Math 1080, Midterm Exam II, Spring 2005

Instructor: D. Swigon

SOLUTIONS

Problem 1: a) (10 points) Determine the relative condition number for the following problem. Are there values of *x* for which the problem is ill-conditioned? Justify your answer.

$$f(x) = \frac{1-e^{-x}}{1+e^{-x}}$$

<u>*SOLUTION:*</u>

The relative condition is given by the formula $\kappa(x) = \left|J(x)\right| \frac{|x|}{|f(x)|}$.

Here

$$J(x) = \frac{df}{dx} = \frac{e^{-x}(1+e^{-x}) - (1-e^{-x})(-e^{-x})}{(1+e^{-x})^2} = \frac{2e^{-x}}{(1+e^{-x})^2},$$

and hence

$$\kappa(x) = \frac{2e^{-x}}{(1+e^{-x})^2} \frac{|x|}{\left|\frac{1-e^{-x}}{1+e^{-x}}\right|} = \frac{2|x|e^{-x}}{\left|1-e^{-2x}\right|}.$$

Using the fact that $e^{x} - e^{-x} = 2\sinh x$ we can write

$$\kappa(x) = \frac{|x|}{|\sinh x|}$$

For values of *x* near 0 we have $\sinh x \approx x$ and hence $\kappa(x) \approx 1$. As $x \to \infty$ we have $\sinh x \to e^{x}/2$ and hence $\kappa(x) \to 0$. As $x \to -\infty$ we have $\sinh x \to -e^{-x}/2$ and hence $\kappa(x) \to 0$. Thus, there are no values of *x* for which the problem is ill-conditioned.

b) (20 points) Determine whether the calculation of $f(x,y)=(1+x)y^2$ by the algorithm $\tilde{f}(x,y)=\left[1\oplus \mathrm{fl}(x)\right]\otimes\left[\mathrm{fl}(y)\otimes\mathrm{fl}(y)\right]$

SOLUTION:

Using the properties of $\varepsilon_{\text{machine}}$ we find that there exist $\varepsilon_i, i=1...6$ such that

$\left|\varepsilon_i\right|\le\varepsilon_{\text{machine}}+O(\varepsilon_{\text{machine}}^2)$ and

$$\begin{aligned}
\tilde{f}(x,y)&=\left[1\oplus \mathrm{fl}(x)\right]\otimes\left[\mathrm{fl}(y)\otimes\mathrm{fl}(y)\right]\\
&=\left[1\oplus x(1+\varepsilon_1)\right]\otimes\left[y(1+\varepsilon_2)\otimes y(1+\varepsilon_2)\right]\\
&=\left(1+x(1+\varepsilon_1)\right)(1+\varepsilon_3)\otimes\left(y(1+\varepsilon_2)\right)^2(1+\varepsilon_4)\\
&=\left(1+x(1+\varepsilon_1)\right)\left(y(1+\varepsilon_2)\right)^2(1+\varepsilon_3)(1+\varepsilon_4)(1+\varepsilon_5)\\
&=\left(1+x(1+\varepsilon_1)\right)\left(y(1+(5/2)\varepsilon_6)\right)^2\\
&=(1+\tilde{x})\tilde{y}^2
\end{aligned}$$

Thus, the algorithm is backward stable.

Problem 2: (20 points) Compute the LU factorization with partial pivoting, **PA** = **LU**, for the following matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -4 \\ 2 & 2 & 0 \\ 1 & 3 & 4 \end{bmatrix}$$

SOLUTION:

$$\mathbf{P}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{P}_1\mathbf{A} = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 2 & -4 \\ 1 & 3 & 4 \end{bmatrix}, \quad \mathbf{L}_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}, \quad \mathbf{L}_1\mathbf{P}_1\mathbf{A} = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & -4 \\ 0 & 2 & 4 \end{bmatrix}$$

$$\mathbf{P}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \ \mathbf{P}_2\mathbf{L}_1\mathbf{P}_1\mathbf{A} = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & -4 \end{bmatrix}, \ \mathbf{L}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{bmatrix},$$

$$\mathbf{U} = \mathbf{L}_2\mathbf{P}_2\mathbf{L}_1\mathbf{P}_1\mathbf{A} = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & -6 \end{bmatrix}$$

$$\mathbf{P} = \mathbf{P}_3\mathbf{P}_2\mathbf{P}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \qquad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 1/2 & 1 \end{bmatrix}$$

Problem 3: (20 points) Check that the coefficient matrix of the following system is symmetric and positive definite. Then, solve the system using Cholesky factorization.

$$\begin{aligned} 9x_1 - 15x_2 + 3x_3 &= 57 \\ -15x_1 + 34x_2 - 11x_3 &= -113 \\ 3x_1 - 11x_2 + 9x_3 &= 31 \end{aligned}$$

SOLUTION:

a) We have the system **Ax** = **b** where

$$\mathbf{A} = \begin{bmatrix} 9 & -15 & 3 \\ -15 & 34 & -11 \\ 3 & -11 & 9 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 57 \\ -113 \\ 31 \end{bmatrix}$$

Clearly, **A** is symmetric. Let us check the necessary conditions for positive definiteness of **A**:

1) $a_{ii} > 0$ for all i : satisfied.

2) $a_{ii} + a_{jj} > 2a_{ij}$ for all i,j : satisfied.

3) the element with the largest magnitude lies on the main diagonal: satisfied.

4) $\det(\mathbf{A}) \neq 0$: difficult to establish.

Since **A** does not fail any of the conditions, there is reasonable chance that it is positive definite. We will proceed with Cholesky factorization.

$$\mathbf{R}_1^{\mathrm{T}} = \begin{bmatrix} 3 & 0 & 0 \\ -5 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \qquad \mathbf{A}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 9 & -6 \\ 0 & -6 & 8 \end{bmatrix} \quad (\text{where } \mathbf{A} = \mathbf{R}_1^{\mathrm{T}}\mathbf{A}_1\mathbf{R}_1)$$

$$\mathbf{R}_2^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}, \qquad \mathbf{A}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad (\text{ where } \mathbf{A}_1 = \mathbf{R}_2^{\mathrm{T}}\mathbf{A}_2\mathbf{R}_2)$$

$$\mathbf{R}_3^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \qquad (\text{where } \mathbf{A}_2 = \mathbf{R}_3^{\mathrm{T}}\mathbf{R}_3)$$

Thus $\mathbf{A} = \mathbf{R}^T\mathbf{R}$ where $\mathbf{R} = \mathbf{R}_3\mathbf{R}_2\mathbf{R}_1 = \begin{bmatrix} 3 & -5 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & 2 \end{bmatrix}$.

The next step is to solve $\mathbf{R}^T\mathbf{y} = \mathbf{b}$ by back substitution

$$\begin{bmatrix} 3 & 0 & 0 \\ -5 & 3 & 0 \\ 1 & -2 & 2 \end{bmatrix}\mathbf{y} = \begin{bmatrix} 57 \\ -113 \\ 31 \end{bmatrix} \quad \text{has the solution} \quad \mathbf{y} = \begin{bmatrix} 19 \\ -6 \\ 0 \end{bmatrix}$$

The last step is to solve $\mathbf{Rx} = \mathbf{y}$ by back substitution

$$\begin{bmatrix} 3 & -5 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & 2 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 19 \\ -6 \\ 0 \end{bmatrix} \quad \text{has the solution} \quad \mathbf{x} = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$$