Fundamental solution:

\[ g(x,t) = \frac{1}{2\sqrt{\pi D t}} e^{-\frac{x^2}{4Dt}} \]

\[ \frac{\partial g}{\partial t} = \frac{-1}{4\pi D} \frac{x^2}{4Dt} e^{-\frac{x^2}{4Dt}} + \frac{1}{2\pi D t} e^{-\frac{x^2}{4Dt}} \left( \frac{x^2}{4Dt} \right)^{-2} g(x,t) \]

\[ \frac{\partial g}{\partial x} = \frac{1}{2\pi D t} e^{-\frac{x^2}{4Dt}} \left( \frac{x}{2Dt} \right) = -\frac{x}{2Dt} g(x,t) \]

\[ \frac{\partial^2 g}{\partial x^2} = -\frac{1}{2Dt} g(x,t) - \frac{x}{2Dt} \frac{\partial g}{\partial x} = \]

\[ = \left( -\frac{1}{2Dt} + \frac{x^2}{4D^2 t^2} \right) g(x,t) \]

\[ \frac{\partial g}{\partial t} - D \frac{\partial^2 g}{\partial x^2} = 0 \Rightarrow g \text{ solves } u_t = Du_{xx} \]

Since \( \frac{1}{2\sqrt{\pi Dt}} > 0 \) and \( e^{-\frac{x^2}{4Dt}} > 0 \) then \( g(x,t) > 0 \) \( \forall x,t \geq 0 \)

As \( t \to 0 \), \( g(x,t) \to 0 \) \( \Rightarrow g(x,t) > 0 \) \( \forall t \geq 0 \)

As \( x \to \pm \infty \), \( g(x,t) \to 0 \) for \( t > 0 \)

By L'Hopital rule, as \( t \to \infty \), \( g(x,t) \to \frac{1}{\infty} = 0 \)
Signal transport

\[ U_t = U_{xx} + U(1-U)(U-\frac{1}{2}) \]

\[ 0 \leq x \leq L \]

\[ u_x(0,t) = 0 \]

\[ u_x(L,t) = 0 \]

(a) Steady state solution -
Assume \( u(x,t) = U(x) \). Let \( V = U' \)

\[ 0 = U'' + U(1-U)(U-\frac{1}{2}) \]

System

\[ U' = V \]

\[ V' = -U(1-U)(U-\frac{1}{2}) = U^3 - \frac{3}{2} U^2 + \frac{1}{2} U \]

(b) Equilibrium points

\( (U^*, V^*) = (0, 0), (\frac{1}{2}, 0), (1, 0) \)

\[ J = \begin{pmatrix} 0 & 1 \\ 3U^2 - 3U + \frac{1}{2} & 0 \end{pmatrix} \]

\[ J(0,0) = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{pmatrix} \]

\( \text{det} J < 0 \)

\( \text{tr} J = 0 \)

\( \Rightarrow \) Saddle

\[ J(\frac{1}{2},0) = \begin{pmatrix} 0 & 1 \\ \frac{1}{4} & 0 \end{pmatrix} \]

\( \text{det} J > 0 \)

\( \text{tr} J = 0 \)

\( \Rightarrow \) Center

\[ J(1,0) = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{pmatrix} \]

\( \Rightarrow \) Saddle

(c) \( H(U, V) = \frac{1}{2} V^2 - \frac{1}{4} U^4 + \frac{1}{2} U^3 - \frac{1}{2} U^2 \)

\[ \frac{\partial H}{\partial U} = -U^3 + \frac{3}{2} U^2 U = -V' \]

\[ \frac{\partial H}{\partial V} = V = U' \]

\( \Rightarrow \) Hamiltonian
\[ H(0,0) = 0 \]
\[ H(1,0) = 0 \]

(a) Steady solutions must satisfy

\[ U_x(0, t) = U_x(L, t) = 0 \quad \text{i.e.,} \quad V(0) = V(L) = 0 \]

They are half-period solutions as shown below.

or multiple period solutions comprising \( n \) rotations about the center point.

As functions of \( x \):

- \( n = 1 \)
- \( n = 2 \)
- \( n = 3 \)

etc.

(b) The solutions represent standing waves.
4.5.7 Linear Transport

\[ U_t + C U_x = 0 \quad U(x,0) = U_0(x) \quad (x) \]

(1) If \( U(x,t) \) represents a surface over \((x,t)\) plane then \( DU = \begin{pmatrix} U_x \\ U_t \end{pmatrix}\) is the direction of steepest ascent along this surface. The equation above can be written as

\[ \begin{pmatrix} C_1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} U_x \\ U_t \end{pmatrix} = 0 \]

Thus the steepest ascent direction is orthogonal to the vector \((C_1,1)\).

(b) Let \( \Gamma_k = \{(t,x(t)) : x(t) = k + C t, \ t \geq 0\} \)

Then, along the curve

\[ \frac{d}{dt} U(x(t),t) = \frac{\partial U}{\partial x} \frac{dx(t)}{dt} + \frac{\partial U}{\partial t} = \frac{\partial U}{\partial x} C + \frac{\partial U}{\partial t} = 0 \]

Thus \( U \) is constant along \( x(t) \)

(c) The solution is given by

\[ U(x(t),t) = \text{constant} \]

\[ U(k + C t, t) = f(k) = U(k,0) = U_0(k) \]

Solution: \( U(x,t) = U_0(x-Ct) \)