6.1.3: Not a linear vector space, axioms 5 and 6 fail to hold.
6.1.5: Not a linear vector space, axiom 8 fails to hold for $c = 1, d = -1$

6.1.27: Not a subspace. Condition a) of Theorem 6.2 fails to hold for $u = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $v = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

6.1.33: Is a subspace.
6.1.36: Not a subspace. Condition a) of Theorem 6.2 fails to hold for $p(x) = 1 + x, q(x) = 1 + x^2$.

6.1.48: a) $U + W$ is the $(x,y)$ plane
6.1.48: b) Suppose $v_1 \in U + W$ and $v_2 \in U + W$. By definition, $v_i = u_i + w_i$ where $u_i \in U$, $w_i \in W$. It follows that $av_1 + bv_2 = au_1 + bu_2 + av_1 + bv_2$, where $au_1 + bu_2 \in U$ and $av_1 + bv_2 \in V$ and hence $av_1 + bv_2 \in U + W$. Thus, $U + W$ is a subspace of $V$.

6.2.7: $c_1x + c_2(x^2) + c_3(x^2 + x^3) = 0$ for all $x$ is equivalent to $c_1 + 2c_2 + 3c_3 = 0$ and $-c_2 + 2c_3 = 0$. These conditions are not sufficient to guarantee $c_1 = c_2 = c_3 = 0$ and hence the vectors $\{x, 2x - x^2, 3x + 2x^2\}$ are linearly dependent. For example,

$3x + 2x^2 = 7(x - 2(2x - x^2))$.

6.2.14: The functions $\{\sin x, \sin 2x, \sin 3x\}$ are linearly independent since the matrix

$$
\begin{bmatrix}
\sin \pi/2 & \sin 2(\pi/2) & \sin 3(\pi/2) \\
\sin \pi/4 & \sin 2(\pi/4) & \sin 3(\pi/4) \\
\sin \pi/3 & \sin 2(\pi/3) & \sin 3(\pi/3)
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & -1 \\
1/\sqrt{2} & 1 & 1/\sqrt{2} \\
\sqrt{3}/2 & \sqrt{3}/2 & 0
\end{bmatrix}
$$
is invertible

6.2.23: $B$ is not a basis because the vectors are linearly dependent: $1 - x - (1 - x^2) + (x - x^2) = 0$

6.2.27: The coordinates satisfy the equation

$$
\begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix}
= a_1 \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} + a_2 \begin{bmatrix}
1 & 1 \\
0 & 0
\end{bmatrix} + a_3 \begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix} + a_4 \begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}
= \begin{bmatrix}
a_1 + a_2 + a_3 + a_4 & a_2 + a_3 + a_4 \\
a_3 + a_4 & a_4
\end{bmatrix}
$$

Solution: $a_1 = -1, a_2 = -1, a_3 = -1, a_4 = 4$.

6.2.37: $B = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}$, Dimension is 3.

6.4.3: $T(aA + bB) = (aA + bB)B = aAB + bCB = aT(A) + bT(C)$. $T$ is a linear transformation.

6.4.10: $T(af(x) + bg(x)) = af(x^2) + bg(x^2) = aT(f(x)) + bT(g(x))$. $T$ is a linear transformation.

6.4.16:

$$
T(a + bx + cx^2) = aT(1) + bT(x) + cT(x^2)
= a(3 - 2x) + b(4x - x^2) + c(2 + 2x^2)
= 3a + 2c + (-2a + 4b)x + (-b + 2c)x^2
$$

$$
T(6 + x - 4x^2) = 10 - 8x - 9x^2
$$
6.4.22: The identity transformation obeys \( id(w) = w \) for all \( w \) in \( V \). Let \( w \) be such a vector. Since \( \{v_1, \ldots, v_n\} \) is a basis of \( V \), we can write \( w = \sum_i a_i v_i \). Then, by linearity of \( T \),

\[
T(w) = T(\sum_i a_i v_i) = \sum_i a_i T(v_i) = \sum_i a_i v_i = w.
\]

6.4.26: We have

\[
(S \circ T)(3 + 2x - x^2) = S(T(3 + 2x - x^2))
\]

\[
= S(2 - 2x)
\]

\[
= 2 - 4x^2
\]

\[
(S \circ T)(a + bx + cx^2) = S(T(a + bx + cx^2))
\]

\[
= S(b + 2cx)
\]

\[
= b + (b + 2c)x + 4cx^2
\]

\[
(T \circ S)(a + bx^2) = T(S(a + bx))
\]

\[
= T(a + (a + b)x + 2bx^2)
\]

\[
= a + b + 4bx
\]
Let $T(A) = AB - BA$ where $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

By definition, $\text{range}(T) = \{ C \mid C = AB - BA, A \in M_{22} \}$. It follows that:

$$C = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} a-b & b-a \\ c-d & d-c \end{bmatrix} - \begin{bmatrix} a-c & b-d \\ c-a & d-b \end{bmatrix} = \begin{bmatrix} c-b & d-a \\ a-d & b-c \end{bmatrix}$$

Thus, $\text{range}(T) = \left\{ \begin{bmatrix} p & q \\ -q & -p \end{bmatrix} \right\} | p, q \in \mathbb{R} \}$, $\text{rank}(T) = \dim(\text{range}(T)) = 2$. By Rank Theorem, $\text{nullity}(T) = \dim(M_{22}) - \text{rank}(T) = 4 - 2 = 2$.

**6.5.17:** Let $T(a + bx + cx^2) = \begin{bmatrix} 2a-b \\ a+b-3c \\ c-a \end{bmatrix}$.

(a) Linear transformation $T$ is one-to-one if and only if $\text{nullity}(T) = 0$.

By definition, $\text{ker}(T) = \{ a + bx + cx^2 \mid 2a-b = 0, a+b-3c = 0, c-a = 0 \}$. The system $2a-b = 0, a+b-3c = 0, c-a = 0$ has a solution $a, b = 2a, c = a$ for all $a$. Thus, $\text{nullity}(T) = \dim(\text{ker}(T)) = 1$ and the linear transformation $T$ is not one-to-one.

(b) Linear transformation $T$ is onto if $\text{range}(T) = \mathbb{R}^3$. Here, in addition, $T : P_2 \to \mathbb{R}^3$, and $\dim(P_2) = \dim(\mathbb{R}^3) = 3$. Thus, $T$ is onto if and only if $T$ is one-to-one. In view of (a), $T$ is not one-to-one and hence is not onto.

**6.5.20:** Let $T\left( \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) = \begin{bmatrix} a+b+c \\ b-2c \\ b-2c \end{bmatrix}$.

(a) Linear transformation $T$ is one-to-one if and only if $\text{nullity}(T) = 0$. 
By definition, \( \ker(T) = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right\} \left[ \begin{bmatrix} a + b + c & b - 2c \\ b - 2c & a - c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \). The system

\[
a + b + c = 0, b - 2c = 0, a - c = 0
\]

has a solution \( a = b = c = 0 \). Thus \( \text{nullity}(T) = \dim(\ker(T)) = 0 \) and the linear transformation \( T \) is one-to-one.

(b) Linear transformation \( T \) is onto if \( \text{range}(T) = W \). Here, in addition, \( T : \mathbb{R}^3 \rightarrow W \), and \( \dim(W) = \dim(\mathbb{R}^3) = 3 \). Thus, \( T \) is onto if and only if \( T \) is one-to-one. In view of (a), \( T \) is one-to-one and hence it is onto.