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On the Dynamics of Flexure and Stretch in the Theory of Elastic Rods

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1. Introduction

We here examine a theory of planar motions of naturally straight elastic rods intended for cases in which the strains relative to an undistorted configuration $\mathfrak{C}^{\mathrm{u}}$, are small, although displacements and rotations may be large. The dynamical equations of the theory, which are complete to within an error of order two in a dimensionless measure of thickness, curvature, and extension, can be derived from Hamilton's principle by assuming that in appropriate units ψ and k, the densities of free energy and kinetic per unit of length in $\mathfrak{C}^{\mathrm{u}}$, have the form:

$$2\psi = (\theta_S)^2 + \varepsilon^2, \tag{1.1}$$

$$2k = (\theta_t)^2 + (x_t)^2 + (y_t)^2. \tag{1.2}$$

Subscripts denote derivatives; $x = x(S, t)$ and $y = y(S, t)$ are Cartesian coordinates at time t of the material point on the rod axis that has arc-length coordinate S when the rod is in the configuration $\mathfrak{C}^{\mathrm{u}}$. We write $s = s(S, t)$ for the arc-length coordinate at t; $\varepsilon = \varepsilon(S, t) = \partial s/\partial S - 1$ is the elongational strain; and $\theta = \theta(S, t)$ is the angle that the tangent to the axial curve makes with the x-axis.

The dynamical equations corresponding to (1.1) and (1.2) can be written

$$((1+\varepsilon)\cos\theta)_{tt} = (T\cos\theta - G\sin\theta)_{SS}, \tag{1.3a}$$

$$((1+\varepsilon)\sin\theta)_{tt} = (T\sin\theta + G\cos\theta)_{SS}, \tag{1.3b}$$

$$\theta_{tt} - \theta_{SS} = (1+\varepsilon)G, \tag{1.3c}$$

where $G = G(S,t)$ is the resultant of the shearing forces and $T = T(S,t)$ the resultant of the normal forces, on the surface $\mathscr{S}(S,t)$ formed at time t by the material points that are on the cross section that has arc-length coordinate S in $\mathfrak{C}^{\mathrm{u}}$. The shearing force G is a reactive parameter not given by constitutive relations; the tension T obeys a familiar constitutive equation that in the units we are using takes the form

$$T = \varepsilon. \tag{1.3d}$$

Thus, the governing equations of the present theory can be written as a system of three equations in three unknowns: θ, ε, G. The theory is the specialization to planar motions of the Kirchhoff-Clebsch theory of rods [1859, 1], [1862, 1], [1876, 1], [1883, 1] without the usual assumption of inextensibility, i.e., of temporal invariance of arc length.

In Sections 2 and 3 below we discuss the origin of the present theory as an approximation to the theory of nonlinearly elastic slender bodies, we present various alternative formulations of the theory, and we give a list of conservation laws implied by the system (1.3).

The remainder of the paper is devoted to the theory of traveling waves. The present study of such waves, like that presented in [1992, 1], was much influenced by the very general investigation of ANTMAN & LIU [1979, 1]. Here, as in [1992, 1], the emphasis is laid on explicit solutions. In Sections 5, 6, and 7 we derive expressions, in terms of elliptic functions and integrals, for the axial curve in each type of traveling-wave solution of (1.3): solitary, inflexional, and noninflexional. In the Appendix we consider a limiting process in which the magnitude of tension decreases to zero, and we discuss the manner in which traveling-wave solutions of (1.3) approach solutions of the system of equations,

$$(\cos\theta)_{tt} = (T\cos\theta - G\sin\theta)_{SS}, \tag{1.4a}$$

$$(\sin\theta)_{tt} = (T\sin\theta + G\cos\theta)_{SS}, \tag{1.4b}$$

$$\theta_{tt} - \theta_{SS} = G, \tag{1.4c}$$

governing planar motions in the Kirchhoff-Clebsch theory when inextensibility is assumed.

In the system (1.4), which is the subject of a growing literature (cf. [1984, 1], [1992, 1, 3], [1994, 1–5]), both the tension T and the resultant shear force G are reactive parameters. The system can be obtained from (1.1), (1.2), and Hamilton's principle by imposing, at each S and t, the constraint $\varepsilon = 0$, which can be written:

$$(x_S)^2 + (y_S)^2 = 1. \tag{1.5}$$

Such a variational formulation of (1.4) and the theory of traveling waves governed by that system are discussed in detail in [1992, 1].

If we were using conventional units, such as centimeters, grams, and seconds, instead of the present dimensionless units, the characteristic speed of unity exhibited by (1.3c) and (1.4c) would appear as $\sqrt{E/\rho}$ with E the tensile modulus and ρ the mass density. Both (1.3) and (1.4) have traveling-wave solutions with wave speeds greater than this characteristic speed. When such solutions were found in the theory of (1.4), although they were difficult to interpret physically, there was not available an argument for their out-of-hand rejection. Here, in Section 4, we show that for each such "supersonic" traveling-wave solution of (1.3) there are places at which $\varepsilon < -1$. It follows that if we are granted the principle that a smooth solution of the dynamical equations of a theory of extensible rods is physically meaningful only if it is free from a change in the ordering of the material points on the rod axis, i.e., only if it is free from intervals of values of S on which $\partial s/\partial S < 0$, then we may reject as inadmissible the supersonic traveling-wave solutions of (1.3). (At places in a rod where $\varepsilon = -1$, the mass density per unit length is infinite; hence in the discussion below we shall use a strengthened form of the principle just stated, and we shall assume that a solution in which ε attains the value -1 should be rejected, even if in it ε does not drop below -1.)

We remark here in passing that in the present study we consider only briefly traveling-wave solutions of (1.3) with characteristic speed; for, as is observed in Section 4, for such waves θ is constant and ε obeys the familiar linear wave equation, $\varepsilon_{tt} = \varepsilon_{SS}$.

The system (1.3) is a refinement of (1.4) that takes into account effects of axial elongation and compression. If we are granted, in addition to the principle asserted above, that each physically meaningful solution of the system (1.4) of equations for inextensible rods should be interpretable as a limit for small tension of solutions of (1.3), then results given in Section 4 and in the Appendix tell us that not only for (1.3), but also for (1.4), we can reject supersonic traveling waves as being without physical meaning.

2. Dynamical Equations for Planar Motions of Rods

We are concerned here with the nonlinear dynamical equations governing planar, twist-free, motions of elastic rods in the limit of small strain, but possibly large displacement. We assume the rods to be homogeneous, naturally straight, and subject to external loads only at their end points. When such a rod, $\mathscr{R}$, is in an undistorted, stress-free, configuration, $\mathfrak{C}^{\mathrm{u}}$, it has the form of a solid cylinder with not necessarily circular cross sections, $\mathscr{S}^{\mathrm{u}}$. We write A and h for the area and diameter of the sets $\mathscr{S}^{\mathrm{u}}$. The locus $\mathscr{C}^{\mathrm{u}}$ of the centroids of the sets $\mathscr{S}^{\mathrm{u}}$ is a straight line or a line segment of length L. In discussions of traveling waves, $\mathscr{C}^{\mathrm{u}}$ is a full line and hence L is infinite; in general, whenever a three-dimensional body $\mathscr{R}$ is called a *rod*, it is presupposed that $h/L \ll 1$. The set $\mathscr{C}^{\mathrm{u}}$ is parametrized with S, the distance from a point on $\mathscr{C}^{\mathrm{u}}$ to a preassigned point, $\mathbf{o}$, of $\mathscr{C}^{\mathrm{u}}$. Thus, for each S in an interval $\mathscr{I}$ of length L, $\boldsymbol{x}^{\mathrm{u}}(S)$ is a point on $\mathscr{C}^{\mathrm{u}}$, and $\mathscr{S}^{\mathrm{u}}(S)$ is the (undistorted) cross section that has $\boldsymbol{x}^{\mathrm{u}}(S)$ as its centroid. For simplicity of exposition, we assume that $\boldsymbol{x}^{\mathrm{u}}(S)$ is in $\mathscr{S}^{\mathrm{u}}(S)$.

At each time t in a motion of $\mathscr{R}$, $\mathscr{C}^{\mathrm{u}}$ is taken into a curve $\mathscr{C}(t)$, called the *axial curve at time t*. The point $\boldsymbol{x}(S,t)$ of $\mathscr{C}(t)$ is the present location in space of the material point of $\mathscr{R}$ that is at $\boldsymbol{x}^{\mathrm{u}}(S)$ in $\mathfrak{C}^{\mathrm{u}}$.

For the motions we consider, $\mathscr{C}(t)$ lies in a fixed plane $\mathscr{P}$ which we coordinatize with a Cartesian system x, y, with natural basis, $\boldsymbol{i}, \boldsymbol{j}$. In order to have a broad class of such planar motions possible, we assume that each cross section $\mathscr{S}^{\mathrm{u}}$ has a line of symmetry which we take to lie in $\mathscr{P}$, and we write I for the moment of inertia of a $\mathscr{S}^{\mathrm{u}}$ about its principal axis normal to $\mathscr{P}$. We denote by $\boldsymbol{F} = \boldsymbol{F}(S,t)$ the resultant of the stress vector field acting at time t on $\mathscr{S}(S,t)$, the surface formed at that time t by the material points that are on $\mathscr{S}^{\mathrm{u}}(S)$ in the configuration $\mathfrak{C}^{\mathrm{u}}$.

As the notation $\boldsymbol{x} = \boldsymbol{x}(S,t)$, or equivalently,

$$x = x(S,t), \quad y = y(S,t), \tag{2.1}$$

indicates, we generally parametrize the curve $\mathscr{C}(t)$ not with its own arc length s, but with the parameter S that gives, for each point on $\mathscr{C}(t)$, the distance it would be from the point $\mathbf{o}$ on $\mathscr{C}^{\mathrm{u}}$ if $\mathscr{R}$ were deformed back to its straight, undistorted, configuration $\mathfrak{C}^{\mathrm{u}}$. One may call S the *material coordinate* or *arc-length coordinate in* $\mathfrak{C}^{\mathrm{u}}$ and call s the *arc-length coordinate at time t*. For the *stretch* $\lambda = \lambda(S,t) = \partial s(S,t)/\partial S$ and the *tangent angle*, i.e., the counterclockwise angle $\theta = \theta(S,t)$ from the x-axis to the vector tangent to $\mathscr{C}(t)$ at the material coordinate S, we have

$$x_S = \lambda\cos\theta, \quad y_S = \lambda\sin\theta \tag{2.2}$$

where $x_S = \partial x(S,t)/\partial S$, etc. Let $\boldsymbol{i}$ and $\boldsymbol{j}$ be unit vectors along the coordinate axes, and let $\boldsymbol{t}$ and $\boldsymbol{n}$ be the unit tangent and normal vectors of $\mathscr{C}(t)$:

$$\boldsymbol{t} = \boldsymbol{i}\cos\theta + \boldsymbol{j}\sin\theta, \quad \boldsymbol{n} = -\boldsymbol{i}\sin\theta + \boldsymbol{j}\cos\theta. \tag{2.3}$$

For the components of $\boldsymbol{F}$ relative to the bases $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{t}, \boldsymbol{n}$, we use the notation

$$\boldsymbol{F} = F^x\boldsymbol{i} + F^y\boldsymbol{j} = T\boldsymbol{t} + G\boldsymbol{n}; \tag{2.4}$$

G is called the *shear resultant* and T the axial force or *tension*. For the magnitude of the resultant moment of the stress field acting on $\mathscr{S}(S,t)$, i.e., for the *bending moment*, we write $M = M(S,t)$.

The rod $\mathscr{R}$ is assumed to be composed of a homogeneous and isotropic and, in general, nonlinear elastic material. We write ρ for the mass density of this material in the undistorted, stress-free, configuration $\mathfrak{C}^{\mathrm{u}}$, and E for the tensile modulus (i.e., Young's modulus) for small deformations from the configuration $\mathfrak{C}^{\mathrm{u}}$.

For each configuration of $\mathscr{R}$, we may define a number δ by

$$\delta = \sup_S \max\{h/L, |\theta_S|h, |\varepsilon|\}, \tag{2.5}$$

where $\varepsilon = \lambda - 1$ is the *elongational strain* at the axis of $\mathscr{R}$. For the planar twist-free motions under consideration, a more general analysis given by KIRCHHOFF [1859, 1], [1876, 1] and CLEBSCH [1862, 1], [1883, 1] and recently discussed from a modern point of view by DILL [1992, 2] (see also [1992, 1]), here yields the conclusion that the dynamical equations expressing the laws of balance linear and angular momentum for the three-dimensional body $\mathscr{R}$ give, to within an error $O(\delta^2)$,

$$\rho A \boldsymbol{x}_{tt} = \boldsymbol{F}_S, \tag{2.6}$$

$$\rho I \theta_{tt} = M_S + \lambda G, \tag{2.7}$$

where, in (2.6), for $T = \boldsymbol{F} \cdot \boldsymbol{t}$ one has

$$T = EA\varepsilon \tag{2.8}$$

and, in (2.7), for M,

$$M = EI\theta_S. \tag{2.9}$$

By differentiating (2.6) with respect to S and making use of (2.8) and (2.9), we obtain the following system of three partial differential equations for θ, λ, and G as functions of S and t:

$$\rho A(\lambda \cos\theta)_{tt} = (EA[\lambda - 1]\cos\theta - G\sin\theta)_{SS}, \tag{2.10a}$$

$$\rho A(\lambda \sin\theta)_{tt} = (EA[\lambda - 1]\sin\theta + G\cos\theta)_{SS}, \tag{2.10b}$$

$$\rho I\theta_{tt} - EI\theta_{SS} = \lambda G. \tag{2.10c}$$

The shear resultant G, whose time-derivative does not occur in the system (2.10), is a reactive parameter not related by a constitutive equation to kinematical variables but rather to be found by solving the dynamical equations for appropriate boundary data.

Since we restrict our attention to motions in which the local stretch ratio λ never attains the value zero, we may use equation (2.10c) to eliminate G from (2.10a) and (2.10b) and thus obtain a system of two equations in two unknown fields, one of which, $\theta = \theta(S, t)$, through its S-derivative, characterizes the flexure of the rod, and the other, $\lambda = \lambda(S, t)$, gives the stretch; that new system is of order two in t-derivatives and of order four in S-derivatives.

As $(I\rho/AE)^{1/2}$ has the dimensions of time, AE the dimension of force, and $(I/A)^{1/2}$ the dimension of length, we now replace t by $(I\rho/AE)^{1/2}\, t$, $\boldsymbol{F}$ by $AE\,\boldsymbol{F}$, and the list $(\boldsymbol{x}, S, s)$ by $(I/A)^{1/2}$ $(\boldsymbol{x}, S, s)$. This replacement leaves λ invariant and induces a change of (T, G) into $AE(T, G)$. In terms of the dimensionless variables so obtained, the system (2.10) reduces to (1.3), i.e.,

$$(\lambda \cos\theta)_{tt} = ([\lambda - 1]\cos\theta - G\sin\theta)_{SS}, \tag{2.11a}$$

$$(\lambda \sin\theta)_{tt} = ([\lambda - 1]\sin\theta + G\cos\theta)_{SS}, \tag{2.11b}$$

$$\theta_{tt} - \theta_{SS} = \lambda G. \tag{2.11c}$$

The first two of these equations, (2.11a) and (2.11b), are equivalent to the vector equation,

$$\boldsymbol{x}_{tt} = \boldsymbol{F}_S, \tag{2.12}$$

whose components in terms of moving basis $\boldsymbol{t}$, $\boldsymbol{n}$ are [1992, 1]

$$x_{tt} \cos\theta + y_{tt} \sin\theta = T_S - G\theta_S, \tag{2.13a}$$

$$y_{tt} \cos\theta - x_{tt} \sin\theta = G_S + T\theta_S, \tag{2.13b}$$

where, in the present dimensionless units,

$$T = \lambda - 1 = \varepsilon. \tag{2.14}$$

3. Conservation Laws and Variational Principles

As is common practice, a relation of the form,

$$\frac{\partial}{\partial t} h(S, t) = \frac{\partial}{\partial S} g(S, t), \tag{3.1}$$

is here called a *conservation law* with h a conserved quantity and $-g$ the associated flux.

Equation (2.12) obviously is a conservation law, for it can be written,

$$\frac{\partial}{\partial t} x_t = \frac{\partial}{\partial S} F^x, \quad \frac{\partial}{\partial t} y_t = \frac{\partial}{\partial S} F^y, \tag{3.2}$$

where (in our dimensionless units) x_t and y_t are the x- and y-components of the density (with respect to S) of linear momentum. It is easily verified that, given equation (2.12), equation (2.11c) is equivalent the conservation law,

$$\frac{\partial}{\partial t} [\theta_t + x y_t - y x_t] = \frac{\partial}{\partial S} [\theta_S + x F^y - y F^x]. \tag{3.3}$$

In the forms (3.2) and (3.3) of balance of linear and angular momentum, F^x and F^y are functions of θ, λ, and G; for by (2.3), (2.4):

$$F^x = T\cos\theta - G\sin\theta, \quad F^y = T\sin\theta + G\cos\theta. \tag{3.4}$$

As equation (2.9) for the bending moment here reduces to $M = \theta_S$, and equation (2.8) reduces to (2.14), we may consider the quantity

$$\psi = \tfrac{1}{2}[(\theta_S)^2 + (\lambda - 1)^2] \tag{3.5}$$

to be a density of free energy. Similarly,

$$k = \tfrac{1}{2}[(\theta_t)^2 + \boldsymbol{x}_t \cdot \boldsymbol{x}_t] \tag{3.6}$$

is a density of kinetic energy, and $w = M\theta_t + F^x x_t + F^y y_t$, i.e.,

$$w = \theta_S \theta_t + F^x x_t + F^y y_t, \tag{3.7}$$

is a density for the rate of working of resultant forces and moments. Indeed, by taking the inner product of equation (2.12) with $\boldsymbol{x}_t$ and multiplying equation (2.11c) by θ_t, one easily verifies that those equations imply the conservation law

$$\frac{\partial}{\partial t}[\psi + k] = \frac{\partial w}{\partial S}, \tag{3.8}$$

which is the expected form of the equation of balance of energy.

One further conservation law may be obtained by taking the inner product of equation (2.12) with $\boldsymbol{x}_S = \lambda \boldsymbol{t}$, multiplying equation (2.11c) by θ_S, and following a calculation first carried out for planar motions of inextensible rods by DICHMANN, MADDOCKS & PEGO [1994, 3], for motions of such rods with torsion and twist by COLEMAN, DILL, LEMBO, LU & TOBIAS [1993, 1], and in a very general context by MADDOCKS & DICHMANN [1994, 5]. This new conservation law here takes the form

$$\frac{\partial}{\partial t}[\theta_t\theta_S + \boldsymbol{x}_t\cdot\boldsymbol{x}_S] = \frac{\partial}{\partial S}[\tfrac{1}{2}((\theta_S)^2 + \boldsymbol{x}_S\cdot\boldsymbol{x}_S + (\theta_t)^2 + \boldsymbol{x}_t\cdot\boldsymbol{x}_t)] \tag{3.9}$$

in which, of course, $\boldsymbol{x}_S\cdot\boldsymbol{x}_S = \lambda^2$. For appropriate boundary conditions, equation (3.9) implies that the quantity

$$\int [\theta_t\theta_S + x_t x_S + y_t y_S]\, dS,$$

called the *linear impulse*, is constant in time. We note that the density whose S-derivative appears on the right in (3.9) exceeds the energy density, $\psi + k$, by an amount, $(2\lambda - 1)/2$, whose S-derivative, $\lambda_S = \varepsilon_S$, is zero in the limiting case of an inextensible rod.

Hamilton's principle here becomes the assertion that for each quadruple (S_1, S_2, t_1, t_2) with $S_1 < S_2$ and $t_1 < t_2$, $\delta\mathscr{A}$, the first variation of

$$\mathscr{A} = \int_{t_1}^{t_2} [K - \Psi]\, dt \tag{3.10}$$

must vanish for all smooth variations $\delta\boldsymbol{x}$ of the basic kinetic field $\boldsymbol{x} = \boldsymbol{x}(S, t)$ for which $\delta\boldsymbol{x} = \boldsymbol{0}$ at $t = t_1$ and $t = t_2$. Here

$$K = K(S_1, S_2, t) = \int_{S_1}^{S_2} k\, dS, \tag{3.11a}$$

$$\Psi = \Psi(S_1, S_2, t) = \int_{S_1}^{S_2} \psi\, dS - [\bar{M}\theta + \bar{\boldsymbol{F}}\cdot\boldsymbol{x}]_{S_1}^{S_2}; \tag{3.11b}$$

ψ and k are as in (3.5) and (3.6); $\bar{M}(S_i, t)$ and $\bar{\boldsymbol{F}}(S_i, t)$ are the bending moment and force applied to the cross section $\mathscr{S}^u(S_i)$, $i = 1, 2$. An argument similar to that given in detail for inextensible rods in [1992, 1] here tells us that if we put $\lambda = [\boldsymbol{x}_S\cdot\boldsymbol{x}_S]^{1/2}$, i.e., employ the relations (2.2) relating $\theta(S, t)$ to $\boldsymbol{x}(S, t)$, and *define* $G = G(S, t)$, $T = T(S, t)$ [and hence $\boldsymbol{F}$ in equation (2.4)] by the relations

$$\lambda G = \theta_{tt} - \theta_{SS}, \quad T = \lambda - 1, \tag{3.12}$$

then Hamilton's principle is satisfied if and only if, for all S_1, S_2 with $S_1 < S_2$ and all t, the equation,

$$\boldsymbol{x}_{tt} = \boldsymbol{F}_S \tag{3.13}$$

holds on the interval (S_1, S_2), while at S_1 and S_2

$$\bar{M} = \theta_S, \quad \boldsymbol{F} = \bar{\boldsymbol{F}}. \tag{3.14}$$

Thus, the equations (2.11c) and (2.12), together, are *equivalent* to Hamilton's principle with the action $\mathscr{A}$ given by (3.10) and (3.11) with k and ψ as in (3.6) and (3.5).

We emphasize this equivalence of the differential equations of the present theory to a natural form of Hamilton's principle for the following reason. One may conjecture that solutions of the full equations of elasticity for $\mathscr{R}$, as a three-dimensional body, are such that G is $\mathrm{O}(\delta)$ and hence that the term λG in equation (2.11c) differs by terms $\mathrm{O}(\delta^2)$ from G itself. If this conjecture be granted, one may consider replacing equation (2.11c) by the equation

$$\theta_{tt} - \theta_{SS} = G, \tag{2.11c)*}$$

familiar in the theory of inextensible rods, albeit we here employ the relation, $T = \lambda - 1$, as a constitutive equation for the tension T, to obtain the form of (2.12) shown in (2.11a), (2.11b). However, the system (2.11a), (2.11b), (2.11c)* does not appear to have a natural variational formulation; nor does it give rise to a conservation law for energy in which ψ has the expected form (3.5) or an acceptable modification of that form.

4. General Theory of Traveling Waves

A solution of the dynamical equations (2.11a, b, c) or, equivalently, of (2.12) and (2.11c), is called a *traveling wave* if for it there is a Cartesian coordinate system on $\mathscr{P}$ for which y and $u = x - S$ are functions of $\xi = S - ct$ with c a constant called the *wave velocity in the material description.* In that coordinate system,

$$y_S(S - ct) = dy(\xi)/d\xi = y'(\xi), \quad y_t(S - ct) = -cy'(\xi), \quad x_t(S, t) = -cu'(\xi), \text{ etc.;}$$

by (2.2), $\lambda(S, t) = \lambda(\xi)$, $\theta(S, t) = \theta(\xi)$; by (2.11c) and (2.14), $G(S, t) = G(\xi)$, $T(S, t) = T(\xi)$; and (2.13a) and (2.13b) reduce to

$$c^2[u'' \cos\theta + y'' \sin\theta] = \lambda' - G\theta', \tag{4.1a}$$

$$c^2[y'' \cos\theta - u'' \sin\theta] = G' + (\lambda - 1)\theta'. \tag{4.1b}$$

As we here have $u' + 1 = \lambda\cos\theta$ and $y' = \lambda\sin\theta$, equations (4.1a, b) and (2.11c) yield the following system of three ordinary differential equations for λ, θ, and G:

$$(1 - c^2)\lambda' = G\theta', \tag{4.2a}$$

$$G' = [1 - \lambda(1 - c^2)]\theta', \tag{4.2b}$$

$$(1 - c^2)\theta'' = -\lambda G. \tag{4.2c}$$

When $|c| = 1$, these last equations become $G\theta' = 0$, $G' = \theta'$, $\lambda G = 0$, and, as λ is nowhere zero, G and θ' vanish everywhere, which implies that (2.11a) and (2.11b) both reduce to $\lambda_{tt} = \lambda_{SS}$. Thus, we make the following observation.

Remark 1. When, in our dimensionless units, $c = \pm 1$ [or, in the units of equations (2.8)–(2.10), $c = \pm\sqrt{E/\rho}$], the only traveling waves are such that the shear resultant G vanishes and the rod remains straight (with θ a constant which we may take to be zero); for such waves both the tensile strain ε and the displacement $u = x - S$ obey the linear wave equation, i.e., $u_{tt} = u_{SS}$ [or $\rho u_{tt} = Eu_{SS}$], which governs longitudinal waves in the linear theory of elasticity.

As waves with $c^2 = 1$ show no flexure and obey a familiar and highly developed theory, we now confine our attention to traveling waves with $c^2 \neq 1$.

In general $\boldsymbol{x}_S = \lambda\boldsymbol{t}$, and, for traveling waves, $\boldsymbol{x}_{tt} = c^2\boldsymbol{x}_{SS}$ and $\boldsymbol{F}' = \boldsymbol{F}_S$. Therefore, equation (2.12) here becomes $c^2(\lambda\boldsymbol{t})' = \boldsymbol{F}'$, and we have $\boldsymbol{F} = c^2\lambda\boldsymbol{t} + \boldsymbol{v}$ with $\boldsymbol{v}$ a constant vector. We assume $\boldsymbol{v} \neq \boldsymbol{0}$, because it is not difficult to show that in a solution of the system (4.2) with $\boldsymbol{v} = \boldsymbol{0}$ the angle θ either is a constant or is linear in $\xi = s - ct$; in the former case the rod is straight; in the latter its axis is a (degenerate) helix flattened onto the (x, y)-plane in such a way that it lies on a circle and rotates about the center of that circle.

Let $\Omega = \cos^{-1}(\boldsymbol{v}\cdot\boldsymbol{i}/v)$ with $v = |\boldsymbol{v}|$. From this point on we consider traveling waves with $\Omega = 0$, i.e., for which $\boldsymbol{v} = v\boldsymbol{i}$. This is justified because the values, y_Ω, u_Ω, G_Ω, T_Ω, λ_Ω, θ_Ω, of y, u, G, T, λ, θ in a traveling wave with $\Omega \neq 0$ can be obtained from the values, y_0, u_0, G_0, T_0, θ_0, of these quantities in a traveling wave with the same velocity c and the same magnitude v of $\boldsymbol{v}$, but with $\Omega = 0$ (in the same Cartesian system), by use of the formulae:

$$y_\Omega(\xi) = (u_0(\xi) + \xi)\sin\Omega + y_0(\xi)\cos\Omega, \tag{4.3a}$$

$$u_\Omega(\xi) = (u_0(\xi) + \xi)\cos\Omega - y_0(\xi)\sin\Omega - \xi, \tag{4.3b}$$

$$G_\Omega(\theta_\Omega) = G_0(\theta_0), \tag{4.3c}$$

$$T_\Omega(\theta_\Omega) = T_0(\theta_0), \tag{4.3d}$$

$$\lambda_\Omega(\theta_\Omega) = \lambda_0(\theta_0), \tag{4.3e}$$

$$\theta_\Omega(\xi) = \theta_0(\xi) + \Omega. \tag{4.3f}$$

It is not difficult to verify that a derivation of the relations (4.3) can be constructed by extending the analysis given below for the case $\Omega = 0$ to the case of $\Omega \neq 0$ and comparing the generalization so obtained with the results we give for $\Omega = 0$.

For $\Omega = 0$ the first integral of the equation of balance of linear momentum here takes the form

$$\boldsymbol{F} = c^2\lambda\boldsymbol{t} + \beta^2\boldsymbol{i}, \quad \beta = \sqrt{v} > 0. \tag{4.4}$$

The parameter β plays an important role in our theory. By (2.4) and (4.4),

$$T = c^2\lambda + \beta^2\cos\theta, \tag{4.5a}$$

$$G = -\beta^2\sin\theta. \tag{4.5b}$$

As $T = \lambda - 1$, we have, for each traveling wave, the following relation between the stretch λ and the local tangent angle θ:

$$\lambda = \frac{1 + \beta^2 \cos\theta}{1 - c^2}. \tag{4.6}$$

Hence, for λ to vanish at a place in the rod it is necessary and sufficient that at that place $\theta = \theta^\dagger$, where

$$\cos\theta^\dagger = -\beta^{-2}. \tag{4.7}$$

Wherever λ does vanish, the density of mass per unit of arc length is infinite, and hence we shall reject as inadmissible solutions in which there are places where $\lambda = 0$.

For traveling waves, equation (2.11c), expressing balance of angular momentum, becomes (4.2c), which, by (4.5b) and (4.6), is

$$(1 - c^2)\theta'' = \frac{\beta^2}{1 - c^2}\sin\theta + \frac{\beta^4}{1 - c^2}\sin\theta\cos\theta. \tag{4.8}$$

Thus (2.11c) has the first integral

$$\frac{1}{2}\frac{(1 - c^2)^2}{\beta^2}(\theta')^2 = b - \cos\theta + \frac{1}{2}\beta^2\sin^2\theta. \tag{4.9}$$

The phase plane for the second-order differential equation (4.8) has at least two equivalence classes, C^* and C^{**}, of critical points, i.e., points where $\theta' = \theta'' = 0$. In C^*, $\theta = \theta^* = 0,\ \pm 2\pi$, etc. and the critical points are saddle points. In C^{**}, $\theta = \theta^{**} = \pm\pi,\ \pm 3\pi$, etc., and the critical points are centers. When $\beta > 1$, equation (4.8) has another class $C^\dagger$ of critical points, namely those for which $\theta = \theta^\dagger$ where $\theta^\dagger$ obeys (4.7). When $\beta = 1$, $C^\dagger$ reduces to C^{**}.

An elementary analysis of the phase-plane diagrams for equation (4.8), i.e., the (θ', θ)-curves determined by (4.9), yields the following remark.

Remark 2. If $\beta \geqq 1$, the class $C^\dagger$ critical points of (4.8) at which (4.7) holds is not empty, each point in $C^\dagger$ is a center, and *every* orbit of equation (4.8) contains a point (θ, θ') at which (4.6) yields $\lambda = 0$. If $\beta < 1$, $C^\dagger$ is empty.

In view of (4.6), the inequalities $c^2 > 1$ and $\beta < 1$ yield $\lambda(\xi) < 0$ for all ξ. Hence, by Remark 2, independently of the value of β we can assert that if $c^2 > 1$, there is at least one place in the rod at which $\lambda \leqq 0$. If $0 \leqq c^2 < 1$ and $\beta < 1$, there is no place at which $\lambda \leqq 0$.

Let us agree to accept the principle that a solution of (2.11) is inadmissible, because it is physically unattainable, if for it there is a pair (S, t) with $\lambda(S, t) \leqq 0$. Then we can assert

Remark 3. In the present theory there are no admissible traveling waves with $|c| > 1$.

Moreover, we have

Remark 4. The traveling waves for which $\lambda(\xi) > 0$ for all ξ correspond to the following set of values of the pair (c^2, β):

$$R = \{(c^2, \beta) | 0 \leqq c^2 < 1, 0 < \beta < 1\}. \tag{4.10}$$

From this point on we assume that (c^2, β) is in R. In view of Remark 4, equation (4.9) can now be written

$$\tfrac{1}{2}(1 - c^2)(a\theta')^2 = b - \cos\theta + \tfrac{1}{2}\beta^2 \sin^2\theta, \tag{4.11}$$

where a is the positive number defined by

$$a^2 = (1 - c^2)\beta^{-2}. \tag{4.12}$$

As $\beta < 1$, the number b in equation (4.11) can be no less than -1. The solutions of (4.8) with $b = 1$ lie on heteroclinic orbits and correspond to solitary waves; those with $-1 < b < 1$ lie on cyclic orbits and correspond to periodic waves that are inflexional in that they have points at which θ' vanishes; and those with $b > 1$, as they lie on orbits outside the region bounded by heteroclinic separatrices, correspond to waves that are noninflexional, i.e., for which there is no value or limit of ξ at which θ' vanishes.

In Figure 1 are shown phase planes for equation (4.8). For Figure 1A, $c^2 = 0.04$, $\beta^2 = 0.092$, and the values of λ lie in the interval [0.9, 1.1] for each choice of b. For Figure 1B, (c^2, β) is not in the set R of (4.10), because $\beta = \sqrt{2}$; in such a case, i.e., for $\beta \geqq 1$, there are critical points in each of the classes C^*, C^{**}, and $C^\dagger$, and each orbit has at least two points with θ in $C^\dagger$ and hence with $\lambda = 0$.

Let

$$\sigma = \frac{\xi - S_0}{a}, \tag{4.13}$$

where S_0 is chosen as explained after equation (4.18) below. It follows from (4.11) that

$$\sigma = \pm\sqrt{1 - c^2} \int_{\theta_0}^{\theta} [2b - 2\cos\psi + \beta^2 \sin^2\psi]^{-1/2} \, d\psi. \tag{4.14}$$

Here one chooses the symbol $\pm$ to be $+$ ($-$) when θ increases (decreases) with σ. If $b \geqq 1$, θ' does not change sign, and the wave is said to be positive or negative in accord with whether θ' is everywhere positive or negative. If $|b| < 1$, θ is periodic in σ and the choice of $+$ or $-$ in (4.14) varies with position on the cyclic orbit determined by (4.11).

From (4.2c), (4.5b), and (2.2) it follows that $(1 - c^2)\theta'' = \beta^2 y'$, and hence for each traveling wave there is a constant η such that

$$\beta^2 y = (1 - c^2)\theta' + \eta \tag{4.15}$$

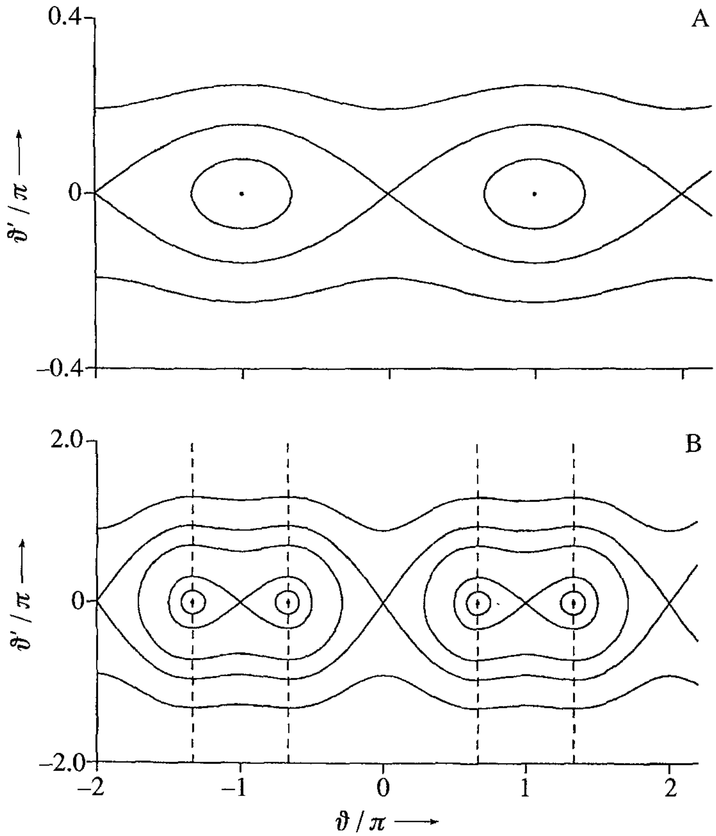

Fig. 1. Phase planes for equation (4.9). In A, $c^2 = 0.04$, $\beta^2 = 0.092$ and hence (c^2, β) is in the set R. In B, $c^2 = 0$, $\beta^2 = 2$; the vertical dashed lines correspond to values of θ in $C^\dagger$.

or, by (4.11) and (4.12),

$$\beta y = \pm[2b - 2\cos\theta + \beta^2 \sin^2\theta]^{1/2} + \eta/\beta. \tag{4.16}$$

Moreover, as $x = S + u$, by (2.2) we have $1 + u' = \lambda\cos\theta$, and, as $u' = \theta' du/d\theta$, (4.6) yields

$$\frac{d\xi}{d\theta} + \frac{du}{d\theta} = \frac{(1 + \beta^2\cos\theta)\cos\theta}{(1 - c^2)\theta'}, \tag{4.17}$$

and hence, by (4.11) and (4.14),

$$u = -a\sigma \pm \frac{1}{\beta}\int_{\theta_0}^{\theta} \frac{(1 + \beta^2\cos\psi)\cos\psi}{[2b - 2\cos\psi + \beta^2\sin^2\psi]^{1/2}} d\psi. \tag{4.18}$$

Let S_0 be the arc-length coordinate of a point on the axis chosen so that $u(S_0 - ct) = 0$ at $t = 0$; θ_0 in (4.14) and (4.18) is the value of $\theta(S - ct)$ at that point and time. When the symbol $\pm$ is given a value, $+$ or $-$, in one of the equations (4.14), (4.16), (4.18), it must be given that value in all three.

Let $\lambda^{\#}$ be the value of the stretch, λ, at any point at which $\theta = \pm 2n\pi$, i.e., at any point where $\cos\theta = 1$ and hence $T = F^x$ and $G = 0$. By (4.6) and (4.12),

$$\lambda^{\#} = \frac{1+\beta^2}{1-c^2} = \frac{1}{1-c^2} + a^{-2}; \tag{4.19}$$

for the elongational strain $\varepsilon = \lambda - 1$ at such a place we have

$$\varepsilon^{\#} = \frac{\beta^2 + c^2}{1-c^2}. \tag{4.20}$$

It is not difficult to show that a pair (c^2, β) is in the set R of (4.10) if and only if $0 < \beta < 1$ and

$$0 \leqq c^2 < \frac{\varepsilon^{\#}}{1+\varepsilon^{\#}} \tag{4.21}$$

with $\varepsilon^{\#}$ as in (4.20). Moreover, (4.20) and the condition that β be less than 1 imply that if $\varepsilon^{\#}$ exceeds 1, then c^2 must be greater than $(\varepsilon^{\#} - 1)/(1 + \varepsilon^{\#})$.

As $T = \varepsilon$ in our present dimensionless units, $\varepsilon^{\#}$ equals the tension at the places where the tangent $\boldsymbol{t}$ is parallel or antiparallel to the x-axis. We note in passing that equation (4.6) yields

$$\lambda = \frac{1}{1-c^2} + \left(\lambda^{\#} - \frac{1}{1-c^2}\right)\cos\theta. \tag{4.22}$$

When $\lambda^{\#}$ and $\varepsilon^{\#}$ are attained, they equal the maximum stretch and tension in the rod.

Only for noninflexional waves and solitary waves are there places or limits where $\theta = \pm 2n\pi$ and hence where $\lambda = \lambda^{\#}$, $\varepsilon = \varepsilon^{\#}$. However, for all traveling waves, including inflexional waves, there are places where $\theta = \pm(2n+1)\pi$, and at those places λ and ε attain their minimum values,

$$\lambda_{\#} = \frac{1-\beta^2}{1-c^2} = \frac{1}{1-c^2} - a^{-2}, \tag{4.23a}$$

$$\varepsilon_{\#} = \frac{c^2 - \beta^2}{1-c^2}. \tag{4.23b}$$

For noninflexional and solitary waves, the average of the supremum and infimum of the stretches is

$$\frac{\lambda^{\#} + \lambda_{\#}}{2} = \frac{1}{1-c^2}. \tag{4.24}$$

We conclude this section with the following remark which follows forthwith from (4.6) and (4.16).

Remark 5. In each traveling wave, $y(\xi)$ and $\lambda(\xi)$ obey the following formal analogue of a relation between $x(\xi)$ and $y(\xi)$ found by COLEMAN & DILL [1992, 1; eq. (101)] for solitary waves in inextensible rods governed by (1.4):

$$\left(\frac{\lambda}{\beta}\right)^2 + \left(\frac{y - \eta/\beta^2}{a}\right)^2 = \frac{\beta^4 + 2b\beta^2 + 1}{\beta^2(1-c^2)^2}. \tag{4.25}$$

5. Solitary Waves

A traveling wave is called a *solitary wave* if $\theta'(\xi) \to 0$ in the two limits $\xi \to \infty$, $\xi \to -\infty$. As we observed in Remark 6, the solitary waves lie on heteroclinic orbits and correspond to the choice $b = 1$ in (4.11). As the saddle points of equation (4.8) lie in the set C^*, and hence have $\theta = \pm 2\pi n$, for solitary waves not only does $\theta'(\pm\infty) = 0$, but also

$$\sin\theta(\pm\infty) = 0, \quad \cos\theta(\pm\infty) = 1, \tag{5.1}$$

and equation (4.6) yields

$$\sup\lambda = \lim_{|\xi|\to\infty} \lambda = \lambda^{\#}, \tag{5.2}$$

with $\lambda^{\#}$ as in (4.19). As $\varepsilon = T$ in the units we are using and $\varepsilon^{\#}$ is equal to $\lambda^{\#} - 1$, in each solitary wave the tension in the limits $\pm\infty$, given by (4.20), is the supremum $\varepsilon^{\#}$ of the tension T in the rod and is positive.

The nonconstant solutions of equation (4.11) with $b = 1$ and $\theta(-\infty) = 0$ have the form

$$\theta = -2\tan^{-1}(\sqrt{1+\beta^2}\,\mathrm{csch}(\pm\sigma\sqrt{\lambda^{\#}})) \tag{5.3}$$

with σ as in (4.13) and $\lambda^{\#}$ as in (4.19). Here the choice $+\sigma$ for the expression $\pm\sigma$ gives a *positive solitary wave* in which θ increases monotonically to 2π as $\xi \to \infty$. The choice $-\sigma$ gives a *negative solitary wave* in which θ decreases to -2π in the same limit. For the former type of wave, $\theta = \pi$ at $\sigma = 0$; for the latter $\theta = -\pi$ at $\sigma = 0$. In both cases the minimum value of λ, i.e., $\lambda_{\#}$, is attained at $\sigma = 0$.

Equation (5.3) yields

$$\sin\theta = \frac{-2\sqrt{1+\beta^2}\sinh(\pm\sigma\sqrt{\lambda^{\#}})}{\cosh^2(\sigma\sqrt{\lambda^{\#}}) + \beta^2}, \tag{5.4}$$

$$\cos\theta = \frac{\sinh^2(\sigma\sqrt{\lambda^{\#}}) - (1+\beta^2)}{\sinh^2(\sigma\sqrt{\lambda^{\#}}) + 1 + \beta^2}, \tag{5.5}$$

which, in view of (4.16), yield, in turn,

$$y = \pm 2\frac{1+\beta^2}{\beta}\cdot\frac{\cosh(\sigma\sqrt{\lambda^{\#}})}{\cosh^2(\sigma\sqrt{\lambda^{\#}}) + \beta^2}. \tag{5.6}$$

We here have put $\eta = 0$, which places the x-axis at such a height that $y \to 0$ as $|\xi| \to \infty$. For a positive solitary wave, $y > 0$, and for a negative solitary wave $y < 0$; in both cases the maximum value of $|y|$ is $2a/\sqrt{1-c^2}$ and occurs at $\sigma = 0$.

For the present case in which $b = 1$, by use of the substitution $z = \operatorname{ctn} \psi/2$, one may evaluate the integral in (4.18) in terms of elementary functions. Upon doing this with $\theta_0 = \pi$ for the positive wave and $-\pi$ for the negative wave, we obtain

$$u = (\xi - S_0)\varepsilon^{\#} - \Gamma(\sigma; \beta). \tag{5.7a}$$

Here

$$\Gamma(\sigma; \beta) = \frac{\sqrt{1+\beta^2}}{\beta} \cdot \frac{\sinh(2\sigma\sqrt{\lambda^{\#}})}{\cosh^2(\sigma\sqrt{\lambda^{\#}}) + \beta^2}; \tag{5.7b}$$

the coefficient $[1+\beta^2]^{1/2}/\beta$ equals $a\sqrt{\lambda^{\#}}$. The term $(\xi - S_0)\varepsilon^{\#}$ is the value the x-displacement would have at $\xi - S_0 = S - S_0 - ct = \sigma a$ if there were no flexure and the stretch were everywhere equal to the stretch in the limits $\pm\infty$. The remaining term, $-\Gamma(\sigma; \beta)$, is the interesting one, for it gives the effect on the x-displacement of the variation of stretch and flexure with ξ. The total loss of span in the x-direction due to the dependence on ξ of stretch and flexure is

$$\lim_{\sigma \to \infty} [\Gamma(\sigma; \beta) - \Gamma(-\sigma; \beta)] = 4\frac{[1+\beta^2]^{1/2}}{\beta} = 4a\sqrt{\lambda^{\#}}. \tag{5.8}$$

The relations (4.22) and (5.5) yield the following expression for $T = T(\sigma) = \lambda(\sigma) - 1$ in a solitary wave:

$$T = \frac{(\cosh^2(\sigma\sqrt{\lambda^{\#}}) - \beta^2)\varepsilon^{\#} - 2\beta^2}{\cosh^2(\sigma\sqrt{\lambda^{\#}}) + \beta^2}. \tag{5.9}$$

Examples of positive and negative solitary waves are shown in Figure 2. The dotted curves illustrate the manner in which graphs of y/a versus x/a approach those corresponding to solutions of (1.4) in the limit in which c^2 and β approach zero, a matter discussed in the Appendix. In that limit, $\varepsilon^{\#} \to 0$ and T approaches zero uniformly in σ.

The graphs of y versus x seen in Figure 3 illustrate the following property of solitary waves with a specified value of $\varepsilon^{\#} > 0$: For c^2 in its appropriate range [see, e.g., the paragraph containing (4.21)], the larger c^2, the larger the loop in the axial curve.

A difference between the present theory and the theory of inextensible rods governed by (1.4) is the following: in the present theory the loops corresponding to solitary waves with different speeds $|c|$, or with different limiting tensions $T(\pm\infty)$, are not related by similarity transformations. The reader will notice that the solitary waves shown in Figure 3 for fixed $\varepsilon^{\#} = T(\pm\infty)$ have loops that are broader the higher the speed.

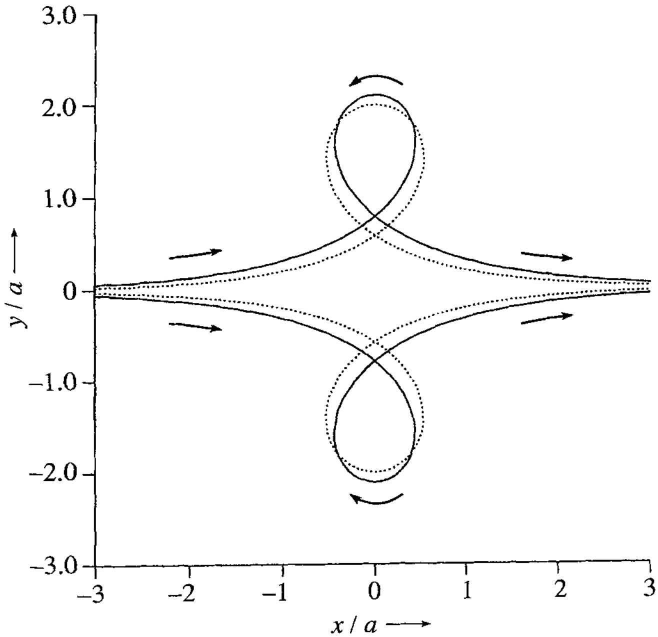

Fig. 2. Graphs of y/a versus x/a for solitary waves described by (5.6) and (5.7). The upper curves are for positive and the lower curves for negative waves. Arrows show the direction of increasing S. Solid curves: $\varepsilon^{\#} = 0.4$, $c^2 = 0.1$, i.e., $\beta^2 = 0.26$. Dotted curves: limit of y/a versus limit of x/a as $c^2 \to 0$ and $\beta \to 0$.

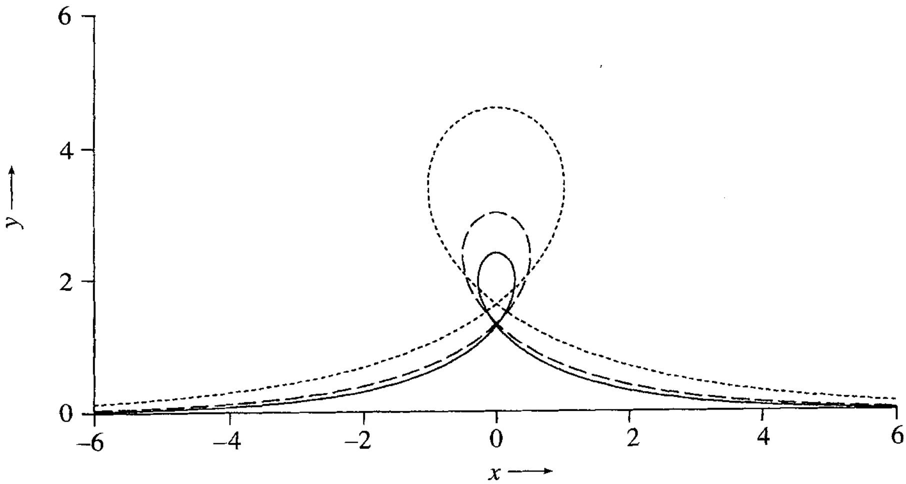

Fig. 3. Graphs of y versus x for positive solitary waves ($b = 1$) with $\varepsilon^{\#} = 0.7$. Short dashes: $c^2 = 0.3$. Long dashes: $c^2 = 0.15$. Solid curve: $c^2 = 0$.

6. Noninflexional Waves

A traveling wave is called *noninflexional* if $(\theta')^2 > 0$ for all ξ and θ' does not have a limit as $\xi \to \pm\infty$. For such waves b in equation (4.11) exceeds 1, and by putting $z = \operatorname{ctn} \psi/2$, equation (4.14) (with $\theta_0 = \pi$) can be written

$$\sigma = \mp \sqrt{\frac{2(1-c^2)}{b-1}} \int_0^w [(z^2 + d_1)(z^2 + d_2)]^{-1/2}\, dz, \tag{6.1}$$

where

$$w = \operatorname{ctn} \theta/2, \tag{6.2}$$

$$d_1 = \frac{b + \beta^2 - \Delta}{b-1}, \quad d_2 = \frac{b + \beta^2 + \Delta}{b-1}, \tag{6.3}$$

$$\Delta = +\sqrt{\beta^4 + 2b\beta^2 + 1}. \tag{6.4}$$

As $b > 1$, we have $d_2 > d_1 > 0$, and for

$$m = \frac{d_2 - d_1}{d_2} = \frac{2\Delta}{b + \beta^2 + \Delta} \tag{6.5}$$

we have $0 < m < 1$. By standard methods we obtain the following implications of (6.1):

$$\sin\theta = \frac{-2\sqrt{d_1}\,\mathrm{sn}(\pm\sigma\gamma|m)\mathrm{cn}(\sigma\gamma|m)}{(d_1 - 1)\mathrm{sn}^2(\sigma\gamma|m) + 1}, \tag{6.6}$$

$$\cos\theta = \frac{(d_1 + 1)\mathrm{sn}^2(\sigma\gamma|m) - 1}{(d_1 - 1)\mathrm{sn}^2(\sigma\gamma|m) + 1}. \tag{6.7}$$

Here γ is the positive number obeying

$$\gamma^2 = \frac{b + \beta^2 + \Delta}{2(1-c^2)} = \frac{(b-1)d_2}{2(1-c^2)}; \tag{6.8}$$

sn and cn are Jacobi elliptic functions. (We use the notation of [1964, 1].) On putting (6.6) and (6.7) into (4.16) with η chosen so that $\min|y| = 0$, we find that

$$\frac{y}{a} = \pm\sqrt{\frac{2(b+1)}{1-c^2}}\left[\frac{\mathrm{dn}(\sigma\gamma|m)}{(d_1 - 1)\mathrm{sn}^2(\sigma\gamma|m) + 1} - \sqrt{\frac{b-1}{b+1}}\right]. \tag{6.9}$$

For a positive wave, $\mp$ has the value $-$ in (6.1) while $\pm$ has the value $+$ in (6.6) and (6.9). With z, w, d_1 and d_2 as in (6.1)–(6.3), one can write (4.18) as

$$\xi + S_0 + u = \frac{-a\sqrt{2}}{\sqrt{(1-c^2)(b-1)}} \int_0^w \frac{[(1+\beta^2)z^2 + 1 - \beta^2](z^2 - 1)}{(1+z^2)^2\sqrt{(z^2 + d_1)(z^2 + d_2)}}\, dz, \tag{6.10}$$

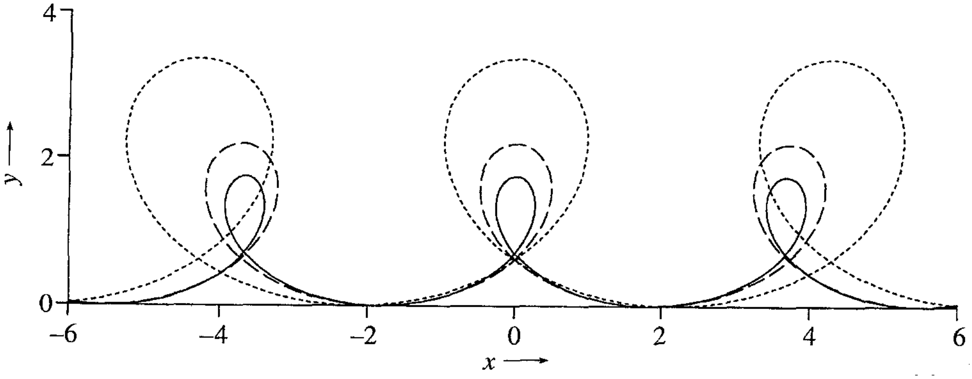

Fig. 4. Graphs of y versus x for positive noninflexional waves with $b = 1.2$ and $\varepsilon^{\#} = 0.7$. Short dashes: $c^2 = 0.3$. Long dashes: $c^2 = 0.15$. Solid curve: $c^2 = 0$.

and, by straightforward but tedious calculation, this can be cast into the form

$$\frac{u}{a} = \left(\frac{b + \beta^2}{1 - c^2} - 1\right)\sigma - 2\gamma \mathrm{E}(\sigma\gamma|m)$$

$$-(d_1 - 1)\sqrt{\frac{d_2 2(b-1)}{1-c^2}}\left[\frac{\mathrm{sn}(\sigma\gamma|m)\mathrm{cn}(\sigma\gamma|m)\mathrm{dn}(\sigma\gamma|m)}{(d_1 - 1)\mathrm{sn}^2(\sigma\gamma|m) + 1}\right], \tag{6.11}$$

where E is an elliptic integral of the second kind (cf. [1964, 1, p. 589]):

$$\mathrm{E}(v|m) = \int_0^u \mathrm{dn}^2(v|m)dv. \tag{6.12}$$

Placement of (6.7) in (4.6) yields an expression for λ as a function of σ in noninflexional waves. For such waves, as for solitary waves, $|\varepsilon|$ is everywhere less than 1 if and only if $2c^2 < 1 - \beta^2$. The maximum tension T is attained at the place where $\theta = \pm 2n\pi$ and equals the number $\varepsilon^{\#}$ in (4.20) and (4.21).

Figure 4 contains graphs of y versus x for noninflexional waves for one value of b; in that figure, as in Figure 3, as $|c|$ varies, β^2 is chosen so that $\varepsilon^{\#}$ equals a fixed value, 0.7.

7. Inflexional Waves

We here consider traveling waves for which there are values of σ at which $\theta' = 0$. For such a wave, θ is periodic in σ and lies on a cyclic orbit about the critical point $(\theta, \theta') = (0, \pi)$. We write θ_* for the smallest value of θ attained on that orbit. Thus the range of θ on the orbit is the interval $[\theta_*, 2\pi - \theta_*]$, and $\theta' = 0$ where $\theta = \theta_*$ and where $\theta = 2\pi - \theta_*$. For such a wave the value of b in (4.11) is in

$(-1, 1)$. Moreover,

$$\cos\theta_* = \frac{\Delta + \beta^2 + 2b - 1}{\Delta + \beta^2 + 1}, \tag{7.1}$$

where Δ is given by (6.4). Let w, d_1, d_2 be as in (6.2) and (6.3), and note that here $d_1 > 0$ and $d_2 < 0$. Equation (4.19), with θ_0 again set equal to π, can be written

$$\sigma = \mp\sqrt{\frac{2(1-c^2)}{1-b}} \int_0^w [(z^2 + d_1)(-d_2 - z^2)]^{-1/2} dz. \tag{7.2}$$

By evaluating the integral in (7.2) on the interval in which w increases from 0 to $\sqrt{-d_2} = \operatorname{ctn}\theta_*/2$ (i.e., in which θ decreases from π to θ_*), one obtains a formula for σ as a function of θ as σ varies over one quarter period of the periodic function $\sigma \mapsto \theta$. Upon doing this and putting, in place of (6.5),

$$\hat{m} = \frac{d_2}{d_2 - d_1} = \frac{\Delta + b + \beta^2}{2\Delta}, \tag{7.3}$$

we find that

$$\sin\theta = \frac{-2\sqrt{\hat{m}d_1}\,\operatorname{sn}(\pm\sigma\hat{\gamma}|\hat{m})\operatorname{dn}(\sigma\hat{\gamma}|\hat{m})}{(d_1 - 1)\hat{m}\operatorname{sn}^2(\sigma\hat{\gamma}|\hat{m}) + 1}, \tag{7.4}$$

$$\cos\theta = \frac{(d_1 + 1)\hat{m}\operatorname{sn}^2(\sigma\hat{\gamma}|\hat{m}) - 1}{(d_1 - 1)\hat{m}\operatorname{sn}^2(\sigma\hat{\gamma}|\hat{m}) + 1}, \tag{7.5}$$

where

$$\hat{\gamma} = +\sqrt{\frac{\Delta}{1-c^2}} \tag{7.6}$$

and $0 < \hat{m} < 1$.

Equations (7.4), (7.5), and (4.16) imply that

$$\frac{y}{a} = \pm\sqrt{\frac{2(b+1)}{1-c^2}}\left[\frac{\operatorname{cn}(\sigma\hat{\gamma}|\hat{m})}{(d_1 - 1)\hat{m}\operatorname{sn}^2(\sigma\hat{\gamma}|\hat{m}) + 1} + \mu\right], \tag{7.7}$$

where $\mu = 1$ when $\min|y| = 0$, as in Figures 5 and 8. The method that gave us equation (6.11) here yields, for inflexional waves,

$$\frac{u}{a} = (\hat{\gamma}^2 - 1)\sigma - 2\hat{\gamma}\operatorname{E}(\sigma\hat{\gamma}|\hat{m})$$

$$-(d_1 - 1)\sqrt{\frac{-d_2 2(1-b)\hat{m}}{1-c^2}}\left[\frac{\operatorname{sn}(\sigma\hat{\gamma}|\hat{m})\operatorname{cn}(\sigma\hat{\gamma}|\hat{m})\operatorname{dn}(\sigma\hat{\gamma}|\hat{m})}{(d_1 - 1)\hat{m}\operatorname{sn}^2(\sigma\hat{\gamma}|\hat{m}) + 1}\right]. \tag{7.8}$$

To obtain λ as a function of σ for inflexional waves, one places (7.5) in (4.6). For inflexional waves, the condition that $|\varepsilon| < 1$ (for all σ) is complicated. If $2c^2 < 1 - \beta^2$, then $|\varepsilon| < 1$ for all b. If $1 - \beta^2 \leq 2c^2 < 1 + \beta^2$, then $|\varepsilon| < 1$ only for b sufficiently near to -1.

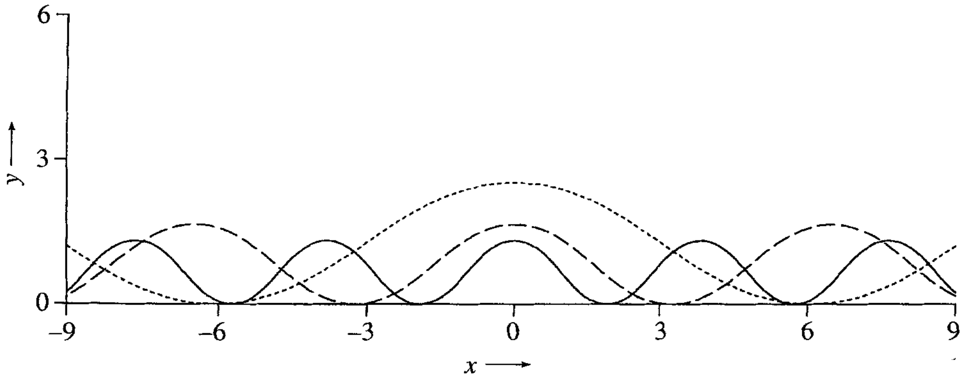

Fig. 5. Graphs of y versus x for positive inflexional waves with $b = -0.85$ and $\varepsilon^{\#} = 0.7$. Short dashes: $c^2 = 0.3$, i.e., $\varepsilon_{\#} = 0.157$. Long dashes: $c^2 = 0.15$, i.e., $\varepsilon_{\#} = -0.347$. Solid curve: $c^2 = 0$, i.e., $\varepsilon_{\#} = -0.7$.

For inflexional waves, the maximum value of the tension T is attained where $\theta = \theta_*$ and $\theta = 2\pi - \theta_*$, and is *less than* $\varepsilon^{\#}$. The minimum tension is attained where $\theta = \pi$ and equals $\varepsilon_{\#}$.

Although ε does not reach the value $\varepsilon^{\#}$ in an inflexional wave, in drawing the graphs shown in Figure 5 for inflexional waves with one value of b and three distinct speeds we have chosen β^2 so that $\varepsilon^{\#}$, here taken to be *defined* by (4.20), is held constant at a value employed for corresponding graphs shown in Figures 3 and 4.

"Figure Eight" Waves

Let $\mathrm{K}(\hat{m})$ and $\mathrm{E}(\hat{m})$ be complete elliptic integrals of the first and second kind defined by

$$\mathrm{K}(\hat{m}) = \int_0^{\pi/2} (1 - \hat{m}\sin^2\varphi)^{-1/2}\, d\varphi, \tag{7.9a}$$

$$\mathrm{E}(\hat{m}) = \int_0^{\pi/2} (1 - \hat{m}\sin^2\varphi)^{1/2}\, d\varphi, \tag{7.9b}$$

and let m_p be the unique solution in (0, 1) of

$$\mathrm{K}(m_\mathrm{p}) = 2\mathrm{E}(m_\mathrm{p}), \tag{7.10}$$

i.e.,

$$m_\mathrm{p} = 0.8261148, \quad k_\mathrm{p} = \sqrt{m_\mathrm{p}} = 0.9089086. \tag{7.11}$$

For a traveling wave with velocity c, it is occasionally of interest to consider a change of coordinates from x, y to x_*, y_*, with

$$x_* = x - ct, \quad y_* = y. \tag{7.12}$$

The relations $x = S + u(\xi)$, $y = y(\xi)$, $\xi = S - ct$ imply that x_* and y_* are functions of ξ, i.e.,

$$x_* = x_*(\xi) = u(\xi) + \xi, \quad y_* = y_*(\xi) = y(\xi), \tag{7.13}$$

and a graph of y_* versus x_* shows the axial curve as seen by an observer moving in the x-direction with velocity c. For an inflexional wave with $\hat{m} = m_p$, it follows from (7.7), (7.8), and (7.13) that not only y_*, but also x_*, is a periodic function of ξ; the minimum period ξ_p of y_* is twice the minimum period of x_* and thus is a common period for y_* and x_*. For this common period we have

$$\xi_p = \frac{4a\mathrm{K}(m_p)}{\hat{\gamma}} \tag{7.14}$$

with a as in (4.12), $\hat{\gamma}$ as in (7.6), and

$$\mathrm{K}(m_p) = 2.321049733. \tag{7.15}$$

In view of (7.3), once we know that $\hat{m} = m_p$, i.e., that y_* and x_* are both periodic in ξ, then β^2 determines b by

$$b = 4\beta^2 m_p(m_p - 1) + (2m_p - 1)\sqrt{4m_p(m_p - 1)\beta^4 + 1}, \tag{7.16}$$

and β^2 and c^2 determine ξ_p by

$$\xi_p = \frac{4\mathrm{K}(m_p)(1 - c^2)}{[\beta^4(2m_p - 1) + \beta^2\sqrt{4m_p(m_p - 1)\beta^4 + 1}]^{1/2}}. \tag{7.17}$$

Clearly such a wave is possible in a rod of length $L = \xi_p$ which is closed, i.e., which, albeit straight in its stress-free reference configuration, has had its ends joined. As the minimum period of y_* is twice the minimum period of x_*, such waves have the form of a "figure eight". The minimum value of ε, $\varepsilon_\# = (c^2 - \beta^2)/(1 - c^2)$, is negative and is attained where $\cos\theta = -1$, that is, at the highest and lowest points of the "figure eight". As we reject, as physically unattainable, configuration with $\varepsilon_\# \leqq -1$, these waves have $\beta^2 < 1$ and, by (7.17), a value of $\xi_p/(1 - c^2)$, i.e., of $L/(1 - c^2)$, greater than

$$\left(\frac{L}{1 - c^2}\right)_{\inf} = \frac{2\sqrt{2}\,\mathrm{K}(m_p)}{\sqrt{2m_p - 1}} = 8.1288519. \tag{7.18}$$

Examples of stationary "figure eight" waves, i.e., equilibrium configurations determined by (7.7), (7.8), and (7.13) with $c = 0$, are shown in Figure 6.

"Figure eight" waves in inextensible rods obeying the system (1.4) were discussed by Coleman & Dill [1992, 1]. Their results yield graphs of y_*/L versus x_*/L that for each L are the same as the dotted curve in Figure 6, which is a limit of graphs of y_*/L versus x_*/L as $L \to \infty$ in the present theory.

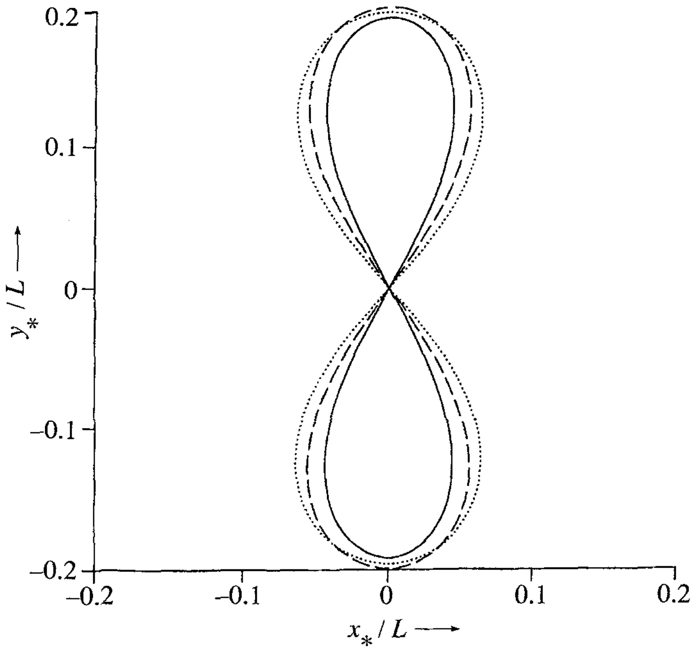

Fig. 6. Graphs y_*/L versus x_*/L for inflexional waves with $\hat{m} = m_p$ and $c = 0$. Here μ is chosen to be zero in (7.7). Solid curve: $\beta^2 = 0.6$, i.e., $b = 0.236109$. Dashed curve: $\beta^2 = 0.3$, i.e., $b = 0.462762$. Dotted curve: limit as $\beta \to 0$, i.e., $L \to \infty$, which corresponds to $b = 0.652230$.

Appendix: Theory of Waves in Inextensible Rods as an Approximation for Small Tension

If, as is customary in the Euler-Kirchhoff-Clebsch theory of rods, one treats the rod as if it were inextensible, then, at each time and place, $s = S$. The constitutive equation (2.8) then does not hold, and both the tension T and the shear resultant G are reactive parameters not given by constitutive equations. As $\lambda = 1$, the relations (2.2) then reduce to

$$x_S = \cos\theta, \quad y_S = \sin\theta, \tag{A.1}$$

and hence, at each time and place

$$(x_S)^2 + (y_S)^2 = 1. \tag{A.2}$$

The constitutive equation (2.9) continues to hold, and the dynamical equations (2.6) and (2.7) become

$$\rho A x_{tt} = F_S^x, \tag{A.3a}$$

$$\rho A y_{tt} = F_S^y, \tag{A.3b}$$

$$\rho I \theta_{tt} = EI\theta_{SS} + G. \tag{A.3c}$$

As $\boldsymbol{F}$ obeys (2.4) and θ is related to $\boldsymbol{x}$ by (A.1), the equations (A.3) may be considered differential equations for the unknowns x, y, F^x, F^y with x and y subject to the constraint (A.2); i.e., we have here four equations for four unknowns, x, y, F^x, F^y, or x, y, T, G. By differentiating (A.3a) and (A.3b) with respect to S, employing (A.1) and the dimensionless variables introduced at the beginning of the paragraph containing (2.11a)–(2.11c), one obtains the system (1.4) shown in the introduction. That system of three equations for three unknowns, θ, T, G, was studied by CAFLISCH & MADDOCKS [1984, 1], COLEMAN & DILL [1992, 1], XU [1992, 3], FALK & XU [1994, 4], COLEMAN & XU [1994, 1], and DICHMANN, MADDOCKS & PEGO [1994, 3]. For (1.4) subject to boundary data appropriate to rods of finite length, CAFLISCH & MADDOCKS [1984, 1] obtained existence and uniqueness theorems and examined the implications of an energy criterion for the stability of equilibrium solutions. COLEMAN & DILL [1992, 1] discussed in detail the relation of the system (1.4) to a form of Hamilton's principle and found the traveling-wave solutions of the system. As we mentioned in Section 3, DICHMANN, MADDOCKS & PEGO [1994, 3] derived the analogue of equation (3.9) for inextensible rods; the set of conservation laws for such rods is discussed in several recent papers [1993, 1], [1994, 3, 4, 5]. An energy preserving second-order finite-difference scheme for numerical solution of (1.4) was developed by XU [1992, 3] and FALK & XU [1994, 4] and employed by COLEMAN & XU [1994, 1] to study the interaction of solitary waves; the numerical evidence indicates that the system (A.4) is not completely integrable in the sense of soliton theory.

We summarize below the results of COLEMAN & DILL [1992, 1] for traveling waves governed by (1.4). For such motions there is a Cartesian system for which θ, y, $u = x - S$, T, and G are functions of $\xi = S - ct$, and (1.4) yields

$$T = c^2 + (\alpha - c^2)\cos\theta, \tag{A.4a}$$

$$G = -(\alpha - c^2)\sin\theta, \tag{A.4b}$$

$$\tilde{a}^2\theta'' = \sin\theta, \tag{A.4c}$$

with $\tilde{a} > 0$ given by

$$\tilde{a}^2 = \frac{1 - c^2}{\alpha - c^2}. \tag{A.5}$$

The integration constant α, which must be such that the right-hand side of (A.5) is positive, equals T at the values (or limits) of ξ for which $\theta = \pm 2n\pi$. When, as is the case for inflexional waves, there are no values of ξ for which $\theta = \pm 2n\pi$, there are values $\xi_\#$ of ξ at which $\theta = \pm(2n + 1)\pi$ and

$$\alpha = 2c^2 - T(\xi_\#). \tag{A.6}$$

The first integral of (A.4c) is

$$\tfrac{1}{2}(\tilde{a}\theta')^2 = b - \cos\theta. \tag{A.7}$$

In analogy with equation (4.13), we now put

$$\tilde{\sigma} = (\xi - S_0)/\tilde{a}, \tag{A.8}$$

and note that (A.7) yields

$$\tilde{\sigma} = \pm \int_{\theta_0}^{\theta} [2b - \cos\psi]^{-1/2} d\psi.$$

We are interested in determining the sense in which traveling-wave solutions of the equations of motion (1.4) for inextensible rods are related to asymptotic limits of traveling-wave solutions of the corresponding equations (2.11) for extensible rods for small values of the magnitude of the tension, which for extensible rods means small values of $|\varepsilon| = |\lambda - 1|$. To this end let us agree to write

$$\bar{y} = y/a, \quad \bar{u} = u/a, \tag{A.9}$$

with a as in (4.12), when y and u are determined by (2.11) and (2.2), i.e., obey (4.16) and (4.18). For comparison, we shall write

$$\tilde{y} = y/\tilde{a}, \quad \tilde{u} = u/\tilde{a}, \tag{A.10}$$

with $\tilde{a}$ as in (A.5), when y and u are determined by (A.4) and (A.1).

The parameter b plays the same role in the equations (4.11) and (A.7). For both the extensible rods governed by (2.11) and the inextensible rods governed by (1.4), traveling waves with b in $(-1, 1)$ are inflexional, those with $b = 1$ are solitary, and those with $b > 1$ are noninflexional. In the theory of the inextensible rods the following expressions for θ, $\tilde{y}$, and $\tilde{u}$ as functions of $\tilde{\sigma}$ hold for these three types of waves:

I. For solitary waves [1992, 1; eqs. (97), (99)]:

$$\theta(\tilde{\sigma}) = 4\tan^{-1}(\exp(\pm\tilde{\sigma})) = -2\tan^{-1}(\operatorname{csch}(\pm\tilde{\sigma})), \tag{A.11}$$

$$\tilde{y}(\tilde{\sigma}) = \pm 2\operatorname{sech}\tilde{\sigma}, \tag{A.12}$$

$$\tilde{u}(\tilde{\sigma}) = -2\tanh\tilde{\sigma}. \tag{A.13}$$

II. For noninflexional waves [1992, 1; eqs. (108), (109)]:

$$\sin\theta = -2\,\mathrm{sn}(\pm\tilde{\sigma}/\tilde{k}|\tilde{m})\mathrm{cn}(\tilde{\sigma}/\tilde{k}|\tilde{m}), \tag{A.14}$$

$$\tilde{y} = \frac{2}{\tilde{k}}[\mathrm{dn}(\tilde{\sigma}/\tilde{k}|\tilde{m}) - (1 - \tilde{m})^{1/2}], \tag{A.15}$$

$$\tilde{u} = 2\left[\frac{1 - \tilde{m}}{\tilde{m}}\right]\tilde{\sigma} - \frac{2}{\tilde{k}}\mathrm{E}(\tilde{\sigma}/\tilde{k}|\tilde{m}), \tag{A.16}$$

where

$$\tilde{m} = \tilde{k}^2 = 2/(b + 1). \tag{A.17}$$

III. For inflexional waves [1992, 1; eqs. (120), (121)]:

$$\sin\theta = -2\tilde{\tilde{k}}\,\mathrm{sn}(\pm\tilde{\sigma}|\tilde{\tilde{m}})\mathrm{dn}(\pm\sigma|\tilde{\tilde{m}}), \tag{A.18}$$

$$\tilde{y} = \pm 2\tilde{\tilde{k}}[1 + \mathrm{cn}(\tilde{\sigma}|\tilde{\tilde{m}})], \tag{A.19}$$

$$\tilde{u} = -2\mathrm{E}(\tilde{\sigma}|\tilde{\tilde{m}}), \tag{A.20}$$

where

$$\tilde{\tilde{m}} = \tilde{\tilde{k}}^2 = (b + 1)/2. \tag{A.21}$$

For each traveling wave of an extensible rod governed by (2.11) let

$$\tau = \sup\{|T(\sigma)|| - \infty < \sigma < \infty\}, \tag{A.22}$$

and note that, by (4.6),

$$\tau = \sup_{\sigma} |\varepsilon(\sigma)| = \sup_{\sigma} \left| \frac{\beta^2 \cos\theta(\sigma) + c^2}{1 - c^2} \right|. \tag{A.23}$$

A natural question that arises is whether, for each fixed value of b, the functions $\tilde{\sigma} \mapsto \theta, \tilde{\sigma} \mapsto \tilde{y}, \tilde{\sigma} \mapsto \tilde{u}$, given above for inextensible rods obeying (1.4), are limits as $\tau \to 0$ of the corresponding expressions for $\sigma \mapsto \theta$, $\sigma \mapsto \bar{y}$, $\sigma \mapsto \bar{u}$, derived in Sections 5, 6, and 7, for extensible rods obeying (2.11).

For $b \geqq 1$, i.e., for a solitary wave and a noninflexional wave, as σ varies over $(-\infty, \infty)$, $\cos\theta$ takes on all values in $(-1, 1)$. Hence when such a wave propagates in a rod obeying (2.11), τ in (A.23) is given by

$$\tau = \varepsilon^{\#} = \frac{\beta^2 + c^2}{1 - c^2}. \tag{A.24}$$

It follows that for each sequence of traveling waves with a fixed $b \geqq 1$,

$$\tau \to 0 \Leftrightarrow \beta \to 0 \quad \text{and} \quad c^2 \to 0. \tag{A.25}$$

For an inflexional wave, i.e., for b in $(-1, 1)$, as σ varies over $(-\infty, \infty)$ the values of $\cos\theta(\sigma)$ cover the interval $[-1, \cos\theta_*]$, where $\cos\theta_*$ is given by (7.1). It follows from (7.1) and (6.4) that for each such b, $\cos\theta_* > -1$ whenever $\beta \neq 0$. As the open interval $[-1, \cos\theta_*]$ is nonempty if $\beta \neq 0$, we may conclude from (A.23) that (A.25) holds also for each sequence of traveling waves with a fixed b in $(-1, 1)$. Thus, by taking the limit of the right-hand side of (4.14) as β and $c^2 \to 0$, we obtain the following conclusion.

Remark 6. For each fixed $b > -1$ and fixed θ_0, as $\tau \to 0$ the functions $\tilde{\sigma} \mapsto \theta$, determined in the theory of (2.11) by (4.14), converge uniformly on compact sets to the function $\tilde{\sigma} \mapsto \theta$, given by (A.9).

In view of this observation and the obvious fact that as $\tau \to 0$ the stretch λ approaches 1 uniformly in σ, we can make the following assertion: as $\tau \to 0$, for solitary waves the expressions (5.3) for θ, (5.6) for $\bar{y}$ and (5.7) for $\bar{u}$, as functions of σ, converge to the expressions (A.11)–(A.13) for θ, $\tilde{y}$, and $\tilde{u}$, as functions of $\tilde{\sigma}$; similarly, for noninflexional waves at fixed b, the expressions (6.6), (6.9), and (6.11) have the expressions (A.14)–(A.16) as limits, while for inflexional waves at fixed b, (7.4), (7.7), (7.8) have (A.18)–(A.20) as limits. Moreover, for each value b, the graphs of $\bar{y}$ versus $\bar{x} = x/a$ approach the graphs of $\tilde{y}$ versus $\tilde{x} = x/\tilde{a}$ as $\tau \to 0$.

Graphs of $\bar{y}$ versus $\bar{x}$ at fixed b and the corresponding limits as $\tau \to 0$ are shown in Figures 2, 7, 8, and 9.

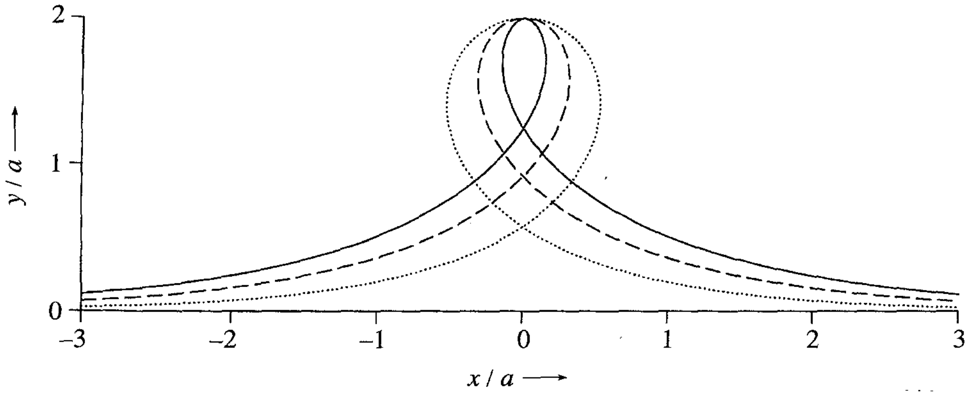

Fig. 7. Graphs of y/a versus x/a for solitary waves that are stationary, i.e., for which $c = 0$. Solid curve: $\varepsilon^{\#} = 0.9$. Dashed curve: $\varepsilon^{\#} = 0.5$. Dotted curve: limit as $\tau = \varepsilon^{\#} \to 0$.

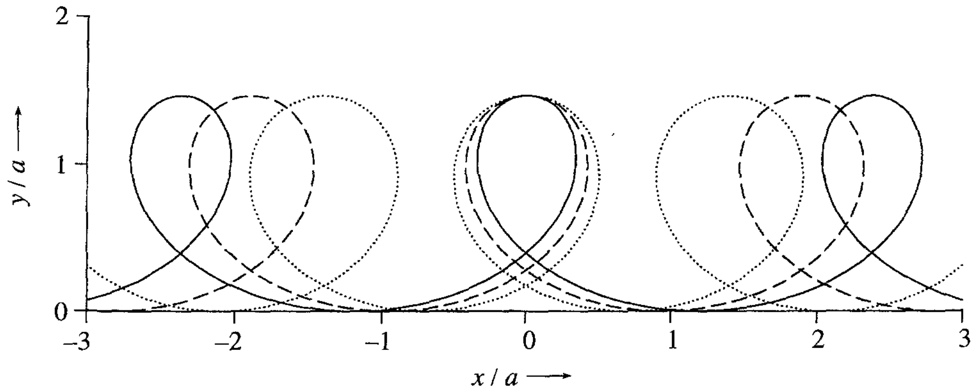

Fig. 8. Graphs of y/a versus x/a for stationary noninflexional waves with $b = 1.2$. Solid curve: $\varepsilon^{\#} = 0.4$. Dashed curve: $\varepsilon^{\#} = 0.2$. Dotted curve: limit as $\tau = \varepsilon^{\#} \to 0$.

In the theory of inextensible rods obeying (1.4) there are traveling waves with $|c| > 1$, and within the confines of that theory there does not appear to be a reason to reject them. However, as we observed in our discussion of Remark 3, traveling waves with $|c| > 1$ in rods obeying (2.11) may be considered to be physically unattainable, because for them there are values of ξ with $\lambda(\xi) < 0$.

The limit of small τ is the only limit in which we expect the theory of inextensible rods to be a useful approximation to the theory of extensible rods, and we have seen that for traveling waves in extensible rods governed by (2.11) the limit of small τ does indeed give expressions for the axial curve which, after appropriate rescaling, have the expected agreement with expressions holding for that curve in the theory of inextensible rods obeying (1.4). As we saw in (A.25), small τ implies small wave speed $|c|$. Thus, if we accept the principle that solutions in the theory of

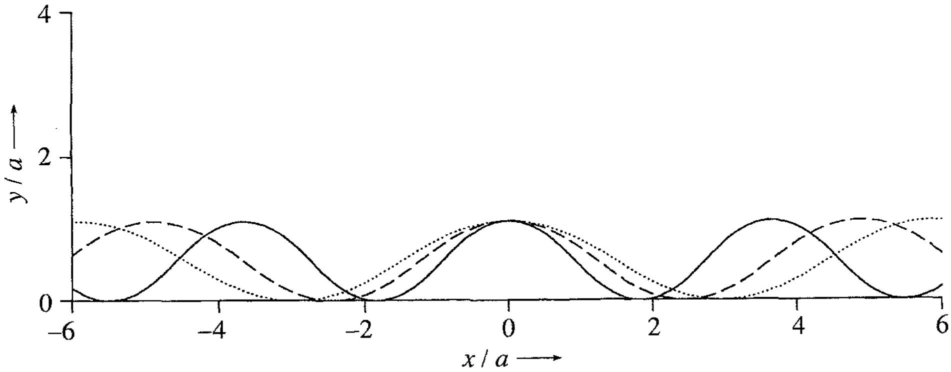

Fig. 9. Graphs of y/a versus x/a for stationary inflexional waves with $b = -0.85$. Solid curve: $\varepsilon^{\#} = 0.6$. Dashed curve: $\varepsilon^{\#} = 0.3$. Dotted curve: limit as $\tau \to 0$.

(1.4) are to be accepted only if they are limits for small τ of solutions in the theory of (2.11), then we have reason to reject as inadmissible supersonic traveling waves not only in the theory of (2.11) but also in the theory of (1.4).

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