# HW #7



## Problem 1

$$P = \begin{bmatrix} 1 & 1/3 & 1/3 \\ 0 & 1/3 & 2/3 \\ 0 & 1/3 & 0 \end{bmatrix}$$

(a) Stochastic matrix obeys $\sum_{i=1}^{3} P_{ij} = 1 \quad \forall j$

$1+0+0=1 \qquad 1/3+1/3+1/3=1 \qquad 1/3+2/3+0=1$

(b) The hypothesis requires that
"Some power of $P$ has all positive entries"

Claim: $P$ does not satisfy the hypothesis because
$$(P^k)_{21} = (P^k)_{31} = 0 \quad \text{for all } k>0$$

Proof: We show that $P^k$ is a block-diagonal matrix for $k=1,2,\dots$

$$P^k = \begin{bmatrix} 1 & a_k^T \\ 0 & B_k \end{bmatrix}$$

By induction $P^1 = P = \begin{bmatrix} 1 & a_1^T \\ 0 & B_1 \end{bmatrix}$ where $a_1^T = (1/3 \;\; 1/3)$, $B_1 = \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & 0 \end{bmatrix}$

Suppose $P^{k-1} = \begin{bmatrix} 1 & a_{k-1}^T \\ 0 & B_{k-1} \end{bmatrix}$

Then $P^k = P P^{k-1} = \begin{bmatrix} 1 & a_1^T \\ 0 & B_1 \end{bmatrix} \begin{bmatrix} 1 & a_{k-1}^T \\ 0 & B_{k-1} \end{bmatrix} =$

$$= \begin{bmatrix} 1 & a_{k-1}^T + a_1^T B_{k-1} \\ 0 & B_1 B_{k-1} \end{bmatrix} = \begin{bmatrix} 1 & a_k^T \\ 0 & B_k \end{bmatrix}$$

where $a_k^T = a_{k-1}^T + a_1^T B_{k-1}$

$B_k = B_1 B_{k-1}$

QED

Conclusion: Theorem 5.1 cannot be applied to prove existence & uniqueness & non-extinction.

(c) For $u_0 = [120 \quad 180 \quad 90]^T$ we have

$u_1 = [210 \quad 120 \quad 60]^T$

$u_2 = [270 \quad 80 \quad 40]^T$

(d) Equilibrium

$$(P - I)u^* = 0$$

$$\begin{bmatrix} 0 & 1/3 & 1/3 \\ 0 & -2/3 & 2/3 \\ 0 & 1/3 & -1 \end{bmatrix} u^* = 0 \qquad u^* = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

RREF(

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} u^* = 0 \Rightarrow$$

# 5.8.1 Forest ecosystem

$$P = \begin{bmatrix} 0.12 & 0.14 & 0.12 & 0.12 & 0.13 \\ 0.12 & 0.05 & 0.08 & 0.28 & 0.27 \\ 0.12 & 0.10 & 0.10 & 0.05 & 0.08 \\ 0.42 & 0.53 & 0.32 & 0.20 & 0.19 \\ 0.22 & 0.18 & 0.38 & 0.35 & 0.33 \end{bmatrix}$$

For $u_0 = [0\ 0\ 0\ 0\ 1]$ (all beech)

we have $u_1 = [0.13\quad 0.27\quad 0.08\quad 0.19\quad 0.33]$

$u_2 = [0.1287\quad 0.1778\quad 0.0865\quad 0.3240\quad 0.283]$

$u_3 = [0.1264\quad 0.1984\quad 0.0807\quad 0.2945\quad 0.3000]$

The relative differences each component are

| | | | | | |
|---|---|---|---|---|---|
| $u_1$: | 25% | 38.1% | −2.0% | 36.5% | 11.1% |
| $u_2$: | 1.4% | 9.0% | 6.0% | 8.3% | −4.7% |
| $u_3$: | 0.4% | 1.5% | −1.1% | −1.5% | 1.1% |

The third iteration $u_3$ has all components within 5% of $u^*$.

5.8.3 $X_i$ are random variables, $Y = \sum X_i$,

Assume $X \in S$, $E(X) = \int_S x\rho(x)\,dx$

$$E(Y) = E\left(\sum_i X_i\right) = \sum_i E(X_i) \quad \text{(by linearity of } E\text{)}$$

$$E(Y) = \int_{x_1+x_2 \cdots + x_n = y} y\,\rho(y)\,dy \qquad \rho(y) = \rho_1(x_1)\rho_2(x_2)\cdots\rho_n\left(y - \sum_{i=1}^{n-1} x_i\right)$$

$$= \int\cdots\int (x_1 + x_2 \cdots + x_n)\,\rho_1(x_1)\,\rho_2(x_2)\cdots\rho_n(x_n)\,dx_1\,dx_2 \cdots dx_n$$

$$= \underbrace{\int x_1\rho(x_1)\,dx_1}_{E(X_1)} \underbrace{\int \rho_2(x_2)\,dx_2}_{1} \cdots \underbrace{\int \rho_n(x_n)\,dx_n}_{1} + \int \rho_1(x_1)\,dx_1 \underbrace{\int x_2\rho_2(x_2)\,dx_2}_{E(X_2)} \int \rho_3(x_3)\,dx_3 \cdots \int \rho_n(x_n)\,dx_n$$

$$= E(X_1) + E(X_2) + \cdots + E(X_n)$$

Similarly $\mathrm{Var}(Y) = E(Y^2) - E(Y)^2$

$$E(Y^2) = E\left(\left(\sum X_i\right)^2\right) = \sum_i E(X_i^2) + \sum_{i,j} E(X_i X_j)$$

$$= \sum_i E(X_i^2) + \sum_{i,j} E(X_i)E(X_j)$$

$$E(Y)^2 = \sum_i E(X_i)^2 + \sum_{i,j} E(X_i)E(X_j)$$

$$\mathrm{Var}(Y) = \sum E(X_i^2) - \sum E(X_i)^2 = \sum \mathrm{Var}(X_i)$$

5.86 Explanations of terms

(a)

Model 1:

$\alpha(x-\lambda)\,p(x-\lambda,t)$ = rate at which individuals are arriving at $x$ from $x-\lambda$

$\alpha(x+\lambda)\,p(x+\lambda,t)$ = rate at which individuals arrive at $x$ from $x+\lambda$

$N(x)\,p(x,t)$ = rate at which individuals remain at $x$

$\alpha(x)$ = spatially dependent rate constant for transitions from $x$ to $x+\lambda$ or $x-\lambda$

$N(x)$ = spatially dependent rate constant for individuals remaining at $x$

Model 2:

$\alpha(x-\lambda/2)\,p(x-\lambda,t)$ = rate at which individuals arrive to $x$ from $x-\lambda$

$\alpha(x+\lambda/2)\,p(x-\lambda,t)$ = rate at which individuals arrive to $x$ from $x+\lambda$

$N(x)\,p(x,t)$ = same as above

$N(x)$ = same as above

$\alpha(x+\lambda/2)$ = spatially dependent rate constant for transitions from $x$ to $x+\lambda$ or from $x+\lambda$ to $x$

5.8.6



b) Model 1 : $p(x, t+\tau) = \alpha(x-\lambda)\, p(x-\lambda, t) + \alpha(x+\lambda)\, p(x+\lambda, t) + \big(1-2\alpha(x)\big)\, p(x,t)$

Let $f(x\pm\lambda, t) = \alpha(x\pm\lambda)\, p(x\pm\lambda, t)$

By Taylor expansion

$$f(x\pm\lambda, t) = f(x,t) \pm \frac{\partial f}{\partial x}(x,t)\,\lambda + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x,t)\,\lambda^2 + O(\lambda^3)$$

$$p(x, t+\tau) = p(x,t) + \frac{\partial p}{\partial t}(x,t)\,\tau + O(\tau^2)$$

Substitute and :

$$\cancel{p(x,t)} + \frac{\partial p}{\partial t}(x,t)\,\tau = \cancel{f(x,t)} \cancel{- \frac{\partial f}{\partial x}\lambda} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\lambda^2 + \cancel{f(x,t)} \cancel{+ \frac{\partial f}{\partial x}\lambda} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\lambda^2 + \cancel{p(x,t)} - \cancel{2f(x,t)}$$

$$\frac{\partial p}{\partial t} = \frac{\lambda^2}{\tau}\,\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2}{\partial x^2}\{A(x)\, p\}$$

where $A(x) = \dfrac{\lambda^2}{\tau}\alpha(x)$

Model 2 :

$$p(x, t+\tau) = \alpha(x-\lambda/2)\, p(x-\lambda, t) + \alpha(x+\lambda/2)\, p(x+\lambda, t) + \big[1 - \alpha(x-\lambda/2) - \alpha(x+\lambda/2)\big]\, p(x,t)$$

Taylor expansion

$$p(x\pm\lambda, t) = p \pm \frac{\partial p}{\partial x}\lambda + \frac{1}{2}\frac{\partial^2 p}{\partial x^2}\lambda^2$$

$$\alpha(x\pm\lambda/2, t) = \alpha \pm \frac{\partial \alpha}{\partial x}\frac{\lambda}{2}$$

$$p(x, t+\tau) = p + \frac{\partial p}{\partial t}\tau$$

Substitute

$$p + \frac{\partial p}{\partial t}\tau = \left(\alpha - \frac{\partial \alpha}{\partial x}\frac{\lambda}{2}\right)\left(p - \frac{\partial p}{\partial x}\lambda + \frac{1}{2}\frac{\partial^2 p}{\partial x^2}\lambda^2\right) +$$

$$+ \left(\alpha + \frac{\partial \alpha}{\partial x}\frac{\lambda}{2}\right)\left(p + \frac{\partial p}{\partial x}\lambda + \frac{1}{2}\frac{\partial^2 p}{\partial x^2}\lambda^2\right) +$$

$$+ \left[1 - \left(\alpha - \frac{\partial \alpha}{\partial x}\frac{\lambda}{2}\right) - \left(\alpha + \frac{\partial \alpha}{\partial x}\frac{\lambda}{2}\right)\right] p$$

$$\cancel{p} + \frac{\partial p}{\partial t}\tau = \cancel{\alpha p} - \cancel{\frac{\lambda}{2}\frac{\partial \alpha}{\partial x} p} - \cancel{\lambda \alpha \frac{\partial p}{\partial x}} + \frac{\lambda^2}{2}\frac{\partial \alpha}{\partial x}\frac{\partial p}{\partial x} + \frac{\lambda^2 \alpha}{2}\frac{\partial^2 p}{\partial x^2} - \cancel{\frac{\lambda^3}{4}\frac{\partial \alpha}{\partial x}\frac{\partial p}{\partial x}}$$

$$+ \cancel{\alpha p} + \cancel{\frac{\lambda}{2}\frac{\partial \alpha}{\partial x} p} + \cancel{\lambda \alpha \frac{\partial p}{\partial x}} + \frac{\lambda^2}{2}\frac{\partial \alpha}{\partial x}\frac{\partial p}{\partial x} + \frac{\lambda^2 \alpha}{2}\frac{\partial^2 p}{\partial x^2} + \cancel{\frac{\lambda^3}{4}\frac{\partial \alpha}{\partial x}\frac{\partial p}{\partial x}}$$

$$+ \cancel{p} - \cancel{2\alpha p}$$

$$\frac{\partial p}{\partial t}\tau = \lambda^2 \frac{\partial \alpha}{\partial x}\frac{\partial p}{\partial x} + \lambda^2 \alpha \frac{\partial^2 p}{\partial x^2}$$

$$= \frac{\partial}{\partial x}\left[\lambda^2 \alpha \frac{\partial p}{\partial x}\right]$$

The equation can be rewritten as

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}\left[B(x)\frac{\partial p}{\partial x}\right] \quad \text{where} \quad B(x) = \frac{\lambda^2 \alpha(x)}{\tau}$$

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The first model can be written as

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}\left[\frac{\partial A(x)}{\partial x} p + A(x)\frac{\partial p}{\partial x}\right]$$

<u>Advection term</u> $\frac{\partial}{\partial x}\left[\frac{\partial A}{\partial x} p\right]$ is missing from the second model. It corresponds to additional flux $J = -\frac{\partial A}{\partial x} p$