

**Math 0250, Midterm I, Fall 2004****Instructor: D. Swigon****SOLUTIONS****Problem 1:** (20 points) Solve the following initial value problem

$$\frac{dy}{dx} = \frac{1+2x^2}{x}y, \quad y(1) = 1$$

Solution

The equation is separable and can be written as

$$\frac{dy}{y} = \frac{1+2x^2}{x}dx$$

which, after integrating both sides, becomes

$$\int \frac{dy}{y} = \int \frac{1}{x}dx + \int 2xdx$$

$$\ln|y| = \ln|x| + x^2 + C$$

$$y = xe^{x^2+C}$$

Application of the initial condition  $y(1) = 1$  gives  $C = -1$  and hence the final answer is:

$$y(x) = xe^{x^2-1}$$

**Problem 2:** (20 points) Suppose that a car starts from rest, its engine providing an acceleration of  $10 \text{ ft/s}^2$ , while air resistance provides deceleration  $(0.001)v^2 \text{ ft/s}^2$  when the car's velocity is  $v$  feet per second.

- (a) Find the car's maximum possible (limiting) velocity  
 (b) Find how long it takes the car to attain 90% of its limiting velocity.  
 (Note that  $\tanh^{-1}(0.9) \approx 1.47$ .)

### Solution

According to the statement of the problem the velocity  $v = v(t)$  of the car obeys the following differential equation:

$$\frac{dv}{dt} = 10 - (0.001)v^2, \quad v(0) = 0$$

(a) The limiting velocity of the car is the stable fixed point of this differential equation. The fixed point is given by the solution of the equation  $10 - (0.001)v^2 = 0$  and is  $v = v_r = 100 \text{ ft/s}$ .

(b) The differential equation is separable and can be written as

$$\frac{dv}{10 - (0.001)v^2} = dt$$

which, after integrating both sides, becomes

$$0.1 \int \frac{dv}{1 - (0.0001)v^2} = t + C$$

$$10 \tanh^{-1}(0.01v) = t + C$$

$$v = 100 \tanh\left(\frac{t}{10} + C_1\right)$$

Application of the initial condition  $v(0) = 0$  gives  $C_1 = 0$ , and hence

$$v(t) = 100 \tanh\left(\frac{t}{10}\right)$$

The car will attain 90% of its limiting velocity at the time  $t$  that obeys the equation  $v(t) = 0.9v_r = 90 \text{ ft/s}$ . Thus

$$t = 10 \tanh^{-1}(0.9) = 14.7 \text{ s}$$

**Problem 3:** (20 points) Find a solution of the initial value problem

$$2x^2y - x^3y' = y^3 \quad y(1) = 1/2$$

Solution

After rewriting the equation so that  $y'$  appears as an explicit term we obtain

$$y' = 2\frac{y}{x} - \frac{y^3}{x^3}$$

We recognize that the equation is homogeneous. We use a substitution  $v = y/x$ , which implies

$$y = vx, \quad \frac{dy}{dx} = x\frac{dv}{dx} + v$$

and rewrite the differential equation in the form

$$x\frac{dv}{dx} + v = 2v - v^3$$

$$\frac{dv}{dx} = \frac{v - v^3}{x}$$

This differential equation is separable and can be solved as follows

$$\frac{dv}{v(1-v^2)} = \frac{dx}{x}$$

$$\int \frac{dv}{v} + \int \frac{v dv}{(1-v^2)} = \int \frac{dx}{x}$$

$$\ln|v| - \frac{1}{2}\ln|1-v^2| = \ln|x| + C$$

$$\frac{v}{\sqrt{1-v^2}} = Kx$$

$$v^2 = K^2x^2(1-v^2)$$

By backsubstituting  $v = y/x$  we obtain

$$y^2 = K^2x^2(x^2 - y^2)$$

$$y^2 = \frac{K^2x^4}{1 + K^2x^2}$$

Finally, the application of the initial condition  $y(1) = 1/2$  implies  $K^2 = 1/3$  and the solution is

$$y^2 = \frac{x^4}{3 + x^2}$$

Alternative solution

After rewriting the equation so that  $y'$  appears as an explicit term we obtain

$$y' - 2\frac{y}{x} = -\frac{y^3}{x^3}$$

We recognize a Bernoulli equation with  $P(x) = -(2/x)$ ,  $Q(x) = -x^{-3}$ , and  $n = 3$ . Hence we substitute  $v = y^{1-n} = y^{-2}$  which implies

$$y = v^{-1/2}, \quad \frac{dv}{dx} = -2y^{-3} \frac{dy}{dx}$$

Substitution from the differential equation for  $dy/dx$  yields

$$\frac{dv}{dx} = -2y^{-3} \left( 2\frac{y}{x} - \frac{y^3}{x^3} \right)$$

$$\frac{dv}{dx} + 4\frac{v}{x} = \frac{2}{x^3}$$

We recognize a linear differential equation for  $v = v(x)$  with  $P(x) = 4/x$  and  $Q(x) = 2x^{-3}$ . The integrating factor is

$$\rho = \exp\left(4\int \frac{1}{x} dx\right) = \exp(4\ln|x|) = x^4$$

Multiplication of the differential equation for  $v = v(x)$  by  $\rho$  yields

$$\frac{d}{dx}(x^4 v) = 2x$$

$$x^4 v = x^2 + C$$

$$v = \frac{x^2 + C}{x^4}$$

Backsubstitution  $v = y^{-2}$  implies that the general solution is

$$y^2 = \frac{x^4}{C + x^2}$$

Finally, the application of the initial condition  $y(1) = 1/2$  implies  $C = 3$  and the solution is

$$y^2 = \frac{x^4}{3 + x^2}$$

**Problem 4:** (20 points) Find the general solution of the differential equation

$$xy' - 12x^4 y^{2/3} = 6y$$

Solution

After rewriting the equation so that  $y'$  appears as an explicit term we obtain

$$y' - \frac{6}{x}y = 12x^3 y^{2/3}$$

We recognize a Bernoulli equation with  $P(x) = -(6/x)$ ,  $Q(x) = 12x^3$ , and  $n = 2/3$ . Hence we substitute  $v = y^{1-n} = y^{1/3}$  which implies

$$y = v^3, \quad \frac{dv}{dx} = \frac{1}{3}y^{-2/3} \frac{dy}{dx}$$

Substituting from the differential equation for  $dy/dx$  gives

$$\begin{aligned} \frac{dv}{dx} &= \frac{1}{3}y^{-2/3} \left( \frac{6}{x}y + 12x^3 y^{2/3} \right) = \frac{2}{x}y^{1/3} + 4x^3 \\ \frac{dv}{dx} - \frac{2}{x}v &= 4x^3 \end{aligned}$$

We recognize a linear differential equation for  $v = v(x)$  with  $P(x) = -(2/x)$  and  $Q(x) = 4x^3$ . The integrating factor is

$$\rho = \exp\left(-\int \frac{2}{x} dx\right) = \exp(-2\ln|x|) = x^{-2}$$

Multiplication of the differential equation for  $v = v(x)$  by  $\rho$  yields

$$\frac{d}{dx}(x^{-2}v) = 4x$$

$$x^{-2}v = 2x^2 + C$$

$$v = 2x^4 + Cx^2$$

Backsubstitution  $y = v^3$  implies that the general solution is

$$y = (2x^4 + Cx^2)^3$$

**Problem 5:** (20 points) Consider a breed of rabbits whose birth and death rates,  $\beta$  and  $\delta$ , are each proportional to the rabbit population  $P = P(t)$  with  $\beta > \delta$ .

(a) Show that

$$P(t) = \frac{P_0}{1 - kP_0t}$$

where  $k$  is a constant and  $P_0 = P(0)$ . Note that  $P(t) \rightarrow \infty$  as  $t \rightarrow 1/(kP_0)$ . This is the doomsday.

(b) If  $P_0 = 6$  and there are 9 rabbits after 10 months, when does doomsday occur?

### Solution

(a) The population of rabbits obeys the differential equation  $dP/dt = (\beta - \delta)P$ . According to the statement of the problem  $\beta = \beta_0P$ ,  $\delta = \delta_0P$  which implies that

$$\frac{dP}{dt} = kP^2$$

where  $k = \beta_0 - \delta_0$  is a constant. This separable equation can be solved as follows

$$\frac{dP}{P^2} = kdt$$

$$-\frac{1}{P} = kt + C$$

$$P = \frac{-1}{C + kt}$$

The assumption  $P(0) = P_0$  implies that  $C = -(1/P_0)$  and hence

$$P = \frac{P_0}{1 - kP_0t}$$

(b) The assumptions  $P_0 = 6$  and  $P(10) = 9$  imply

$$9 = \frac{6}{1 - 60k}$$

$$k = 1/180$$

Hence the doomsday occurs when  $t = 1/(kP_0) = 180/6 = 30$ , i.e., after 30 months.