Math 0250, Midterm I, Fall 2004

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SOLUTIONS

Problem 1: (20 points) Solve the following initial value problem

$$\frac{dy}{dx} = \frac{1+2x^2}{x} y, \qquad y(1) = 1$$

Solution

The equation is separable and can be written as

$$\frac{dy}{y} = \frac{1+2x^2}{x} dx$$

which, after integrating both sides, becomes

$$\int \frac{dy}{y} = \int \frac{1}{x} dx + \int 2x dx$$

$$\ln|y| = \ln|x| + x^2 + C$$

$$y = xe^{x^2+C}$$

Application of the initial condition $y(1) = 1$ gives $C = -1$ and hence the final answer is:

$$y(x) = xe^{x^2-1}$$

Problem 2: (20 points) Suppose that a car starts from rest, its engine providing an acceleration of 10 ft/s^2, while air resistance provides deceleration (0.001) v^2 ft/s^2 when the car's velocity is v feet per second.

(a) Find the car's maximum possible (limiting) velocity

(b) Find how long it takes the car to attain 90% of its limiting velocity. (Note that $\tanh^{-1}(0.9) \approx 1.47$.)

Solution

According to the statement of the problem the velocity $v = v(t)$ of the car obeys the following differential equation:

$$\frac{dv}{dt} = 10 - (0.001)v^2, \qquad v(0) = 0$$

(a) The limiting velocity of the car is the stable fixed point of this differential equation. The fixed point is given by the solution of the equation $10 - (0.001)v^2 = 0$ and is $v = v_\tau = 100$ ft/s.

(b) The differential equation is separable and can be written as

$$\frac{dv}{10 - (0.001)v^2} = dt$$

which, after integrating both sides, becomes

$$0.1\int \frac{dv}{1 - (0.0001)v^2} = t + C$$

$$10 \tanh^{-1}\left(0.01v\right) = t + C$$

$$v = 100 \tanh\left(\frac{t}{10} + C_1\right)$$

Application of the initial condition $v(0) = 0$ gives $C_1 = 0$, and hence

$$v(t) = 100 \tanh\left(\frac{t}{10}\right)$$

The car will attain 90% of its limiting velocity at the time t that obeys the equation $v(t) = 0.9v_\tau = 90$ ft/s. Thus

$$t = 10 \tanh^{-1}(0.9) = 14.7 \text{ s}$$

Problem 3: (20 points) Find a solution of the initial value problem

$$2x^2 y - x^3 y' = y^3 \qquad y(1) = 1/2$$

Solution

After rewriting the equation so that y' appears as an explicit term we obtain

$$y' = 2\frac{y}{x} - \frac{y^3}{x^3}$$

We recognize that the equation is homogeneous. We use a substitution $v = y/x$, which implies

$$y = vx\,, \qquad \frac{dy}{dx} = x\frac{dv}{dx} + v$$

and rewrite the differential equation in the form

$$x\frac{dv}{dx} + v = 2v - v^3$$

$$\frac{dv}{dx} = \frac{v - v^3}{x}$$

This differential equation is separable and can be solved as follows

$$\frac{dv}{v(1-v^2)} = \frac{dx}{x}$$

$$\int\frac{dv}{v} + \int\frac{vdv}{(1-v^2)} = \int\frac{dx}{x}$$

$$\ln|v| - \tfrac{1}{2}\ln\left|1-v^2\right| = \ln|x| + C$$

$$\frac{v}{\sqrt{1-v^2}} = Kx$$

$$v^2 = K^2 x^2 (1-v^2)$$

By backsubstituting $v = y/x$ we obtain

$$y^2 = K^2 x^2 (x^2 - y^2)$$

$$y^2 = \frac{K^2 x^4}{1 + K^2 x^2}$$

Finally, the application of the initial condition $y(1) = 1/2$ implies $K^2 = 1/3$ and the solution is

$$y^2 = \frac{x^4}{3 + x^2}$$

Alternative solution

After rewriting the equation so that y' appears as an explicit term we obtain

$$y' - 2\frac{y}{x} = -\frac{y^3}{x^3}$$

We recognize a Bernoulli equation with $P(x) = -(2/x)$, $Q(x) = -x^{-3}$, and $n = 3$. Hence we substitute $v = y^{1-n} = y^{-2}$ which implies

$$y = v^{-1/2}, \qquad \frac{dv}{dx} = -2y^{-3}\frac{dy}{dx}$$

Substitution from the differential equation for dy/dx yields

$$\frac{dv}{dx} = -2y^{-3}\left(2\frac{y}{x} - \frac{y^3}{x^3}\right)$$

$$\frac{dv}{dx} + 4\frac{v}{x} = \frac{2}{x^3}$$

We recognize a linear differential equation for $v = v(x)$ with $P(x) = 4/x$ and $Q(x) = 2x^{-3}$. The integrating factor is

$$\rho = \exp\left(4\int \frac{1}{x}dx\right) = \exp(4\ln|x|) = x^4$$

Multiplication of the differential equation for $v = v(x)$ by ρ yields

$$\frac{d}{dx}\left(x^4 v\right) = 2x$$

$$x^4 v = x^2 + C$$

$$v = \frac{x^2 + C}{x^4}$$

Backsubstitution $v = y^{-2}$ implies that the general solution is

$$y^2 = \frac{x^4}{C + x^2}$$

Finally, the application of the initial condition $y(1) = 1/2$ implies $C = 3$ and the solution is

$$y^2 = \frac{x^4}{3 + x^2}$$

Problem 4: (20 points) Find the general solution of the differential equation

$$xy' - 12x^4 y^{2/3} = 6y$$

<u>Solution</u>

After rewriting the equation so that y' appears as an explicit term we obtain

$$y' - \frac{6}{x} y = 12x^3 y^{2/3}$$

We recognize a Bernoulli equation with $P(x) = -(6/x)$, $Q(x) = 12x^3$, and $n = 2/3$. Hence we substitute $v = y^{1-n} = y^{1/3}$ which implies

$$y = v^3, \qquad \frac{dv}{dx} = \frac{1}{3} y^{-2/3} \frac{dy}{dx}$$

Substituting from the differential equation for dy/dx gives

$$\frac{dv}{dx} = \frac{1}{3} y^{-2/3} \left(\frac{6}{x} y + 12x^3 y^{2/3} \right) = \frac{2}{x} y^{1/3} + 4x^3$$

$$\frac{dv}{dx} - \frac{2}{x} v = 4x^3$$

We recognize a linear differential equation for $v = v(x)$ with $P(x) = -(2/x)$ and $Q(x) = 4x^3$.
The integrating factor is

$$\rho = \exp\left(-\int \frac{2}{x} dx \right) = \exp\left(-2 \ln|x| \right) = x^{-2}$$

Multiplication of the differential equation for $v = v(x)$ by ρ yields

$$\frac{d}{dx}\left(x^{-2} v \right) = 4x$$

$$x^{-2} v = 2x^2 + C$$

$$v = 2x^4 + Cx^2$$

Backsubstitution $y = v^3$ implies that the general solution is

$$y = (2x^4 + Cx^2)^3$$

Problem 5: (20 points) Consider a breed of rabbits whose birth and death rates, β and δ, are each proportional to the rabbit population $P = P(t)$ with $\beta > \delta$.

(a) Show that

$$P(t) = \frac{P_0}{1 - kP_0 t}$$

where k is a constant and $P_0 = P(0)$. Note that $P(t) \to \infty$ as $t \to 1/(kP_0)$. This is the doomsday.

(b) If $P_0 = 6$ and there are 9 rabbits after 10 months, when does doomsday occur?

<u>Solution</u>

(a) The population of rabbits obeys the differential equation $dP/dt = (\beta - \delta)P$. According to the statement of the problem $\beta = \beta_0 P$, $\delta = \delta_0 P$ which implies that

$$\frac{dP}{dt} = kP^2$$

where $k = \beta_0 - \delta_0$ is a constant. This separable equation can be solved as follows

$$\frac{dP}{P^2} = kdt$$

$$-\frac{1}{P} = kt + C$$

$$P = \frac{-1}{C + kt}$$

The assumption $P(0) = P_0$ implies that $C = -(1/P_0)$ and hence

$$P = \frac{P_0}{1 - kP_0 t}$$

(b) The assumptions $P_0 = 6$ and $P(10) = 9$ imply

$$9 = \frac{6}{1 - 60k}$$

$$k = 1/180$$

Hence the doomsday occurs when $t = 1/(kP_0) = 180/6 = 30$, i.e., after 30 months.