Math 1080, Midterm Exam, Spring 2011
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SOLUTIONS

Problem 1: (25 points)

a) Find the orthogonal projector **P** onto range(**A**) where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ -1 & 1 \end{bmatrix}$$

b) Show that **Q** = **I** – 2**P** is an orthogonal matrix.

SOLUTION:

a) The projector is given by $\mathbf{P} = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}$

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix},$$

$$(\mathbf{A}^T\mathbf{A})^{-1} = \frac{1}{9}\begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix},$$

$$\mathbf{P} = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T = \frac{1}{9}\begin{bmatrix} 5 & 2 & -4 \\ 2 & 8 & 2 \\ -4 & 2 & 5 \end{bmatrix}$$

b) Since **P** is orthogonal projector, it obeys $\mathbf{P}^2 = \mathbf{P} = \mathbf{P}^T$. Therefore

$$\begin{aligned} \mathbf{Q}\mathbf{Q}^T &= (\mathbf{I} - 2\mathbf{P})(\mathbf{I} - 2\mathbf{P})^T \\ &= \mathbf{I} - 2\mathbf{P} - 2\mathbf{P}^T + 4\mathbf{P}\mathbf{P}^T \\ &= \mathbf{I} - 4\mathbf{P} + 4\mathbf{P}^2 \\ &= \mathbf{I} \end{aligned}$$

Problem 2: (30 points)

(a) Compute the QR factorization of the following matrix. Show all the steps.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 4 \\ 2 & 5 & 6 \end{bmatrix}$$

(b) Use the factorization to solve the system $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{b} = [-1 \quad 1 \quad 1]^T$.

SOLUTION:

(a) Using classical Gram-Schmidt orthogonalization procedure we obtain

$$r_{11} = \|\mathbf{a}_1\| = 3, \quad \mathbf{q}_1 = \frac{\mathbf{a}_1}{r_{11}} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix},$$

$$r_{12} = \mathbf{q}_1^{\mathrm{T}}\mathbf{a}_2 = 6, \quad \mathbf{v}_2 = \mathbf{a}_2 - r_{12}\mathbf{q}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad r_{22} = \|\mathbf{v}_2\| = \sqrt{2}, \quad \mathbf{q}_2 = \frac{\mathbf{v}_2}{r_{22}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix},$$

$$r_{13} = \mathbf{q}_1^{\mathrm{T}}\mathbf{a}_3 = 8, \quad r_{23} = \mathbf{q}_2^{\mathrm{T}}\mathbf{a}_3 = \sqrt{2}, \quad \mathbf{v}_3 = \mathbf{a}_3 - r_{13}\mathbf{q}_1 - r_{23}\mathbf{q}_2 = \frac{1}{3}\begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}, \quad r_{33} = \|\mathbf{v}_3\| = \sqrt{2},$$

$$\mathbf{q}_3 = \frac{\mathbf{v}_3}{r_{33}} = \frac{1}{3\sqrt{2}}\begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}$$

The solution is $\mathbf{Q} = \begin{bmatrix} \frac{1}{3} & 0 & \frac{4}{3\sqrt{2}} \\ \frac{2}{3} & \frac{-1}{\sqrt{2}} & \frac{-1}{3\sqrt{2}} \\ \frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{-1}{3\sqrt{2}} \end{bmatrix}$ and $\mathbf{R} = \begin{bmatrix} 3 & 6 & 8 \\ 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & \sqrt{2} \end{bmatrix}$.

(b) The solution of the system $\mathbf{Ax} = \mathbf{b}$ is obtained by solving $\mathbf{Rx} = \mathbf{Q}^T\mathbf{b}$ which here becomes

$$\begin{bmatrix} 3 & 6 & 8 \\ 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & \sqrt{2} \end{bmatrix} \mathbf{x} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{4}{3\sqrt{2}} & \frac{-1}{3\sqrt{2}} & \frac{-1}{3\sqrt{2}} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -\sqrt{2} \end{bmatrix}$$

Solution is $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Problem 3: (25 points)

a) Find the Householder reflector $\mathbf{H}$ that takes the vector $\mathbf{x} = [3 \quad 0 \quad 4]^{\mathrm{T}}$ into $\|\mathbf{x}\|\mathbf{e}_1$.
b) How many operations (flops) did this computation require?

SOLUTION:

a) The Householder reflector is defined as $\mathbf{H} = \mathbf{I} - 2\frac{\mathbf{v}\mathbf{v}^{\mathrm{T}}}{\mathbf{v}^{\mathrm{T}}\mathbf{v}}$, where $\mathbf{v} = \mathbf{x} - \|\mathbf{x}\|\mathbf{e}_1$.

Here

$$\|\mathbf{x}\|\mathbf{e}_1 = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \mathbf{x} - \|\mathbf{x}\|\mathbf{e}_1 = \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}, \quad \mathbf{v}^{\mathrm{T}}\mathbf{v} = 20, \quad \mathbf{v}\mathbf{v}^{\mathrm{T}} = \begin{bmatrix} 4 & 0 & -8 \\ 0 & 0 & 0 \\ -8 & 0 & 16 \end{bmatrix},$$

and hence

$$\mathbf{H} = \mathbf{I} - \frac{1}{10}\mathbf{v}\mathbf{v}^{\mathrm{T}} = \frac{1}{5}\begin{bmatrix} 3 & 0 & 4 \\ 0 & 5 & 0 \\ 4 & 0 & -3 \end{bmatrix}.$$

b) Execution of $\mathbf{v} = \mathbf{x} - \|\mathbf{x}\|\mathbf{e}_1$ required 3 subtractions, 6 operations for the norm and 3 multiplications, total of 12. Execution of $\mathbf{H} = \mathbf{I} - 2\frac{\mathbf{v}\mathbf{v}^{\mathrm{T}}}{\mathbf{v}^{\mathrm{T}}\mathbf{v}}$ requires 9 subtractions, 5 operations for $\mathbf{v}^{\mathrm{T}}\mathbf{v}$, 9 operations for $\mathbf{v}\mathbf{v}^{T}$, 9 divisions and one multiplication, a total of 33. The grand total is 45 operations.

Problem 4: (20 points) Find a second column of Q

$$Q = \begin{bmatrix} 2/\sqrt{5} & \\ -1/\sqrt{5} & \end{bmatrix}$$

so that the resulting matrix is orthogonal and corresponds to

(a) a rotation

(b) a reflection.

SOLUTION:

a) The rotation is given by $Q = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$. The matrix Q has no invariant vector. The angle between a unit vector $\mathbf{x}$ and its image $Q\mathbf{x}$ obeys $\cos\theta = \mathbf{x}^T Q\mathbf{x} = 2/\sqrt{5}$ and is independent of **x.**

(b) The reflection is given by $Q = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ -1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}$. The vector $\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5}-1 \end{bmatrix}$ is invariant and defines the axis of reflection.