibrium prices, i.e. $0.86022 \leq \mathbf{P}_{eq} \leq 1$, the solution trajectories quickly move to equilibrium. Thus, the starting and ending prices are

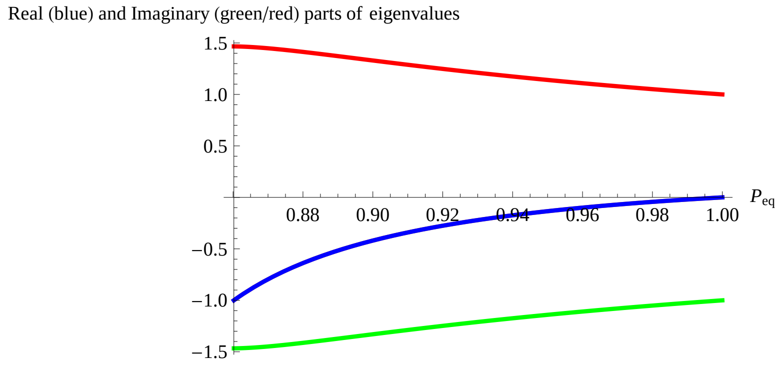


Figure 4.3. *Eigenvalues of the Jacobian of the system in Case 1 versus equilibrium price. Three of these eigenvalues are* $\{0, -1, -1\}$*, and the remaining two form a complex conjugate pair. The real (blue) and imaginary (green and red) parts of this pair are plotted in this figure. As the real parts (blue curve) of these eigenvalues are less than zero, all equilibrium points in this scenario are stable.*

essentially identical. However, by changing the parameter values these trajectories behave in a much different manner.

Case 2. $q_1 = 1$, $q_2 = 5$, $c_1 = c_2 = 5$

Figure 4.4 shows the (red/green) curve of (unstable/stable) equilibrium points for the parameter set: $q_1 = 1$, $q_2 = 5$, and $c_1 = c_2 = 5$. In this case all five trajectories start near unstable equilibrium points and terminate near stable equilibria.

Figure 4.5 shows a stability transition price, $\mathbf{P}_{eq}^{tr}$, of approximately 0.835. This is consistent with the point where the red and green points meet in Figure 4.4. Note that in Figure 4.5 the curve rises along the line $y = x$ and then monotonically decreases as the equilibrium points about which the trajectories are starting become unstable.

Unlike the dynamics in Case 1, the trajectories in Figure 4.4 do not stay near the curve of equilibria. Instead they deviate significantly from this curve. The maximum and minimum prices attained along each trajectory are plotted in Figure 4.6. In this figure the black and red points correspond to maximum and minimum prices attained when the initial price movement was positive, i.e. $\mathbf{P}(2) > \mathbf{P}(1)$. The blue and green points correspond to maximum and minimum prices when the initial price movement was negative. Note the following points:

1. Initially, the maximum and minimum values coincide until approximately the stability transition price, $\mathbf{P}_{eq}^{tr} = 0.835$, is reached. These points correspond to stable equilibria.

2. For the initial price greater than 0.835 the minimum (red and green) values appear to monotically decrease along the same curve. However, the maximum values do not appear to adhere to a discernible pattern. Rather they appear to follow a distribution. Contrast this figure with Figure 4.11 from Case 3.

Figure 4.7 plots the complex pair of eigenvalues for this case. The final eigenvalue is -5. Thus, consistent with the other figures for this case the stability transition price appears to be approximately 0.835 as this is where the blue curve in Figure 4.7 becomes positive.

The four subfigures within Figure 4.8 show the evolution of Group 1's assets over time. Each plot contains five curves which correspond to the five trajectories in Figure 4.4 (color

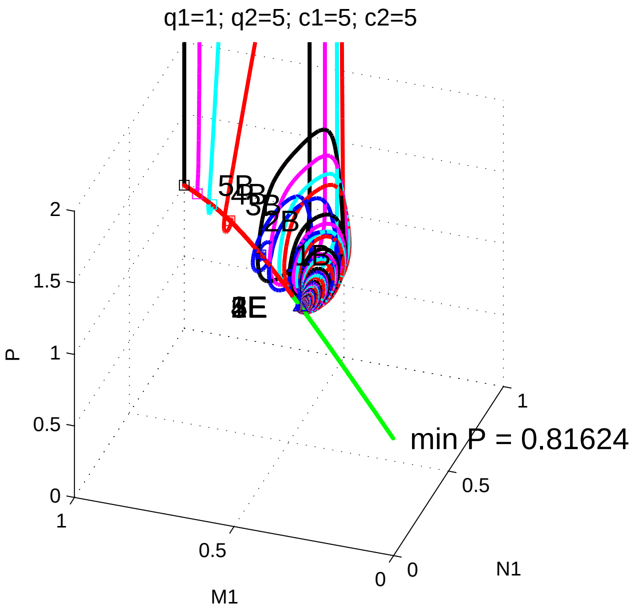


Figure 4.4. *Examples of five trajectories for Case 2 that begin near unstable equilibria. The green curve shows the location of stable equilibria while the red curve shows the location of unstable equilibria. The permissible range of equilibrium prices is* $0.81624 \leq \mathbf{P}_{eq} < 1$.

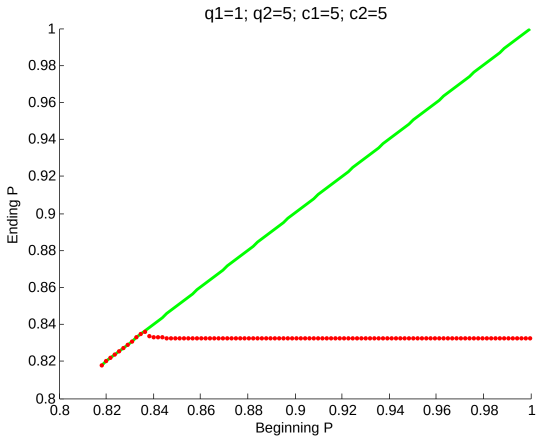


Figure 4.5. *Plot of average trajectory ending price (red dots) versus starting price. In this case the transition price between stable and unstable equilibrium points,* $\mathbf{P}_{eq}^{tr}$*, is approximately 0.835. For* $\mathbf{P}(0) < \mathbf{P}_{eq}^{tr}$ *the trajectories start near stable equilibria and their end points lie along the line* $y = x$ *(denoted in green). For* $\mathbf{P}(0) > \mathbf{P}_{eq}^{tr}$ *the trajectories start near unstable equilibria, and their end points decrease monotonically with* $\mathbf{P}(0)$.

coded). It appears that a larger initial outlay of cash for Group 1 leads to a larger maximum price (consistent with experimental results, see [11]). Also, it seems that as the price increases,

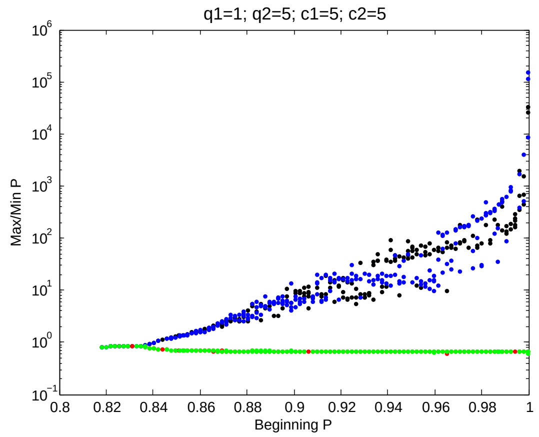


Figure 4.6. *Plot of maximum and minimum prices attained along solution trajectories. The black and red points (often overlapped by green points) correspond to maximum and minimum values, respectively, provided the initial price movement was positive, i.e.* $\mathbf{P}'(0) > 0$*. Similarly, the blue and green points correspond to maximum and minimum values, respectively, provided the initial price movement was negative.*

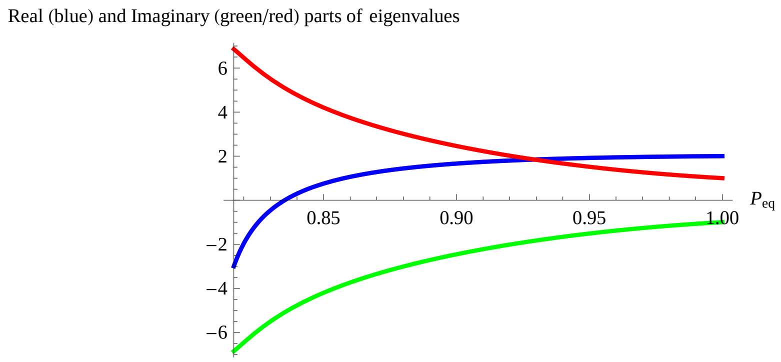
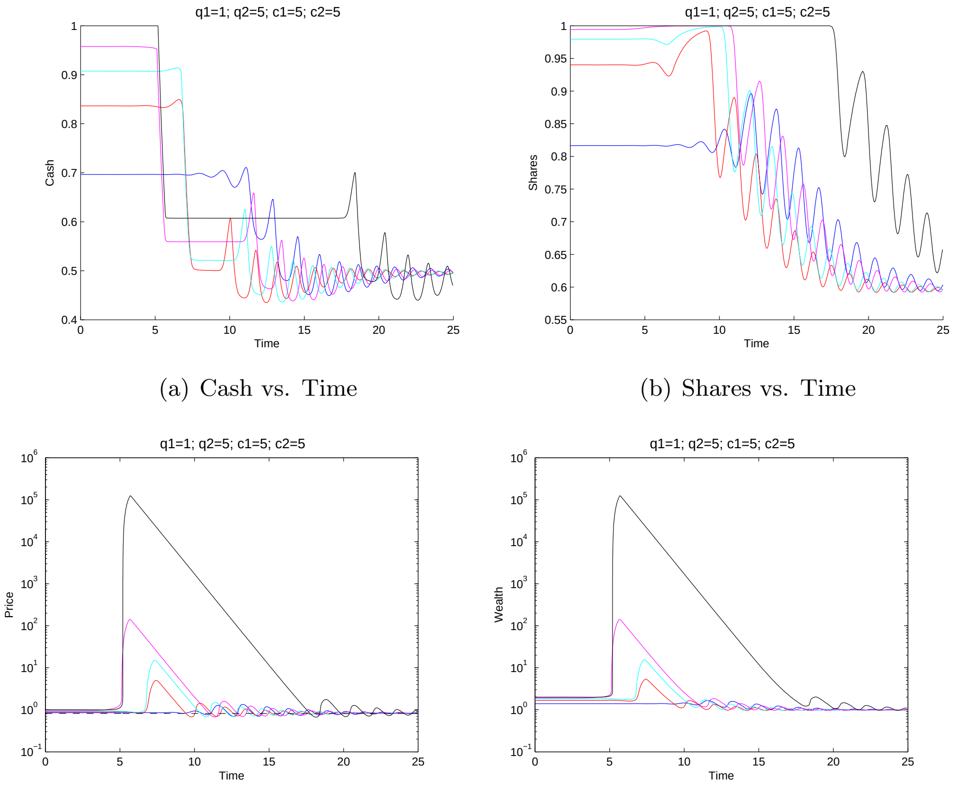


Figure 4.7. *Eigenvalues of the Jacobian of the system in Case 2 versus equilibrium price. The blue curve corresponds to the real parts of the eigenvalues, while the red and green curves correspond to the imaginary parts. Consistent with Figures 4.4, 4.5, and 4.6 it appears the real parts of the eigenvalues become positive at approximately* $\mathbf{P}_{eq} = 0.835$*. As the remaining three eigenvalues are* $\{0, -1, -1\}$*, this indicates stability for* $\mathbf{P}_{eq} < 0.835$ *and instability for* $\mathbf{P}_{eq} > 0.835$*.*

the cash supply of Group 1 decreases while its share level increases. As the price falls, however, both the cash and number of shares of Group 1 decrease in an oscillatory fashion. Thus, while Group 1's wealth is at a maximum near the peak of the price trajectory, it falls as the price decreases to its new equilibrium value.

Case 3. $q_1 = 5$, $q_2 = 5$, $c_1 = c_2 = 5$

Figure 4.9 shows the (red/green) curve of (unstable/stable) equilibrium points for the parameter set: $q_1 = 5$, $q_2 = 5$, and $c_1 = c_2 = 5$. Similar to Figure 4.4 all five trajectories start near unstable equilibrium points and terminate near stable equilibria. The stability



(a) Cash vs. Time (b) Shares vs. Time

(c) Log(Price) vs. Time (d) Log(Wealth) vs. Time

Figure 4.8. *Plots of Group 1's assets and wealth versus time along with the price versus time. Each figure contains five color coded curves corresponding to the five solution trajectories in Figure 4.4. Each subfigure only includes the first 25 time units. After this the trajectory is close to equilibrium.*

transition price, $\mathbf{P}_{eq}^{tr}$, is approximately 0.818.

In Figure 4.10 the curve very briefly rises along the line $y = x$ and then monotonically decreases as the equilibrium points about which the trajectories originate become unstable. Note that most permissible equilibria in this scenario are unstable.

The maximum and minimum prices attained along each trajectory are plotted in Figure 4.11. In this figure the black and red points correspond to maximum and minimum prices attained when the initial price movement was positive, i.e. $\mathbf{P}(2) > \mathbf{P}(1)$. The blue and green points correspond to maximum and minimum prices when the initial price movement was negative. Note the following points:

1. Initially, the maximum and minimum values coincide until one nearly reaches the stability transition price, $\mathbf{P}_{eq}^{tr} = 0.818$. These points correspond to stable equilibria.

2. From $\mathbf{P} = 0.818$ to approximately 0.835 the maximum and minimum values separate, though there is no discernible difference between the black and blue points (both corresponding to maximum values) or the red and green points (corresponding to minimum values).

3. For the initial price greater than 0.835 the maximum values separate into two curves.

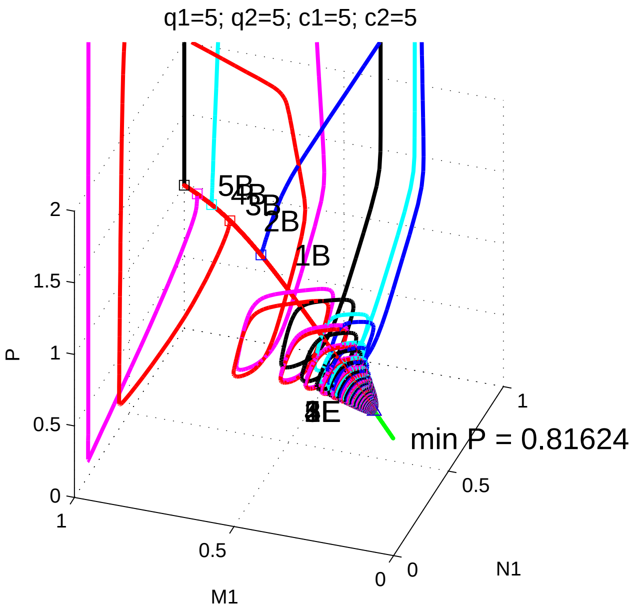


Figure 4.9. *Examples of five trajectories for Case 3 that begin near unstable equilibria. The green curve shows the location of stable equilibria while the red curve shows the location of unstable equilibria. The permissible range of equilibrium prices is* $0.81624 \leq \mathbf{P}_{eq} < 1$.

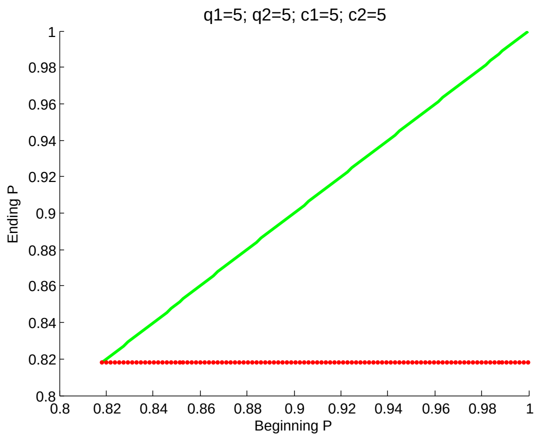


Figure 4.10. *Plot of average trajectory ending price (red dots) versus starting price. Note that in the range shown no trajectories start near a stable equilibrium point. The line* $y = x$ *is denoted in green.*

Similarly, the minimum values also separate. Note that the larger (or higher) maximum and minimum curves correspond to a positive initial price movement. Alternately, the blue and green curves, corresponding to the maximum and minimum values for a negative initial price movement, are lower. Compare this figure with Figure 4.6 in Case 2.

As in the above cases a plot of the pair of complex eigenvalues is presented in Figure 4.12.

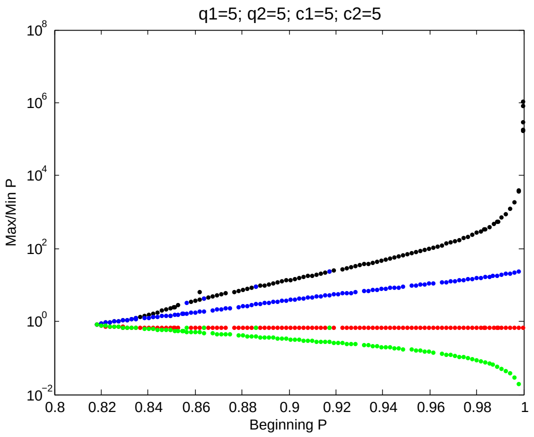


Figure 4.11. *Plot of maximum and minimum prices attained along solution trajectory. The black and red points correspond to maximum and minimum values, respectively, provided the initial price movement was positive, i.e.* $\mathbf{P}'(0) > 0$*. Similarly, the blue and green points correspond to maximum and minimum values, respectively, provided the initial price movement was negative.*

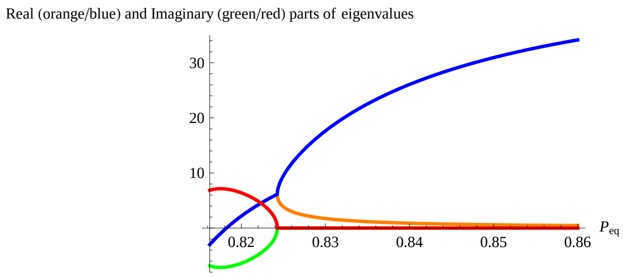
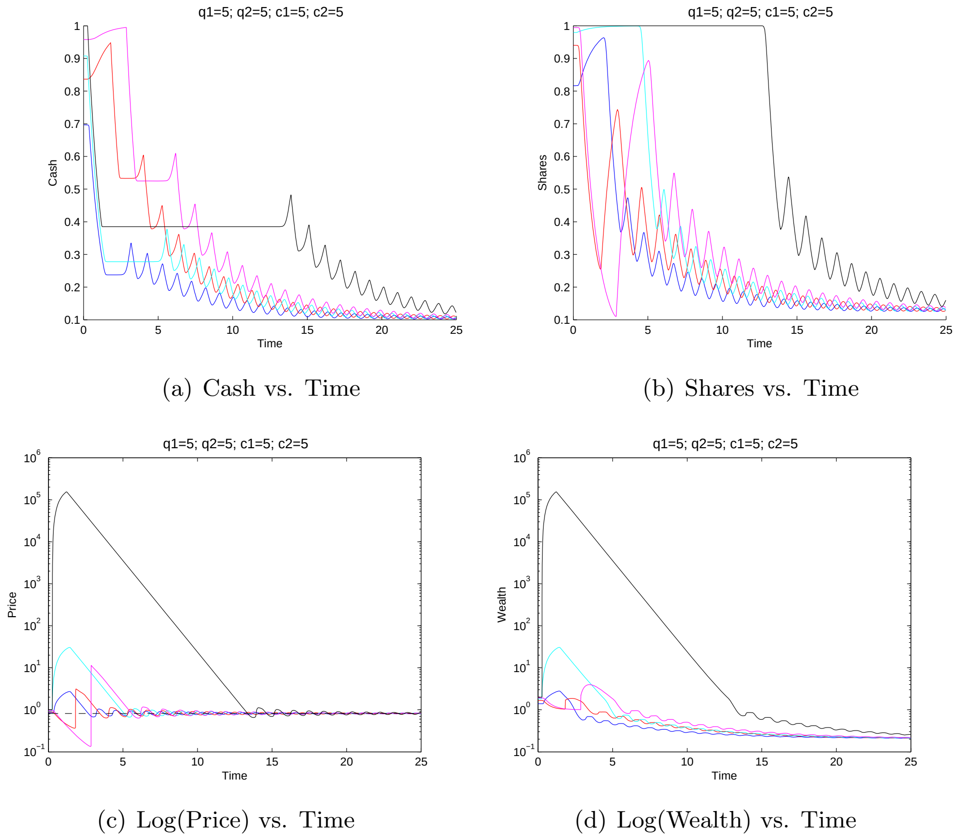


Figure 4.12. *Eigenvalues of the Jacobian of the system in Case 3 versus equilibrium price. The red and green curves correspond to the imaginary parts of the two eigenvalues. Note that as* $\mathbf{P}_{eq}$ *increases through approximately* 0.824 *the imaginary parts become zero leaving a pair of real, positive eigenvalues (blue and orange curves). These curves coincide until approximately* $\mathbf{P}_{eq} = 0.824$*. As the remaining three eigenvalues are* $\{0, -1, -1\}$*, this plots suggests we have stability for* $\mathbf{P}_{eq} < 0.818$*, which is consistent with Figures 4.9, 4.10, and 4.11.*

Similar to Figure 4.8, the four subfigures within Figure 4.13 show the evolution of Group 1's assets over time. Each plot contains five curves which correspond to the five trajectories in Figure 4.9 (color coded). Again, larger initial cash levels for Group 1 correspond to larger maximum prices. Also, as in Case 2, Group 1's wealth is at a maximum near the peak of the price trajectory and falls as the price settles toward a new, lower equilibrium value.

Thus we have observed two distinct dynamical behaviors for the solution trajectories of this system[11].

[11] The three parameter regimes considered in this section were chosen because they illustrated the different dynamical behaviors of the system. Other parameter combinations yield variations of these dynamics. For



(a) Cash vs. Time

(b) Shares vs. Time

(c) Log(Price) vs. Time

(d) Log(Wealth) vs. Time

Figure 4.13. *Plots of Group 1's assets and wealth versus time along with the price versus time. Each figure contains five color coded curves corresponding to the five solution trajectories in Figure 4.9. Each subfigure only includes the first 25 time units. After this the trajectory is close to equilibrium.*

5. Conclusion.

A key difference between our models and those of classical finance involves the nature of equilibrium and associated stability properties. Within our asset flow approach, there is a multi-dimensional manifold of equilibria for each set of valuations, $\mathbf{P}_a^{(i)}$. In particular, Theorem 3.3 shows that given a set of valuations $\mathbf{P}_a^{(i)}$, for any distribution of cash endowments, $\mathbf{M}_{eq}^{(i)}$, there will be a unique equilibrium solution that specifies the price, $\mathbf{P}_{eq}$, and the share distribution, $\mathbf{N}_{eq}^{(i)}$. Thus there exists a curve of equilibria even if all participants agree on valuation. If all participants are fundamental traders, then each equilibrium is stable. Unstable equilibrium is possible with the introduction of momentum traders. This is a central conclusion that underscores the importance of trading motivations.

The differences in formulation between our approach and classical finance include the

example, shortening the timescale of the trend-based trading group (e.g., setting $c_1 = 100$) in Cases 2 and 3 does not produce a new dynamical behavior. Indeed, the parameter combinations (i) $q_1 = 1$, $q_2 = 1$, $c_1 = 100$, $c_2 = 1$,(ii) $q_1 = 1$, $q_2 = 5$, $c_1 = 100$, $c_2 = 5$, and (iii) $q_1 = 5$, $q_2 = 5$, $c_1 = 100$, $c_2 = 5$ result in behaviors similar to Case 3 above, i.e. plots of maximum and minimum prices along solution trajectories and ending price versus beginning price are similar to those attained in Case 3.

finiteness of assets, disparate views on valuation, and distinct strategies implemented by groups of traders. By varying the endowment of the different groups and the parameters governing their strategies, we obtain a spectrum of solution trajectories. As shown in the numerical studies, if the initial conditions correspond to an unstable equilibrium, there will be an excursion that ultimately ends at a stable equilibrium. The computations show the dependence of the excursions on the parameters of the system. It is also clear from the computations and theoretical results that varying the parameters can change a stable equilibrium to an unstable one.

The stability of markets has been of great interest to market participants and government regulators in recent years. The methods of classical finance provide few clues into factors that might de-stabilize markets. Our methodology is capable of addressing these problems. The results can be compared with experimental asset markets (e.g., [11], [19], and [21]) which can test the conclusions. Comparison with world markets would be possible with estimates of the asset endowments and strategies of market participants. Theory, computations and asset market experiments can be instrumental in terms of reducing the amount or type of information needed. The goal would be to confine the information needed on the particular world market to publicly available numbers. For example, for mutual funds and exchange traded funds, the ratio of cash to assets is generally a known quantity, as is the inflow of cash into various sectors of these funds. On the other hand, the motivations of individuals, even in the aggregate, are more difficult to quantify. In principle it would be very useful to estimate the parameters involving motivations, since the inflows of cash are slightly lagging the changes in motivations. In recent months, the "risk appetite" has appeared as a key factor in markets. When there is relative calm in the financial world, investors who are tired of bond dividends below inflation tend to flock to riskier assets such as stocks, oil, etc. Thus the riskier assets have tended to move together even in the absence of correlations in the underlying fundamentals. As sovereign debt issues, the US budget and debt ceiling impasses have surfaced, investors lost their appetite for risk and moved toward relatively risk free assets such as short-term US Treasuries. Thus, the investor sentiment that has changed rapidly with news is manifested in the inflows about a week later.

The rules and regulations for trading in financial markets have become a focus of attention in recent years, partly because of several "flash crashes" where prices fell sharply in the absence of any significant new information. Unfortunately, there is little scientific theory that can assist government and industry leaders in formulating rules that would foster stability. Classical finance does not address this issue, mainly because of the assumption of infinite arbitrage capital. Thus a theory in which assets are finite is crucial to obtaining an understanding of market dynamics in these situations. This leads naturally to a system of differential equations, where a large amount of theory of stability of solutions can be utilized. Consequently, the asset flow differential equation approach can be very useful in terms of understanding the basic issues underlying regulatory concerns and market stability.

Our results show that within many parameter regimes, simply shortening the time scale, $1/c_1$, of the trend investors is sufficient to shift the equilibrium from stable to unstable. How much it must be shortened in order to make this transition depends on the particular point on the equibrium curve, and the distance from the stable point to the unstable. In some cases reducing $1/c_1$ by a factor of five or less is adequate to move the point from stable to unstable.

The resulting excursion (i.e., maximum deviation in price from the starting point) can be quite large. In the high frequency trading that prevails today, the scale $1/c_1$ is generally very small. The reason we do not see even more flash crashes is probably due to the existence of layers of limit orders (by value based managers) that support prices at various levels below the trading prices.

The parameters c_i and q_i can be evaluated using optimization methods [14] so that one can obtain an objective measure of the aggregate or average time scale $1/c_1$ among the trend based investors within a market. One can thus utilize the parameters together with other information in order to determine stability and estimate the anticipated excursions in the case of instablility.

We have used values of parameters that are typical of experiments and the optimization studies that have been performed on particular asset classes. In practical application, for example of the aggregate US market, one can use the data from SPY to optimize and estimate the parameters that will determine the curve of equilibria as well as the stable and unstable points. With this information one can also estimate how much of a change in the time scale would result in a transition from stable to unstable. Furthermore, a shift in investor motivations (e.g. an increased emphasis on the trend), which would be accompanied by a shift in the fraction of assets owned by the momentum group, could also precipitate instability (see [9]). Moreover, one could also approximate the resulting excursion.

We have considered purely deterministic equations as a direct way to understand stability issues. Stochastic issues in finance are often studied through a default equation such as (1.1) in which a constant level of randomness is inserted into price as an empirical equation. Using the asset flow approach one can model randomness through a term in the cash supply, valuation or other variables in the system and determine how these are manifested in terms of the price. A related question is the resulting price distribution that one would obtain as a result. In other words, does a normal distribution in the cash supply, for example, result in a normal distribution in the price? Are there fat tails that are observed in many markets?

Equation (1.1) has been at the heart of modern options theory. Any conclusion that one can obtain through the asset flow equations about the variance in price would be instrumental in obtaining more realistic equations for options theory. Thus, the asset flow equations would yield a variance, $\sigma^2(t)$, that would be used in equation (1.1) as the basis for the derivation of the options equations.

Appendix A. Details for Theorem 3.5.

In this appendix we provide details for the proof of Theorem 3.5 (in which the $k^{(i)}$ are not constants) that were omitted above. Specifically, we show how the matrix A may be transformed into $\tilde{A}$.

The system (2.6)-(2.8) is linearized using the following relations:

$$\left.\frac{\partial \mathbf{F}}{\partial \mathbf{M}^{(i)}}\right|_{eq} = \frac{k^{(i)}}{\beta}$$

$$\left.\frac{\partial \mathbf{F}}{\partial \mathbf{N}^{(i)}}\right|_{eq} = -\frac{\alpha}{\beta^2}\left(1 - k^{(i)}\right)$$

$$\left.\frac{\partial \dot{\mathbf{M}}^{(i)}}{\partial \mathbf{M}^{(j)}}\right|_{eq} = -k^{(i)}\delta_{ij} + v^{(i)}k^{(j)}$$

$$\left.\frac{\partial \dot{\mathbf{M}}^{(i)}}{\partial \mathbf{N}^{(j)}}\right|_{eq} = \frac{\alpha}{\beta}\left(1-k^{(i)}\right)\delta_{ij} - \frac{\alpha}{\beta}v^{(i)}\left(1-k^{(j)}\right)$$

$$\left.\frac{\partial \dot{\mathbf{N}}^{(i)}}{\partial x}\right|_{eq} = -\frac{1}{\mathbf{P}_{eq}}\left.\frac{\partial \dot{\mathbf{M}}^{(i)}}{\partial x}\right|_{eq} = -\frac{\beta}{\alpha}\left.\frac{\partial \dot{\mathbf{M}}^{(i)}}{\partial x}\right|_{eq}$$

and

$$\left.\frac{\partial \dot{\mathbf{P}}}{\partial x}\right|_{eq} = \left.\frac{\partial \mathbf{F}}{\partial x}\right|_{eq}$$

In addition to the relations above we have also

$$\left.\frac{\partial \mathbf{F}}{\partial \zeta^{(i)}}\right|_{eq} = \left.\frac{\partial \mathbf{F}}{\partial k^{(i)}}\right|_{eq}\left.\frac{\partial k^{(i)}}{\partial \zeta^{(i)}}\right|_{eq} = \frac{\mathbf{W}^{(i)}\theta^{(i)}}{\beta},$$

$$\left.\frac{\partial \dot{\mathbf{M}}^{(i)}}{\partial \zeta^{(j)}}\right|_{eq} = \left(\mathbf{W}^{(i)}\delta_{ij} + v^{(i)}\mathbf{W}^{(j)}\right)\theta^{(j)},$$

The matrix A has the form shown in (3.24) where the block matrices have the form

$$A^{(j,j)} = \begin{bmatrix} -k^{(j)}\left(1-v^{(j)}\right) & \frac{\alpha}{\beta}\left(1-k^{(j)}\right)\left(1-v^{(j)}\right) & -\left(1-v^{(j)}\right)\mathbf{W}^{(j)}\theta^{(j)} \\ \frac{\beta}{\alpha}k^{(j)}\left(1-v^{(j)}\right) & -\left(1-k^{(j)}\right)\left(1-v^{(j)}\right) & \frac{\beta}{\alpha}\left(1-v^{(j)}\right)\mathbf{W}^{(j)}\theta^{(j)} \\ \frac{\partial\psi^{(j)}}{\partial\mathbf{F}}\frac{k^{(j)}}{\beta} & -\frac{\partial\psi^{(j)}}{\partial\mathbf{F}}\frac{\alpha}{\beta^2}\left(1-k^{(j)}\right) & \frac{\partial\psi^{(j)}}{\partial\mathbf{F}}\frac{\mathbf{W}^{(j)}\theta^{(j)}}{\beta} + \frac{\partial\psi^{(j)}}{\partial\zeta^{(j)}} \end{bmatrix},$$

$$A^{(i,j)} = \begin{bmatrix} k^{(j)}v^{(i)} & -\frac{\alpha}{\beta}\left(1-k^{(j)}\right)v^{(i)} & v^{(i)}\mathbf{W}^{(j)}\theta^{(j)} \\ -\frac{\beta}{\alpha}k^{(j)}v^{(i)} & \left(1-k^{(j)}\right)v^{(i)} & -\frac{\beta}{\alpha}\mathbf{W}^{(j)}\theta^{(j)}v^{(i)} \\ \frac{\partial\psi^{(i)}}{\partial\mathbf{F}}\frac{k^{(j)}}{\beta} & -\frac{\partial\psi^{(i)}}{\partial\mathbf{F}}\frac{\alpha}{\beta^2}\left(1-k^{(j)}\right) & \frac{\partial\psi^{(i)}}{\partial\mathbf{F}}\frac{\mathbf{W}^{(j)}\theta^{(j)}}{\beta} \end{bmatrix},$$

$$A^{(G+1,j)} = \begin{bmatrix} k^{(j)}/\beta & -\alpha\left(1-k^{(j)}\right)/\beta^2 & \mathbf{W}^{(j)}\theta^{(j)}/\beta \end{bmatrix}, \text{ and}$$

$$A^{(j,G+1)} = \begin{bmatrix} 0 \\ 0 \\ \frac{\partial\psi^{(i)}}{\partial p} \end{bmatrix}.$$

We utilize the same change of coordinates as above, i.e.

$$Q = \begin{bmatrix} Q^{(1)} & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & Q^{(j)} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

with

$$Q^{(j)} = \begin{bmatrix} \beta & \alpha & 0 \\ -\beta k^{(j)} & \alpha\left(1-k^{(j)}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This yields

$$\tilde{A} = \begin{bmatrix} \tilde{A}^{(1,1)} & \dots & \tilde{A}^{(1,j)} & \dots & \tilde{A}^{(1,G+1)} \\ \vdots & & \vdots & & \vdots \\ \tilde{A}^{(j,1)} & \dots & \tilde{A}^{(j,j)} & \dots & \tilde{A}^{(j,G+1)} \\ \vdots & & \vdots & & \vdots \\ \tilde{A}^{(G+1,1)} & \dots & \tilde{A}^{(G+1,j)} & \dots & -1 \end{bmatrix}$$

with

$$\tilde{A}^{(1,1)} = Q^{(1)}A^{(1,1)}(Q^{(1)})^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\left(1-v^{(j)}\right) & \beta\left(1-v^{(j)}\right)\mathbf{W}^{(j)}\theta^{(j)} \\ 0 & -\frac{\partial\psi^{(j)}}{\partial\mathbf{F}}\frac{1}{\beta^2} & \frac{\partial\psi^{(j)}}{\partial\mathbf{F}}\frac{\mathbf{W}^{(j)}\theta^{(j)}}{\beta} + \frac{\partial\psi^{(j)}}{\partial\zeta^{(j)}} \end{bmatrix},$$

$$\tilde{A}^{(i,j)} = Q^{(i)}A^{(i,j)}(Q^{(j)})^{-1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & v^{(i)} & -\beta v^{(i)}\mathbf{W}^{(j)}\theta^{(j)} \\ 0 & -\frac{\partial\psi^{(i)}}{\partial\mathbf{F}}\frac{1}{\beta^2} & \frac{\partial\psi^{(i)}}{\partial\mathbf{F}}\frac{\mathbf{W}^{(j)}\theta^{(j)}}{\beta} \end{bmatrix},$$

$$\tilde{A}^{(G+1,j)} = A^{(G+1,j)}(Q^{(j)})^{-1}\begin{bmatrix} 0 & -1/\beta^2 & \mathbf{W}^{(j)}\theta^{(j)}/\beta \end{bmatrix},$$

and

$$\tilde{A}^{(j,G+1)} = Q^{(j)}A^{(j,G+1)}\begin{bmatrix} 0 \\ 0 \\ \frac{\partial\psi^{(i)}}{\partial\mathbf{P}} \end{bmatrix}.$$

REFERENCES

[1] Z. Bodie, A. Kane, and A.J. Marcus, *Investments*, 7^{th} edition, McGraw-Hill Education, Boston, 2008.

[2] G. Caginalp and D. Balenovich, *Asset flow and momentum: Deterministic and stochastic equations*, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. , 357 (1999), pp. 2119–2133.

[3] G. Caginalp and M. DeSantis, *Stock price dynamics: nonlinear trend, volume, volatility, resistance and money supply*, Quant. Finance, 11 (2011), pp. 849-861.

[4] —, *Nonlinearity in the dynamics of financial markets*, Nonlinear Anal. Real World Appl., 12 (2011), pp. 1140-1151.

[5] —, *Multi-group asset flow equations and stability*, Discrete Contin. Dyn. Syst. Ser. B, 16 (2011), pp. 109-150.

[6] G. Caginalp, M. DeSantis and D. Swigon, *Are flash crashes caused by instabilities arising from rapid trading?*, Wilmott magazine, July (2011), pp. 46-47.

[7] G. Caginalp and B. Ermentrout, *A kinetic thermodynamics approach to the psychology of fluctuations in financial markets*, Appl. Math. Lett. 3 (1990), pp. 17-19.

[8] G. Caginalp and V. Ilieva, *Hybrid Methodology for Technical Analysis*, Nonlinear Anal. Hybrid Syst., 2 (2008), pp. 1144-51.

[9] —, *The dynamics of trader motivations in asset bubbles*, Journal of Economic Behavior & Organization, 66 (2008), pp. 641–656.

[10] G. Caginalp and H. Merdan, *Asset price dynamics with heterogeneous groups*, Phys. D, 225 (2007), pp. 43–54.
[11] G. Caginalp, D. Porter and V.L. Smith, *Initial cash/asset ratio and asset prices: an experimental study*, Proc. Natl. Acad. Sci. USA, 95 (1998), pp. 756-761.
[12] M. DeSantis, *Dynamics of asset price changes: Statistical and differential equations models*, (Ph.D. thesis, Department of Mathematics, University of Pittsburgh, Pittsburgh, PA, 2011).
[13] R.C. Dorf and R.H. Bishop, *Modern Control Systems*, 11^{th} edition, Pearson Prentice-Hall, Upper Saddle River, NJ, 2008.
[14] A. Duran and G. Caginalp, *Parameter optimization for differential equations in asset price forecasting*, Optim. Methods Softw., 23 (2008), pp. 551-574.
[15] D. Grether, *Bayes rule as a descriptive model: The representativeness heuristic*, Quarterly Journal of Economics, 95 (1980), pp. 537-557.
[16] J.M. Henderson and R.E. Quandt, *Microeconomic theory, a mathematical approach*, 3^{rd} edition, McGraw-Hill, 1980.
[17] C. Jeffries, *Qualitative stability and digraphs in model ecosystems*, Ecology, 55 (1974), pp. 1415-1419
[18] R. May, *Stability and Complexity in Model Ecosystems*, Princeton University Press, Princeton, NJ, 1973.
[19] D. Porter and V.L. Smith, *Stock market bubbles in the laboratory*, Appl. Math. Finance, 1 (1994), pp. 111-28.
[20] J. Quirk and R. Rupert, *Qualitative economics and the stability of equilibrium*, Rev. Econ. Stud., 32 (1965), pp. 311-326.
[21] V.L. Smith, G. L. Suchanek, and A. W. Williams, *Bubbles, crashes and endogenous expectations in experimental spot asset markets*, Econometrica, 56 (1988), pp. 1119-1151.
[22] D.S. Watson and M. Getz, *Price Theory and Its Uses*, 5^{th} edition, University Press of America, Lanham, MD, 1993.