

Midterm Exam

Instructor: D. Swigon

Problem 1: Solve the Cauchy problem:

$$yu_y - xu_x = 2xyu, \quad x = s, y = s, z = s$$

Problem 2: Find a weak solution of the initial value problem.

$$u_y + uu_x = 0 \quad u(x,0) = \begin{cases} 1 & x \leq 0 \\ 1-x & 0 < x \leq 1 \\ 0 & x \geq 1 \end{cases}$$

- Sketch characteristic lines in the xy -plane.
- Determine solution for $0 \leq y \leq 1$
- For $R(u) = u$ and $S(u) = u^2/2$ find the corresponding integral equation.
- Find the position of the shock for $1 \leq y$.

Problem 3: Show that the equation:

$$u_{xx} + 4u_{xy} + 3u_{yy} + 3u_x - u_y + 2u = 0$$

is of hyperbolic type. Find its characteristic curves and write the equation in canonical form. Furthermore, using the transformation

$$u(\xi, \eta) = v(\xi, \eta)e^{\alpha\xi + \beta\eta}$$

determine the constants α, β so as to eliminate the terms containing the first derivatives in the resulting canonical form.

Problem 4: Let $u \in C^2(\Omega)$ be subharmonic, i.e.,

$$u(\xi) \leq \frac{1}{\omega_n \rho^{n-1}} \int_{S(\xi, \rho)} u(x) dS \quad \text{for all } \xi \in \Omega, \rho > 0$$

Show that $\Delta u \geq 0$.**Problem 5:** Let $n = 2$, Ω be the half plane $x_2 > 0$. For $\xi = (\xi_1, \xi_2)$ define $\xi^* = (\xi_1, -\xi_2)$

- Show that $G(x, \xi) = K(x, \xi) - K(x, \xi^*)$ is a Green's function for Ω .
- Show that the analogy of Poisson's formula is

$$u(\xi) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\xi_2 f(x_1)}{(x_1 - \xi_1)^2 + \xi_2^2} dx_1 \quad \text{for all } \xi \in \Omega$$

- Show that the formula in b) actually represents a bounded continuous solution of the Dirichlet problem ($\Delta u = 0$ in Ω , $u = f$ on $\partial\Omega$) if f is bounded and continuous.

BONUS PROBLEM: Show that a bounded harmonic function defined in all of R^n is constant. (Hint, use a bound for the derivative $\partial u / \partial x_k$ from the book.)