\[ P_{n+1} = P_n + \delta P_n - \mu P_n = (1 + \delta - \mu) P_n \]

\[ P_0 \text{ to Dec. 31, 1998} \]

\[ P_0 = 82,037,000 \]

\[ \Delta P_0 = P_1 - P_0 = 770,744 \text{ births} - 846,330 \text{ deaths} \]

\[ \delta P_0 \quad \mu P_0 \]

\[ \delta = \frac{770,744}{P_0} = 0.0094 \]

\[ \mu = \frac{846,330}{P_0} = 0.0103 \]

Future population

\[ P_n = (1 + \delta - \mu)^n P_0 = 0.9991^n P_0 \]

The population will decrease exponentially to 0.

Realistic model should account for:
- Time dependence of \( \delta \) and \( \mu \)
- Immigration (influx of population)
\[ a_{n+1} = a_n + k \left( M - a_n \right) \left( a_n - m \right) \]

Assume \( M > m \)

Population movement
\[ \Delta a_n = k \left( M - a_n \right) \left( a_n - m \right) \]

\[ \Delta a_n > 0, \text{ i.e. population increases} \]
if \( m < a_n < M \)

\[ \Delta a_n < 0, \text{ i.e. population decreases} \]
if \( a_n > M \) or \( a_n < m \)

The model corresponds to the description.

\[ x^* = x^* + k \left( M - a^* \right) \left( a^* - m \right) \]

Fixed points \( x^* = m \) or \( x^* = M \)

Assume
\[ M = 5000 \]
\[ m = 100 \]
\[ k = 0.0001 \]

\[ f'(a) = 1 - k \left( a - m \right) + k \left( M - a \right) \]

\[ f'(M) = 1 - k \left( M - m \right) = 0.51 \]

\[ \alpha = M \] is stable

\[ f'(m) = 1 + k \left( M - m \right) = 1.49 \] \( \alpha = m \) is unstable

See next page
See next page

When \( a_0 > M \) then the increment \( \Delta a_n \)
is negative and may become larger in magnitude than \( a_n \)
which would lead to \( a_{n+1} < 0 \) \( \Leftrightarrow \) not acceptable.
When $a_0 < m$ then $a_{n+1} < a_n$ and eventually $a_n$ becomes so small that $|\Delta a_n| < a_n$, $\Delta a_n < 0$ which will result in $a_{n+1} < 0 \not\in \text{acceptable}$ (For $a_0 = 99$ this happens at $t = 11$)

(f) The model must be corrected so that $\Delta a_i < a_i$ for all $a_i$.

For example, $a_{n+1} = a_n + \frac{2}{a_n^2} (1-a) (a_n - m)$ has this property for $\frac{2}{(1-a) (m-a)} < \max f(a)$.

\[ f(a) \]
\[ \text{new } f(a) \]
\[ \text{original } f(a) \]
2.4.7 There are many ways to solve this problem. Here is an example.

(a) Examples of $f_4(x)$ dynamics for $r = 2.5, 3, 3.5, 4$.

(Graphs obtained using XPP app for iPad).
The plot of $f^4(x)$ shows that the four period orbit appears by period doubling bifurcation within the range $3.44 < r < 3.46$.

Bifurcation plot obtained using MATLAB gives the same results.

```matlab
syms r x
f = r*x*(1-x)
f2 = subs(f,x,f)
f3 = subs(f2,x,f)
f4 = subs(f3,x,f)
figure
subplot(1,2,1)
ezplot(f4-x,[2 4 0 1])
title('x=f^4(x)');
subplot(1,2,2)
ezplot(f4-x,[3.44 3.46 0 1])
title('x=f^4(x)');
```
(c) Cobweb diagram of \( f_4(x) \) shows that the four period orbit becomes unstable within the range \( 3.54 < r < 3.55 \).

Stability plot obtained using MATLAB gives the same results: Here green shows the bifurcation diagram and red the values of \((r,x)\) at which \(|df_4(x)/dx| = 1\).

```matlab
syms r x
f = r*x*(1-x)
f2 = subs(f,x,f)
f3 = subs(f2,x,f)
f4 = subs(f3,x,f)
figure
subplot(1,2,1)
ezplot(f4-x,[3.4 3.6 0.3 0.55]); hold on
h = ezplot(abs(diff(f4))-1,[3.4 3.6 0.3 0.55]);
set(h,'Color',[1 0 0])
title('x=f^4(x) (green); |df^4(x)/dx|=1 (red)');
subplot(1,2,2)
ezplot(f4-x,[3.54 3.56 0.3 0.55]); hold on
h = ezplot(abs(diff(f4))-1,[3.4 3.6 0.3 0.55]);
set(h,'Color',[1 0 0])
title('x=f^4(x) (green); |df^4(x)/dx|=1 (red)');
```
\[
f(x) = \begin{cases} \mu x & \text{for } 0 \leq x \leq 0.5 \\ \mu (1-x) & \text{for } 0.5 \leq x \leq 1 \end{cases}
\]

(6) Steady States

\[
x' = \mu x^* \quad \text{for } \quad 0 \leq x \leq 0.5 \Rightarrow x^* = \frac{\mu}{\mu + 1} \quad \text{for } \mu < 1
\]

\[
x' = \mu (1-x^*) \quad \text{for } \quad 0.5 \leq x \leq 1
\]

\[
x' = \mu - \mu x^* \quad 0.5 \leq \frac{\mu}{1+\mu} \leq 1
\]

\[
\left(1-\mu\right) \leq \mu \leq 1+\mu \Rightarrow \mu \geq \frac{1}{2}
\]

<table>
<thead>
<tr>
<th>[\mu &lt; 1]</th>
<th>[\mu = 1]</th>
<th>[\mu &gt; 1]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x' = 0$</td>
<td>$0 \leq x^* \leq 0.5$</td>
<td>$x' = 0$</td>
</tr>
<tr>
<td>$x' = \frac{\mu}{1+\mu}$</td>
<td>$x' = 0$</td>
<td>$x' = \frac{\mu}{1+\mu}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Stable</th>
<th>Unstable</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f'(x) = \mu - \mu x^*$</td>
<td>$f'(x) = \mu$</td>
</tr>
</tbody>
</table>

\[
\frac{1}{2} \leq \frac{\mu}{2}
\]
\[ f^2(x) = \begin{cases} 
\mu f(x) & \text{for } 0 \leq f(x) \leq 0.5 \\
\mu (1-f(x)) & \text{for } 0.5 \leq f(x) \leq 1
\end{cases} \]

\[ \text{and } \mu > 1 \Rightarrow f(x) \leq 0.5 \]

\[ \forall x = \mu^2 x^* \Rightarrow \text{some a fixed point } x^* = 0 \]

\[ \mu > 1 \]

\[ f^2(x) = \begin{cases} 
\mu^2 x & \text{for } 0 \leq x \leq \frac{1}{2\mu} \\
\mu (1-\mu x) & \frac{1}{2\mu} \leq x \leq \frac{1}{\mu} \\
\mu^2 (1-\mu x) & \frac{1}{\mu} \leq x \leq 1
\end{cases} \]

\[ \text{for } \mu > 1 \]

\[ \text{Observe:} \]

1. \[ x = \mu^2 x^* \text{ for } 0 \leq x \leq \frac{1}{2\mu} \Rightarrow \text{original fixed point.} \]

2. \[ x = \mu (1-\mu x) \text{ for } \frac{1}{2\mu} \leq x \leq \frac{1}{\mu} \]

\[ x = \frac{\mu}{1-\mu^2} \]

\[ 0 \leq \mu \leq 1 \]

\[ 1 = \frac{2\mu^2}{1-\mu^2} \leq \mu \]

\[ 1-\mu^2 \leq 2\mu^2 \leq \mu + \mu^2 \]

\[ 0 \leq 1-\mu^2 \leq 1 \]

\[ (1, \mu^2) \]
(c) \[ x = \mu^2(1 - 2x) \]
\[ x = \mu^2 - \mu^2 + \mu^2 x \]
\[ x = \mu \frac{1 - \mu}{1 + \mu^2} = \frac{\mu}{1 + \mu^2} \rightarrow \text{fixed point} \]
\[ \frac{1}{2} \leq x \leq 1 - \frac{1}{2\mu} \]

(d) \[ x = \mu^2(1 - x) \]
\[ x = \frac{\mu^2}{1 + \mu^2} \text{ corresponds to } \frac{\mu}{1 + \mu^2} \]

2 periodic orbits oscillates between \[ \frac{\mu}{1 + \mu^2} \text{ and } \frac{\mu^2}{1 + \mu^2} \]

It exists for \( \mu > 1 \) and is unstable.

\[ f^3(x) \text{ for } \mu = 2 \]

- Fixed point: \( x = \frac{2}{3} \)
- There are \( \frac{3}{2} \) periodic orbits

\[ \circ : x = 2 - 8x \Rightarrow \left( \begin{array}{c} \frac{2}{9} \\ \frac{4}{9} \\ \frac{8}{9} \end{array} \right) \]

\[ \circ : x = -2 + 8x \Rightarrow \left( \begin{array}{c} \frac{2}{7} \\ \frac{4}{7} \\ \frac{6}{7} \end{array} \right) \]