Practice problems II

① (a)

$$u(t+\Delta t, a) = u(t, a-\Delta a) - \mu a\, u(t,a)\,\Delta t$$

$$u(t+\Delta t, a) - u(t,a) = u(t, a-\Delta a) - u(t,a) - \mu a\, u(t,a)$$

$$\frac{u(t+\Delta t, a) - u(t,a)}{\Delta t} = \frac{\Delta a}{\Delta t}\,\frac{u(t,a-\Delta a) - u(t,a)}{\Delta a} - \mu a\, u(t,a)$$

take limits $\Delta a \to 0$ $\Delta t \to 0$ Note $\frac{\Delta a}{\Delta t} = 1$

$$\frac{\partial u}{\partial t} = -\frac{\partial u}{\partial a} - \mu a u \qquad (*)$$

(b) Suppose $u(t,a) = e^{\lambda t} w(a)$

$$\frac{\partial u}{\partial t} = \lambda e^{\lambda t} w(a)$$

$$\frac{\partial u}{\partial a} = e^{\lambda t} w'(a)$$

Substitute into $(*)$

$$\lambda e^{\lambda t} w(a) = -e^{\lambda t} w'(a) - \mu a\, w(a)\, e^{\lambda t}$$

Simplify

$$w'(a) = -(\lambda + \mu a) w(a)$$

solve

$$\frac{dw}{w} = -(\lambda + \mu a)\,da \qquad \ln w = -\lambda a - \frac{\mu a^2}{2} + C$$

$$w(a) = w(0)\, e^{-\lambda a - \frac{\mu a^2}{2}}$$

(c) ~~[illegible]~~

Since $u(t,0) = \beta = \text{const}$,

$$u(t,0) = e^{\lambda t} w(0) = \beta \quad \Rightarrow \quad \lambda = 0$$

$$u(t,a) = \beta e^{-\frac{\mu a^2}{2}}$$ represents a stationary solution

---

(2)

$$\frac{\partial m}{\partial t} = D \frac{\partial^2 m}{\partial x^2} - \chi \frac{\partial}{\partial x}\left(m \frac{\partial c}{\partial x}\right) + rm$$

(a) Steady-state solution: $m(x,t) = M(x)$, $\chi = 2, D = 1$, $c(x) = x$

Substitute

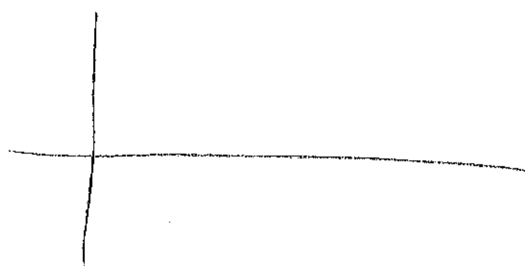
$$0 = D M''(x) - \chi \left(M(x) c'(x)\right)' + r M(x)$$

$$0 = M'' - 2M' + rM$$

Phase plane

let $M' = N$

$N' = +2N - rM$



Linear system $J = \begin{pmatrix} 0 & 1 \\ -r & 2 \end{pmatrix}$ $\operatorname{tr} J = 2$

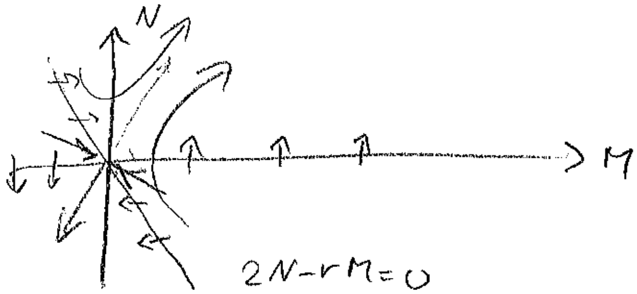
$\det J = r$ saddle for $r < 0$

$\operatorname{tr}^2 J - 4\det = 4 - 4r = 4(1-r)$

unstable node for $1 > r$

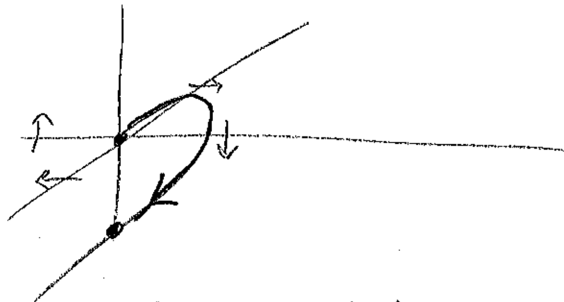
unstable spiral for $1 < r$

__saddle__
$r<0$



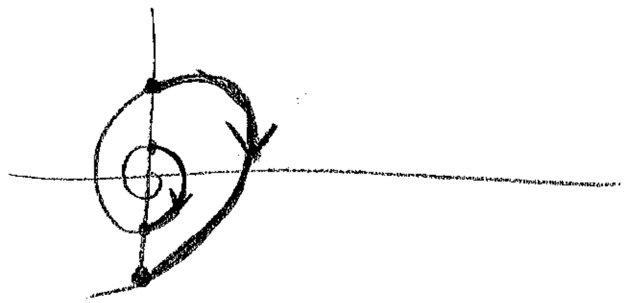
No solutions that start and end with $M=0$

__Unstable node__
$0<r<1$



there are solutions that approach $M=0$ as $x\to-\infty$
No solution of the ~~Dirichlet~~ Dirichlet problem

__Unstable spiral__
$\boxed{r>1}$



__Solutions exist.__

(3)

$$\frac{\partial u}{\partial t} = k\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) + u(1-u)$$

Traveling wave $\quad u(x,t) = g(x-ct) = g(z)$

$$-cg' = k(gg')' + g(1-g)$$

$$-cg' = k(g')^2 + kgg'' + g(1-g)$$

$$g'' = \frac{-cg' - k(g')^2 - g(1-g)}{kg}$$

Phase plane

$$\frac{dg}{dz} = h$$

$$\frac{dh}{dz} = \frac{-ch - kh^2}{kg} - \frac{1}{k}(1-g)$$

Change independent variable

$g\,dw = dz$

$$\frac{dg}{dw} = gh$$

$$\frac{dh}{dw} = \frac{-ch - kh^2 - g(1-g)}{k}$$

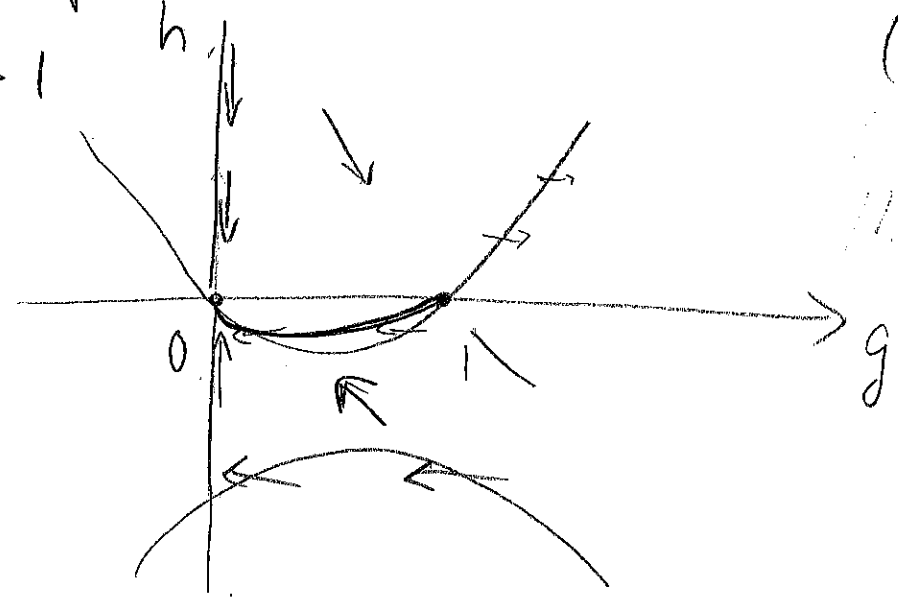
Equilibrium points

$(g^*, h^*) = (1,0)$ – saddle

$(g^*, h^*) = (0,0)$ – half stable

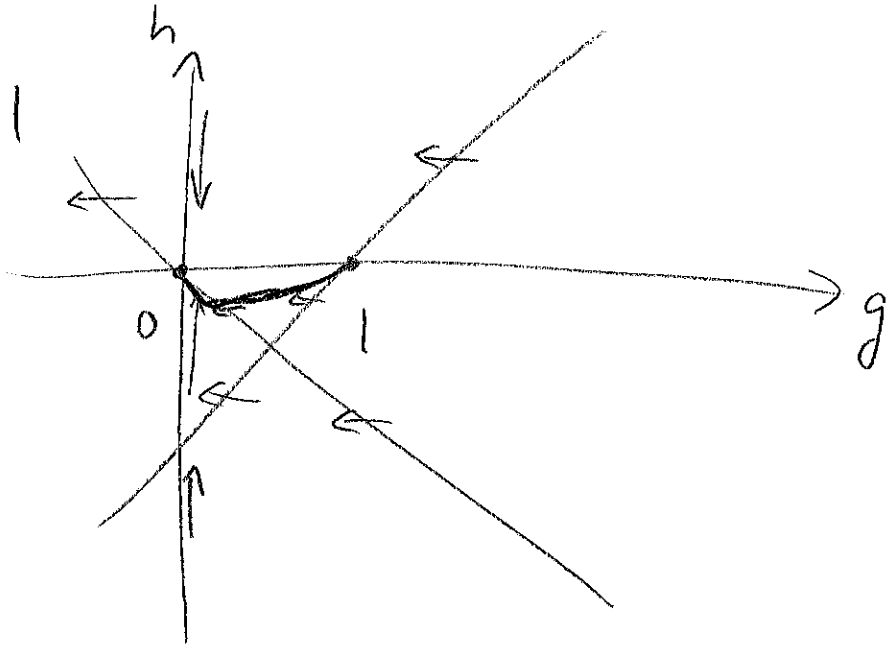
Phase portraits

$c > 1$



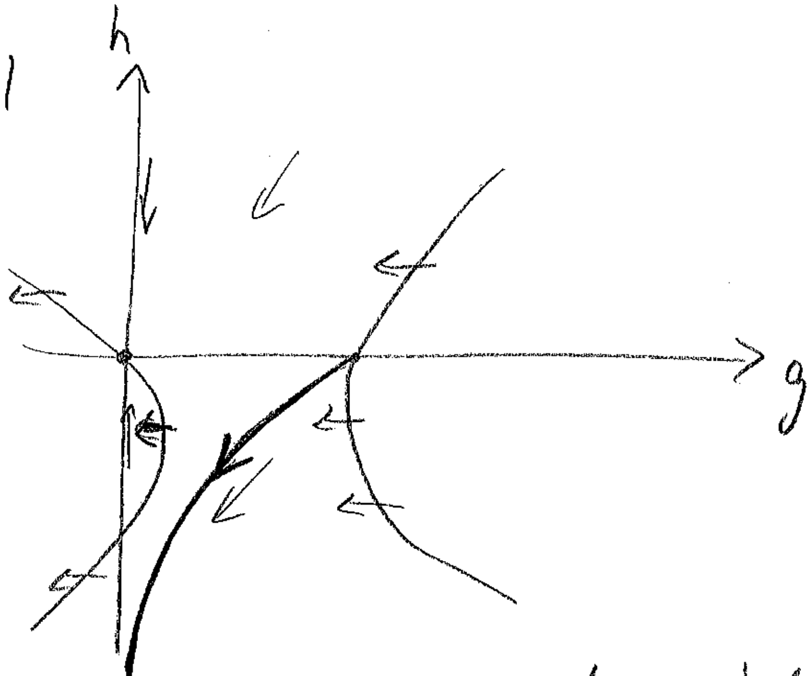
Traveling wave exists

$C = 1$



Traveling wave exists

$C < 1$



Traveling wave may not exist if the trajectory starting at (1,0) does not reach (0,0)

(4)

$$P = \begin{bmatrix} 1/2 & 1/3 & 0 & 0 \\ 0 & 1/3 & 2/4 & 0 \\ 1/2 & 1/3 & 1/4 & 1/2 \\ 0 & 0 & 1/4 & 1/2 \end{bmatrix}$$

Equilibrium distribution

$$(P - I)u^* = 0$$

Row reduction

$$\begin{bmatrix} -1/2 & 1/3 & 0 & 0 \\ 0 & -2/3 & 1/2 & 0 \\ 1/2 & 1/3 & -3/4 & 1/2 \\ 0 & 0 & 1/4 & -1/2 \end{bmatrix} \xrightarrow{R_1 + R_3} \begin{bmatrix} -1/2 & 1/3 & 0 & 0 \\ 0 & -2/3 & 1/2 & 0 \\ 0 & 2/3 & -3/4 & 1/2 \\ 0 & 0 & 1/4 & -1/2 \end{bmatrix} \xrightarrow{R_2 + R_3}$$

$$\begin{bmatrix} -1/2 & 1/3 & 0 & 0 \\ 0 & -2/3 & 1/2 & 0 \\ 0 & 0 & -1/4 & 1/2 \\ 0 & 0 & 1/4 & -1/2 \end{bmatrix} \rightarrow \begin{bmatrix} -1/2 & 1/3 & 0 & 0 \\ 0 & -2/3 & 1/2 & 0 \\ 0 & 0 & -1/4 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

free

$$u^* = \begin{bmatrix} c \\ 3/2\, c \\ 2c \\ c \end{bmatrix}$$

Find C

$$\sum_i u_i^* = \frac{11}{2} c = 1$$

$$c = \frac{2}{11}$$

$$u^* = \begin{bmatrix} 2/11 \\ 3/11 \\ 4/11 \\ 2/11 \end{bmatrix}$$

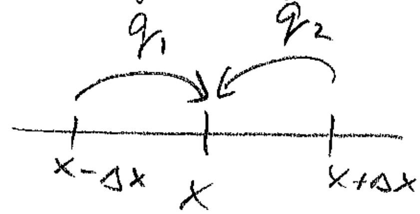
⑤

$$\rho(x, t+\Delta t) = q_1\, \rho(x-\Delta x, t) + (1-q_1-q_2)\, \rho(x,t) + q_2\, \rho(x+\Delta x, t)$$

(a) term 1 $\rightarrow$ influx of probability due to jumps from $x-\Delta x$

term 2 $\rightarrow$ probability of remaining at x,

term 3 $\rightarrow$ influx of probability due to jumps from $x+\Delta x$



$$\rho(x\pm\Delta x, t) = \rho(x,t) \pm \frac{\partial \rho}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 \rho}{\partial x^2}\Delta x^2$$

$$\rho(x, t+\Delta t) = \rho(x,t) + \frac{\partial \rho}{\partial t}\Delta t$$

$$\cancel{\rho} + \frac{\partial \rho}{\partial t}\Delta t = q_1\left[\rho - \frac{\partial \rho}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 \rho}{\partial x^2}\Delta x^2\right] + \cancel{(1-q_1-q_2)\rho} + q_2\left[\rho + \frac{\partial \rho}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 \rho}{\partial x^2}\Delta x^2\right]$$

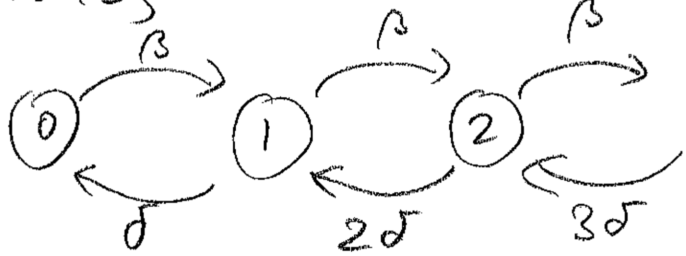
$$\boxed{\frac{\partial \rho}{\partial t} = \frac{(q_2-q_1)\Delta x}{\Delta t}\frac{\partial \rho}{\partial x} + \frac{(q_1+q_2)\Delta x^2}{2\Delta t}\frac{\partial^2 \rho}{\partial x^2}}$$

$$\frac{\partial \rho}{\partial t} = \alpha\frac{\partial \rho}{\partial x} + \beta\frac{\partial^2 \rho}{\partial x^2}$$

where $\alpha = \dfrac{(q_2-q_1)\Delta x}{\Delta t}$

$\beta = \dfrac{(q_1+q_2)\Delta x^2}{2\Delta t}$

(6) Process



(a) Master equation

$$\frac{dp_k(t)}{dt} = \beta\, p_{k-1}(t) + \delta(k+1)\,p_{k+1}(t) - (\beta+\delta k)\,p_k(t)$$

(b) Mean

$$M_1(t) = \sum k\, p_k(t)$$

$$\frac{dM_1}{dt} = \sum k \frac{dp_k}{dt} = \beta \sum k\, p_{k-1} + \delta \sum k(k+1)\,p_{k+1} - \beta \sum k\, p_k - \delta \sum k^2 p_k$$

$$= \sum \left[\beta(n+1) + \delta(n-1)n - \beta n - \delta n^2\right] p_n$$

$$= \sum \beta p_n - \sum \delta n p_n$$

$$= \beta \sum p_n - \delta \sum n p_n$$

$$\frac{dM_1}{dt} = \beta - \delta M_1$$

Solution $$\frac{dM_1}{M_1 - \beta/\delta} = -\delta\, dt$$

$$\ln|M_1 - \beta/\delta| = -\delta t + C$$

$$\boxed{M_1(t) = \beta/\delta + \left[M_1(0) - \beta/\delta\right] e^{-\delta t}}$$

Variance $\sigma^2 = M_2 - M_1^2$ $\qquad M_2 = \sum k^2 p_k$

$$\frac{dM_2}{dt} = \sum k^2 \frac{dp_k}{dt} = \beta \sum k^2 p_{k-1} + \delta \sum k^2 (k+1) p_{k+1} - \sum (\beta + \delta k) k^2 p_k$$

$$= \sum \left[\beta (n+1)^2 + \delta (n-1)^2 n - \beta n^2 - \delta n^3 \right] p_n$$

$$= \sum \left[\beta (2n+1) + \delta (-2n+1) n \right] p_n$$

$$= \beta + 2\beta M_1 - 2\delta M_2 + \delta M_1 = \beta - 2\delta M_2 + (2\beta + \delta) M_1$$

$$\frac{d\sigma^2}{dt} = \frac{dM_2}{dt} - 2M_1 \frac{dM_1}{dt} =$$

$$= \beta - 2\delta M_2 + (2\beta + \delta) M_1 - 2M_1(\beta - \delta M_1)$$

$$= \beta - 2\delta (M_2 - M_1^2) + \delta M_1$$

$$= -2\delta \sigma^2 + \beta + \delta M_1$$

Thus $$\frac{d\sigma^2}{dt} + 2\delta \sigma^2 = 2\beta + [M_1(0)\delta - \beta] e^{-\delta t}$$

integrating factor $\rho = e^{2\delta t}$

$$\frac{d}{dt}\left(e^{2\delta t} \sigma^2(t) \right) = 2\beta e^{2\delta t} + [M_1(0)\delta - \beta] e^{+\delta t}$$

$$e^{2\delta t} \sigma^2(t) = \beta/\delta \, e^{2\delta t} + [M_1(0) - \beta/\delta] e^{+\delta t}$$

$$\sigma^2(t) = \beta/\delta + [M_1(0) - \beta/\delta] e^{-\delta t} + [\sigma^2(0) - M_1(0)] e^{-2\delta t}$$