Math 0250, Midterm Exam II, Fall 2004
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NAME: ____

There are 5 problems, each worth 20 points.
Show all your work.
If you run out of space continue on the back of the page.
Notes and calculators are not allowed.

Problem 1: (20 points) Find the inverse $\mathbf{A}^{-1}$ of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 2 \\ 3 & 0 & 5 \\ 1 & -1 & 2 \end{bmatrix}$$

Solution: Use elementary row operations to convert **A** into reduced echelon form and with the same sequence operations convert **I** to $\mathbf{A}^{-1}$.

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 3 & 0 & 5 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_1 + R_3]{-3R_1 + R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 6 & -1 & -3 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right] \xrightarrow{swap(R_2, R_3)}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 6 & -1 & -3 & 1 & 0 \end{array}\right] \xrightarrow[-6R_2 + R_3]{2R_2 + R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 0 & 2 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 3 & 1 & -6 \end{array}\right] \xrightarrow[-R_3]{2R_3 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 2 & -10 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -3 & -1 & 6 \end{array}\right]$$

Thus

$$\mathbf{A}^{-1} = \begin{bmatrix} 5 & 2 & -10 \\ -1 & 0 & 1 \\ -3 & -1 & 6 \end{bmatrix}$$

Problem 2: (20 points) Find all solutions of the system of linear equations

$$\begin{aligned} x_1 + 3x_2 + 3x_3 + 3x_4 &= 2 \\ 2x_1 + 7x_2 + 5x_3 - \ \ x_4 &= 1 \\ 2x_1 + 8x_2 + 4x_3 - 4x_4 &= 2 \end{aligned}$$

Solution: Use Gauss-Jordan elimination to convert the augmented matrix [**A b**] into reduced echelon form.

$$[\mathbf{A} \quad \mathbf{b}] = \left[\begin{array}{cccc|c} 1 & 3 & 3 & 3 & 2 \\ 2 & 7 & 5 & -1 & 1 \\ 2 & 8 & 4 & -4 & 2 \end{array}\right] \xrightarrow[-2R_1 + R_3]{-2R_1 + R_2}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 3 & 3 & 2 \\ 0 & 1 & -1 & -7 & -3 \\ 0 & 2 & -2 & -10 & -2 \end{array}\right] \xrightarrow[-2R_2 + R_3]{-3R_2 + R_1}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 6 & 24 & 11 \\ 0 & 1 & -1 & -7 & -3 \\ 0 & 0 & 0 & 4 & 4 \end{array}\right] \xrightarrow{R_3 / 4}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 6 & 24 & 11 \\ 0 & 1 & -1 & -7 & -3 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[7R_3 + R_2]{-24R_3 + R_1} \left[\begin{array}{cccc|c} 1 & 0 & 6 & 0 & -13 \\ 0 & 1 & -1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Thus the solution of the system can be written:

$$\mathbf{x} = \begin{bmatrix} -13 - 6t \\ 4 + t \\ t \\ 1 \end{bmatrix}$$

Problem 3: (20 points) Calculate the determinant of **A**

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 4 & 8 \\ 0 & 1 & -2 & 2 \\ 3 & 3 & 1 & 4 \\ 0 & 1 & -3 & 2 \end{bmatrix}$$

Solution: There are several ways you can solve this problem:

1. By subtracting 2 times column 2 from column 4 and expanding about column 4

$$\det \mathbf{A} = \begin{vmatrix} 1 & 4 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 3 & 3 & 1 & -2 \\ 0 & 1 & -3 & 0 \end{vmatrix} = -(-2)\begin{vmatrix} 1 & 4 & 4 \\ 0 & 1 & -2 \\ 0 & 1 & -3 \end{vmatrix} = 2\begin{vmatrix} 1 & -2 \\ 1 & -3 \end{vmatrix} = 2(-3+2) = -2$$

2. By subtracting row 2 from row 4 and expanding about row 4

$$\det \mathbf{A} = \begin{vmatrix} 1 & 4 & 4 & 8 \\ 0 & 1 & -2 & 2 \\ 3 & 3 & 1 & 4 \\ 0 & 0 & -1 & 0 \end{vmatrix} = -(-1)\begin{vmatrix} 1 & 4 & 8 \\ 0 & 1 & 2 \\ 3 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} + \begin{vmatrix} 4 & 8 \\ 1 & 2 \end{vmatrix} = -2+0 = -2$$

3. By using Gauss elimination

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 4 & 8 \\ 0 & 1 & -2 & 2 \\ 3 & 3 & 1 & 4 \\ 0 & 1 & -3 & 2 \end{bmatrix} \xrightarrow{-3R_1 + R_3} \begin{bmatrix} 1 & 4 & 4 & 8 \\ 0 & 1 & -2 & 2 \\ 0 & -9 & -11 & -20 \\ 0 & 1 & -3 & 2 \end{bmatrix} \xrightarrow[-R_2 + R_4]{9R_2 + R_3}$$

$$\begin{bmatrix} 1 & 4 & 4 & 8 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & -29 & -2 \\ 0 & 0 & -1 & 0 \end{bmatrix} \xrightarrow{-30R_4 + R_3} \begin{bmatrix} 1 & 4 & 4 & 8 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 0 \end{bmatrix} \xrightarrow{R_3 + R_4} \begin{bmatrix} 1 & 4 & 4 & 8 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -2 \end{bmatrix} = \mathbf{B}$$

Because all row operations we used were of the type $aR_i + R_j$,

$$\det \mathbf{A} = \det \mathbf{B} = -2$$

Problem 4: (20 points) Show that the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ form a basis of $\mathbb{R}^3$: Express $\mathbf{y}$ as a linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{w}$

$$\mathbf{u} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} -7 \\ 0 \\ -5 \end{bmatrix}$$

Solution: The vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ form a basis of $\mathbb{R}^3$ if they are linearly independent, which is the case if and only if $\det \mathbf{V} \neq 0$, where $\mathbf{V} = [\mathbf{u} \quad \mathbf{v} \quad \mathbf{w}]$, which is true if and only if the system

$$\mathbf{V}\mathbf{c} = \mathbf{y}$$

has a unique solution.

We find that solution using Gauss-Jordan elimination:

$$[\mathbf{V} \quad \mathbf{y}] = \left[\begin{array}{ccc|c} 1 & 4 & -3 & -7 \\ 4 & 2 & 3 & 0 \\ 5 & 5 & 1 & -5 \end{array}\right] \xrightarrow[-5R_1 + R_3]{-4R_1 + R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -3 & -7 \\ 0 & -14 & 15 & 28 \\ 0 & -15 & 16 & 30 \end{array}\right] \xrightarrow{-R_3 + R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -3 & -7 \\ 0 & 1 & -1 & -2 \\ 0 & -15 & 16 & 30 \end{array}\right] \xrightarrow[15R_2 + R_3]{-4R_2 + R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_3 + R_2]{-R_3 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Thus, $\det \mathbf{V} \neq 0$ and the system $\mathbf{V}\mathbf{c} = \mathbf{y}$ has a unique solution $\mathbf{c} = (1, -2, 0)$ and hence

$$\mathbf{y} = \mathbf{u} - 2\mathbf{v}$$

Problem 5: (20 points) Use Cramer's rule to solve the following system for x_3:

$$\begin{aligned} 3x_1 - x_2 - 5x_3 &= 3 \\ 4x_1 - 4x_2 - 3x_3 &= -4 \\ x_1 - 5x_3 &= 2 \end{aligned}$$

Solution: Cramer's rule states that

$$x_3 = \frac{\det[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{b}]}{\det \mathbf{A}}$$

Thus

$$x_3 = \frac{\begin{vmatrix} 3 & -1 & 3 \\ 4 & -4 & -4 \\ 1 & 0 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & -1 & -5 \\ 4 & -4 & -3 \\ 1 & 0 & -5 \end{vmatrix}} = \frac{\begin{vmatrix} -1 & 3 \\ -4 & -4 \end{vmatrix} + 2\begin{vmatrix} 3 & -1 \\ 4 & -4 \end{vmatrix}}{\begin{vmatrix} -1 & -5 \\ -4 & -3 \end{vmatrix} + (-5)\begin{vmatrix} 3 & -1 \\ 4 & -4 \end{vmatrix}} = \frac{16-16}{23} = 0$$