**Math 1080: Spring 2011**

**Homework #5**

**SOLUTIONS**

**Problem 1:** Compute LU factorization of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 & 4 \\ -2 & -2 & -1 & -4 \\ 0 & -2 & -2 & 5 \\ -2 & 2 & -6 & 7 \end{bmatrix}$$

$$\mathbf{L}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{L}_1\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 2 & -3 & 4 \\ 0 & -2 & -2 & 5 \\ 0 & 6 & -8 & 15 \end{bmatrix}$$

$$\mathbf{L}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix}, \qquad \mathbf{L}_2\mathbf{L}_1\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 2 & -3 & 4 \\ 0 & 0 & -5 & 9 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\mathbf{L}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/5 & 1 \end{bmatrix}, \qquad \mathbf{U} = \mathbf{L}_3\mathbf{L}_2\mathbf{L}_1\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 2 & -3 & 4 \\ 0 & 0 & -5 & 9 \\ 0 & 0 & 0 & 24/5 \end{bmatrix}$$

$$\mathbf{L} = (\mathbf{L}_3\mathbf{L}_2\mathbf{L}_1)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -2 & 3 & -1/5 & 1 \end{bmatrix}$$

Problem 2:

Solve the following system of equations by LU factorization and QR factorization:

$$\begin{aligned} 2x_1 + 8x_2 + 3x_3 &= 2 \\ x_1 + 3x_2 + 2x_3 &= 5 \\ 2x_1 + 7x_2 + 4x_3 &= 8 \end{aligned}$$

*SOLUTION*:

The system of equations can be written as **Ax** = **b**, where

$$\mathbf{A} = \begin{bmatrix} 2 & 8 & 3 \\ 1 & 3 & 2 \\ 2 & 7 & 4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

**By LU factorization:** The first step of the solution is to find the LU factorization of **A**:

$$\mathbf{L}_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad \mathbf{L}_1\mathbf{A} = \begin{bmatrix} 2 & 8 & 3 \\ 0 & -1 & 1/2 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\mathbf{L}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad \mathbf{U} = \mathbf{L}_2\mathbf{L}_1\mathbf{A} = \begin{bmatrix} 2 & 8 & 3 \\ 0 & -1 & 1/2 \\ 0 & 0 & 1/2 \end{bmatrix}, \quad \mathbf{L} = \mathbf{L}_1^{-1}\mathbf{L}_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

The second step is to find the solution **y** of the equation **Ly** = **b** by back substitution:

$$\mathbf{Ly} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\mathbf{y} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

$$y_1 = 2, \quad y_2 = 5 - y_1/2 = 4, \quad y_3 = 8 - y_1 - y_2 = 2$$

The last step is to find the solution **x** of the equation **Ux** = **y** by back substitution:

$$\mathbf{Ux} = \begin{bmatrix} 2 & 8 & 3 \\ 0 & -1 & 1/2 \\ 0 & 0 & 1/2 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$$

$$x_3 = 4, \quad x_2 = -(4 - x_3/2) = -2, \quad x_1 = (2 - 8x_2 - 3x_3)/2 = 3;$$

$$\mathbf{x} = \begin{bmatrix} 3 & -2 & 4 \end{bmatrix}^{\mathrm{T}}$$

**By QR factorization:** The first step of the solution is to find the QR factorization of **A**:

$$\mathbf{A} = \begin{bmatrix} 2 & 8 & 3 \\ 1 & 3 & 2 \\ 2 & 7 & 4 \end{bmatrix}$$

Using the classical Gram-Schmidt orthogonalization we compute:

$$r_{11} = \|a_{11}\| = 3, \quad \mathbf{q}_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$r_{12} = \mathbf{q}_1{}^T \mathbf{a}_2 = 11, \quad \mathbf{v} = \mathbf{a}_2 - r_{12}\mathbf{q}_1 = \frac{1}{3}\begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}, \quad r_{22} = \|\mathbf{v}\| = 1, \quad \mathbf{q}_1 = \frac{1}{3}\begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$$

$$r_{13} = \mathbf{q}_1{}^T \mathbf{a}_2 = -16/3, \quad r_{23} = \mathbf{q}_2{}^T \mathbf{a}_2 = -2/3 \quad \mathbf{v} = \mathbf{a}_3 - r_{13}\mathbf{q}_1 - r_{23}\mathbf{q}_2 = \frac{1}{9}\begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix},$$

$$r_{33} = \|\mathbf{v}\| = 1/3, \qquad \mathbf{q}_1 = \frac{1}{3}\begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$$

Thus, the resulting matrices **Q** and **R** are

$$\mathbf{Q} = \frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ 1 & -2 & -2 \\ 2 & -1 & 2 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} -3 & -11 & -16/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1/3 \end{bmatrix}$$

The last step is to find the solution **x** of the equation $\mathbf{Rx} = \mathbf{Q}^T\mathbf{b}$ by back substitution:

$$\mathbf{Rx} = \begin{bmatrix} -3 & -11 & -16/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1/3 \end{bmatrix}\mathbf{x} = \mathbf{Q}^T\mathbf{b} = \begin{bmatrix} -25/3 \\ -14/3 \\ 4/3 \end{bmatrix}$$

$$x_3 = 4, \quad x_2 = -14/3 + (2/3)x_3 = -2, \quad x_1 = (-25/3 + (16/3)x_3 + 11x_2)/(-3) = 3;$$

$$\mathbf{x} = \begin{bmatrix} 3 & -2 & 4 \end{bmatrix}^{\mathrm{T}}$$

**Computer Assignment 3:**

a) Write a MATLAB function **`[L,U]=gauss(A)`** that computes the LU factorization of a square $m \times m$ matrix **A** using Gaussian elimination. The output variables are a lower triangular $m \times m$ matrix **L** and an upper triangular $m \times m$ matrix **U**.

b) Compute the LU factorization of the following matrix

$$\mathbf{A} = \begin{bmatrix} 3 & -8 & -5 & -6 & -7 \\ 3 & -1 & 7 & -6 & 6 \\ 4 & 4 & -6 & -10 & -7 \\ -1 & 7 & 6 & -9 & -7 \\ 1 & -5 & 8 & 2 & 9 \end{bmatrix}$$

*PROGRAM:*

```
function [L,U] = gauss(A)
m = length(A);
L = eye(m);
for k = 1:m-1
    for j = k+1:m
        L(j,k) = A(j,k)/A(k,k);
        A(j,k:m) = A(j,k:m) - L(j,k)*A(k,k:m);
    end
end
U = A;
```

*OUTPUT:*

```
>> A = [3 -8 -5 -6 -7; 3 -1 7 -6 6; 4 4 -6 -10 -7; -1 7 6 -9 -7;
1 -5 8 2 9]

A =

     3    -8    -5    -6    -7
     3    -1     7    -6     6
     4     4    -6   -10    -7
    -1     7     6    -9    -7
     1    -5     8     2     9

>> [L,U] = gauss(A)
```

```
L =

    1.0000         0         0         0         0
    1.0000    1.0000         0         0         0
    1.3333    2.0952    1.0000         0         0
   -0.3333    0.6190    0.1265    1.0000         0
    0.3333   -0.3333   -0.5584   -0.2683    1.0000


U =

    3.0000   -8.0000   -5.0000   -6.0000   -7.0000
         0    7.0000   12.0000         0   13.0000
         0   -0.0000  -24.4762   -2.0000  -24.9048
         0         0         0  -10.7471  -14.2315
         0         0   -0.0000   -0.0000   -2.0574
```