Math 1080: Spring 2011

Homework #4

Problem 1:

Determine the relative condition number for the following mathematical problems:

a) $f(x) = \cos(x)$

b) $f(\mathbf{x}) = \|\mathbf{x}\| = \sqrt{\sum_{i=1}^{n} x_i^2}$

c) $f(x) = \dfrac{1}{1+x^2}$

SOLUTION:

Bu definition, for a differentiable scalar function *f* of vector argument **x**:

$$\kappa(\mathbf{x}) = \|\mathbf{J}(\mathbf{x})\| \frac{\|\mathbf{x}\|}{\|f(\mathbf{x})\|}, \quad \text{where} \quad \mathbf{J}(\mathbf{x}) = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} & \dfrac{\partial f}{\partial x_2} & \cdots & \dfrac{\partial f}{\partial x_n} \end{bmatrix}$$

a)

$$J(x) = \frac{d}{dx} f(x) = -\sin x$$

$$\kappa(x) = |-\sin x| \frac{|x|}{|\cos x|} = |x \tan x|$$

b)

$$J_i(\mathbf{x}) = \frac{\partial f}{\partial x_i} = \frac{x_i}{\sqrt{\sum_{i=1}^{n} x_i^2}}, \qquad \|\mathbf{J}(\mathbf{x})\| = \frac{1}{|f(\mathbf{x})|}\|\mathbf{x}\| = 1$$

$$\kappa(\mathbf{x}) = 1 \frac{\|\mathbf{x}\|}{|f(\mathbf{x})|} = 1 \ .$$

c)

$$J(x) = \frac{d}{dx} f(x) = \frac{-2x}{(1+x^2)^2}$$

$$\kappa(x) = \frac{2|x|}{(1+x^2)^2} \frac{|x|}{(1+x^2)^{-1}} = \frac{2x^2}{1+x^2}$$

Problem 2:

Determine whether the following algorithms are backward stable, stable, or unstable:

a) Computation of $f(x) = \dfrac{1}{1+x}$ as $\tilde{f}(x) = 1 \oslash (1 \oplus \mathrm{fl}(x))$

b) Computation of $f(x, y) = x^2 - y^2$ as $\tilde{f}(x, y) = [\mathrm{fl}(x) \otimes \mathrm{fl}(x)] \ominus [\mathrm{fl}(y) \otimes \mathrm{fl}(y)]$

c) Computation of $f(x, y) = x^2 - y^2$ as $\tilde{f}(x, y) = [\mathrm{fl}(x) \oplus \mathrm{fl}(y)] \otimes [\mathrm{fl}(x) \ominus \mathrm{fl}(y)]$

SOLUTION:

a) Using the properties of $\varepsilon_{\text{machine}}$ we find that there exist ε_i, $i = 1...5$ such that $|\varepsilon_i| \le \varepsilon_{\text{machine}} + O(\varepsilon_{\text{machine}}^2)$ and

$$
\begin{aligned}
\tilde{f}(x) &= 1 \oslash (1 \oplus \mathrm{fl}(x)) \\
&= 1 \oslash \big(1 \oplus x(1+\varepsilon_1)\big) \\
&= 1 \oslash \big(1 + x(1+\varepsilon_1)\big)(1+\varepsilon_2) \\
&= \frac{(1+\varepsilon_3)}{\big(1 + x(1+\varepsilon_1)\big)(1+\varepsilon_2)} \\
&= \frac{(1+\varepsilon_3)(1+\varepsilon_4)}{1 + x(1+\varepsilon_1)} \\
&= \frac{(1+2\varepsilon_5)}{1+\tilde{x}}
\end{aligned}
$$

Therefore,

$$
\frac{|\tilde{f}(x) - f(\tilde{x})|}{|f(\tilde{x})|} \le 2\varepsilon_{\text{machine}} \qquad \text{for some } \tilde{x} \text{ such that } \qquad \frac{|\tilde{x} - x|}{|x|} \le \varepsilon_{\text{machine}}
$$

and the algorithm is stable but not backward stable.

b) There exist ε_i, $i = 1...7$ such that $|\varepsilon_i| \le \varepsilon_{\text{machine}} + O(\varepsilon_{\text{machine}}^2)$ and

$$
\begin{aligned}
\tilde{f}(x, y) &= [\mathrm{fl}(x) \otimes \mathrm{fl}(x)] - [\mathrm{fl}(y) \otimes \mathrm{fl}(y)] \\
&= \{(x(1+\varepsilon_1)x(1+\varepsilon_1))(1+\varepsilon_2) - (y(1+\varepsilon_3)y(1+\varepsilon_3))(1+\varepsilon_4)\}(1+\varepsilon_5) \\
&= \big(x^2(1+\varepsilon_1)^2(1+\varepsilon_2)(1+\varepsilon_5)\big) - \big(y^2(1+\varepsilon_3)^2(1+\varepsilon_4)(1+\varepsilon_5)\big) \\
&= \big(x(1+2\varepsilon_6)\big)^2 - \big(y(1+2\varepsilon_7)\big)^2 \\
&= \tilde{x}^2 - \tilde{y}^2
\end{aligned}
$$

Therefore,

$$\tilde{f}(x) = f(\tilde{x}) \qquad \text{for some } \tilde{x} \text{ such that } \quad \frac{|\tilde{x}-x|}{|x|} \le 2\varepsilon_{\text{machine}}$$

and the algorithm is backward stable.

c) There exist $\varepsilon_i \;, i = 1...7$ such that $|\varepsilon_i| \le \varepsilon_{\text{machine}} + O(\varepsilon^2_{\text{machine}})$ and

$$\begin{aligned}
\tilde{f}(x,y) &= [\text{fl}(x) \oplus \text{fl}(y)] \otimes [\text{fl}(x) - \text{fl}(y)] \\
&= \{(x(1+\varepsilon_1) + y(1+\varepsilon_2))(1+\varepsilon_3)(x(1+\varepsilon_1) - y(1+\varepsilon_2))(1+\varepsilon_4)\}(1+\varepsilon_5) \\
&= \{(x(1+\varepsilon_1))^2 - (y(1+\varepsilon_2))^2\}(1+\varepsilon_3)(1+\varepsilon_4)(1+\varepsilon_5) \\
&= (x(1+5/2\varepsilon_6))^2 - (y(1+5/2\varepsilon_7))^2 \\
&= \tilde{x}^2 - \tilde{y}^2
\end{aligned}$$

Therefore,

$$\tilde{f}(x) = f(\tilde{x}) \qquad \text{for some } \tilde{x} \text{ such that } \quad \frac{|\tilde{x}-x|}{|x|} \le \frac{5}{2}\varepsilon_{\text{machine}}$$

and the algorithm is backward stable.

Problem 3:

Determine the accuracy of the algorithms b) and c) of Problem 2. Which is more accurate ?

SOLUTION:

The relative conditioning number for the problem $f(x,y) = x^2 - y^2$ is

$$\kappa(\mathbf{x}) = \|\mathbf{J}(\mathbf{x})\| \frac{\|\mathbf{x}\|}{\|f(\mathbf{x})\|}$$

We have

$$\mathbf{J}(\mathbf{x}) = \begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & -2y \end{bmatrix},$$

$$\|\mathbf{J}(\mathbf{x})\| = 2\|\mathbf{x}\|$$

$$\kappa(\mathbf{x}) = 2\|\mathbf{x}\| \frac{\|\mathbf{x}\|}{|x^2 - y^2|} = 2\frac{x^2 + y^2}{|x^2 - y^2|}$$

Because both algorithms are backward stable, their accuracy is given by $\Delta \le C\kappa(\mathbf{x})\varepsilon_{\text{machine}}$.
Since the constant C is equal 2 for b) and 5/2 for c), the algorithm in b) is slightly more accurate.