HW3

3.1.10

To find the minimum and maximum growth rate, use the first derivative test on the growth rate dP/dt. That is, set $d^2P/dt^2 = 0$.

Obtain: $$\frac{d^2P}{dt^2} = rP'\left(1 - \frac{2P}{k}\right)$$

Thus, dP/dt reaches an extremum when $P' = 0$ and when $P = k/2$.

We want to show that dP/dt is a maximum when $P = \frac{k}{2}$. So we check that d^2P/dt^2 changes sign from positive to negative at $P = k/2$. When $P = k/2$, $P' = r(k/2)(1 - (k/2)/k) = rk/4$ which is positive. So any possible change in sign of d^2P/dt^2 is in the third factor $(1 - 2P/k)$. This factor does change sign from positive to negative at $P = k/2$. Therefore dP/dt reaches a maximum at $P = k/2$.

3.1.13

The carrying capacity $K = 20,000$, and the initial condition $P_0 = 1000$, and it is given that $P(8) = 1200$. Using equation (1.13), we obtain:

$$1200 = \frac{(20000)(1000)}{1000 + (19000)e^{-8r}}$$

$$\Rightarrow r = \left(-\frac{1}{8}\right)\ln\left(18800/(12)(19000)\right) \approx 0.0241$$

Now, find t so that $P(t) = 3K/4 = 15000$. This gives, $e^{-rt} = \frac{1}{57}$.

That is, $t = (\ln 57)/r \approx \boxed{167.6713}$

3.3.3

Let $P(t)$ represent the balance in the account t years after the initial investment. Let r represent the annual rate, d the yearly deposit, and P_0 the initial investment. Then,

$$P' = rP + d \quad , \quad P(0) = P_0$$

This equation is linear, with integrating factor e^{-rt}.

Consequently, $(e^{-rt}P)' = de^{-rt}$

$$e^{-rt}P = -\frac{d}{r}e^{-rt} + C$$

$$P = -\frac{d}{r} + Ce^{rt}$$

Use $P(0) = P_0$ to produce $C = P_0 + d/r$ and $P(t) = -\frac{d}{r} + \left(P_0 + \frac{d}{r}\right)e^{rt}$

Thus, the amount in the account after 10 years is

$$P(10) = -\frac{1200}{0.06} + \left(5000 + \frac{1200}{0.06}\right)e^{0.06(10)}$$

$$\approx \$\boxed{25,553}$$

3.3.5

Let $P(t)$ represent the balance in the account after t years. Let r represent the annual rate, w the yearly withdrawal, and P_0 the amount of the inheritance. Then $P' = rP - w$, $P(0) = P_0$

The equation is linear with integrating factor e^{-rt}. Consequently,

$$(e^{-rt}P)' = -we^{-rt}$$

$$e^{-rt}P = \frac{w}{r}e^{-rt} + C$$

$$P = \frac{w}{r} + Ce^{rt}$$

Use $P(0) = P_0$ to produce $C = P_0 - \frac{w}{r}$ and

$$P(t) = \frac{w}{r} + \left(P_0 - \frac{w}{r}\right)e^{rt}$$

Now to find when the funds are depleted, set $P(t) = 0$

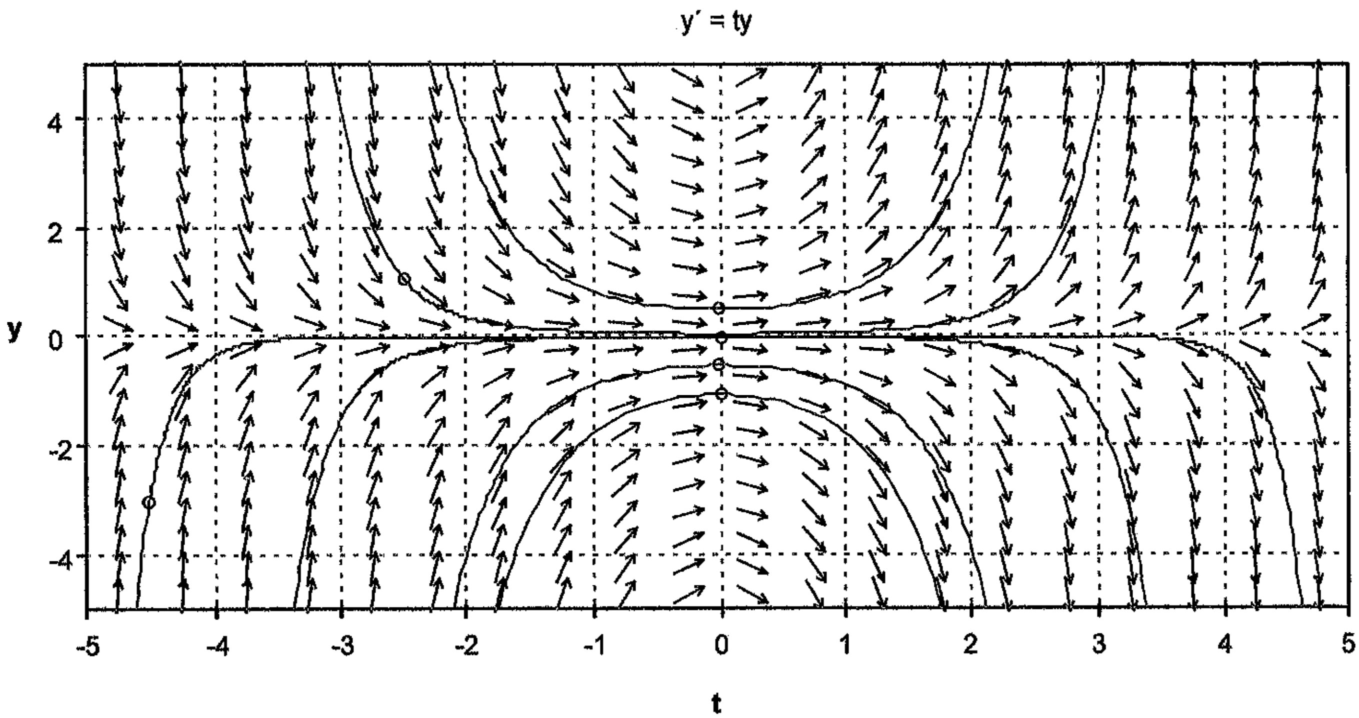
$$0 = \frac{w}{r} + \left(P_0 - \frac{w}{r}\right)e^{rt}$$

$$e^{rt} = \frac{w/r}{w/r - P_0}$$

$$t = \frac{1}{r}\ln\frac{w/r}{w/r - P_0}$$

Thus, the account will be depleted in

$$t = \frac{1}{0.05}\ln\frac{8000/0.5}{8000/0.5 - 50000} \approx \boxed{7.5} \text{ years}$$



y' = ty
y
t
4
2
0
-2
-4
-5
-4
-3
-2
-1
0
1
2
3
4
5