Problem 1: Let $f(x)$ and $g(x)$ have compact support. Show that the solution $u(x,t)$ of
\[ u_t - c^2 u_{xx} = 0, \quad u(x,0) = f(x), \quad u_t(x,0) = g(x) \]
has compact support in $x$ for each fixed $t$. Show that the functions $F, G$ in the decomposition $u(x,t) = F(x+ct) + G(x-ct)$ have compact support only if
\[ \int_{-\infty}^{\infty} g(x) dx = 0. \]

Problem 2: Solve the initial-boundary-value problem
\[
\begin{align*}
&u_t = u_{xx}, \quad \text{for } 0 < x < \pi, \ 0 < t \\
&u = 0, \quad \text{for } x = 0, \pi; \ 0 < t \\
&u = 1, \quad u_t = 0 \quad \text{for } 0 < x < \pi; \ t = 0
\end{align*}
\]

Problem 3: (Legendre transformation) Let $u(x,y)$ be a solution of a quasi-linear equation
\[ a(u_x, u_y)u_{xx} + 2b(u_x, u_y)u_{xy} + c(u_x, u_y)u_{yy} = 0 \]
Introduce new independent variables $\xi, \eta$ and a new function $\phi$ such that
\[ \xi = u_x(x,y), \quad \eta = u_y(x,y), \quad \phi = xu_x + yu_y - u \]
Show that $\phi = \phi(\xi,\eta)$ satisfies $x = \phi_x, y = \phi_\eta$ and the linear differential equation
\[ a(\xi,\eta)\phi_{\eta\eta} - 2b(\xi,\eta)\phi_{\xi\eta} + c(\xi,\eta)\phi_{\xi\xi} = 0 \]