Math 1080: Spring 2005

Homework #3

Problem 1:

Show that the Householder reflector $\mathbf{H} = \mathbf{I} - 2\mathbf{w}\mathbf{w}^{\mathrm{T}}$, with $\|\mathbf{w}\| = 1$, is symmetric and orthogonal. Find the eigenvalues and eigenvectors of **H**.

SOLUTION:

Symmetry:

$$\mathbf{H}^{\mathrm{T}} = \left(\mathbf{I} - 2\mathbf{w}\mathbf{w}^{\mathrm{T}}\right)^{\mathrm{T}} = \mathbf{I} - 2\left(\mathbf{w}\mathbf{w}^{\mathrm{T}}\right)^{\mathrm{T}} = \mathbf{I} - 2\mathbf{w}\mathbf{w}^{\mathrm{T}} = \mathbf{H}$$

Orthogonality:

$$\begin{aligned}\mathbf{H}^{\mathrm{T}}\mathbf{H} = \mathbf{H}^2 &= \left(\mathbf{I} - 2\mathbf{w}\mathbf{w}^{\mathrm{T}}\right)\left(\mathbf{I} - 2\mathbf{w}\mathbf{w}^{\mathrm{T}}\right) \\ &= \mathbf{I} - 4\mathbf{w}\mathbf{w}^{\mathrm{T}} + 4\mathbf{w}\mathbf{w}^{\mathrm{T}}\mathbf{w}\mathbf{w}^{\mathrm{T}} = \mathbf{I} - 4\mathbf{w}\mathbf{w}^{\mathrm{T}} + 4\mathbf{w}\mathbf{I}\mathbf{w}^{\mathrm{T}} = \mathbf{I}\end{aligned}$$

Eigenvalues: Let us look for vectors $\mathbf{v}_i$ and numbers λ_i obeying $\mathbf{H}\mathbf{v}_i = \lambda_i\mathbf{v}_i$. First, suppose $\mathbf{v} \parallel \mathbf{w}$:

$$\mathbf{H}\mathbf{v} = \mathbf{v} - 2\mathbf{w}\mathbf{w}^{\mathrm{T}}\mathbf{v} = \mathbf{v} - 2\mathbf{v} = -\mathbf{v}$$

Hence the matrix **H** has one eigenvalue −1 with the corresponding eigenvector being parallel to **w**.

Now, suppose $\mathbf{v} \perp \mathbf{w}$:

$$\mathbf{H}\mathbf{v} = \mathbf{v} - 2\mathbf{w}\mathbf{w}^{\mathrm{T}}\mathbf{v} = \mathbf{v}$$

Hence any vector $\mathbf{v} \perp \mathbf{w}$ is an eigenvectorof **H**, and hence **H** has an eigenvalue 1 of multiplicity $n - 1$ where n is the dimension of **H**.

Problem 2:

Use Householder orthogonalization procedure to find the QR factorization of

$$A = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

SOLUTION:

Step 1:

$$B = A = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & -1 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v} = -\|\mathbf{b}_1\|\mathbf{e}_1 - \mathbf{b}_1 = \begin{bmatrix} -\sqrt{2} - 1 \\ 0 \\ -1 \end{bmatrix}, \ \mathbf{v}^T\mathbf{v} = 2\sqrt{2}(\sqrt{2} + 1)$$

$$F = I - 2\mathbf{v}\mathbf{v}^T/(\mathbf{v}^T\mathbf{v}) = \begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \qquad Q_1 = F = \begin{bmatrix} -\sqrt{1/2} & 0 & -\sqrt{1/2} \\ 0 & 1 & 0 \\ -\sqrt{1/2} & 0 & \sqrt{1/2} \end{bmatrix}$$

$$A_1 = Q_1 A = \begin{bmatrix} -\sqrt{2} & -2\sqrt{2} & 0 \\ 0 & 2 & 1 \\ 0 & -2\sqrt{2} & -\sqrt{2} \end{bmatrix}$$

Step 2:

$$B = \begin{bmatrix} 2 & 1 \\ -2\sqrt{2} & -\sqrt{2} \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 2 \\ -2\sqrt{2} \end{bmatrix}, \quad \mathbf{v} = -\|\mathbf{b}_1\|\mathbf{e}_1 - \mathbf{b}_1 = \begin{bmatrix} -2(\sqrt{3}+1) \\ 2\sqrt{2} \end{bmatrix},$$

$$\mathbf{v}^T\mathbf{v} = 8\sqrt{3}(\sqrt{3}+1), \quad F = I - 2\mathbf{v}\mathbf{v}^T/(\mathbf{v}^T\mathbf{v}) = \begin{bmatrix} -\sqrt{1/3} & \sqrt{2/3} \\ \sqrt{2/3} & \sqrt{1/3} \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\sqrt{1/3} & \sqrt{2/3} \\ 0 & \sqrt{2/3} & \sqrt{1/3} \end{bmatrix}$$

The results are

$$R = Q_2 A_1 = \begin{bmatrix} -\sqrt{2} & -2\sqrt{2} & 0 \\ 0 & -2\sqrt{3} & -\sqrt{3} \\ 0 & 0 & 0 \end{bmatrix}$$

$$Q = Q_1 Q_2 = \begin{bmatrix} -\sqrt{1/2} & -\sqrt{1/3} & -\sqrt{1/6} \\ 0 & -\sqrt{1/3} & \sqrt{2/3} \\ -\sqrt{1/2} & \sqrt{1/3} & \sqrt{1/6} \end{bmatrix}$$

Problem 3:

Find the *reduced* QR factorization of the matrix

$$A = \begin{bmatrix} 2 & 2 \\ -2 & 3 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$r_{11} = \|\mathbf{a}_1\| = 3, \quad \mathbf{q}_1 = \mathbf{a}_1 / r_{11} = \frac{1}{3}\begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \quad r_{12} = \mathbf{q}_1^{\mathrm{T}} \mathbf{a}_2 = \frac{-1}{3}, \quad \mathbf{v}_2 = \mathbf{a}_2 - r_{12}\mathbf{q}_1 = \frac{1}{9}\begin{bmatrix} 20 \\ 25 \\ 9 \\ 10 \end{bmatrix},$$

$$r_{22} = \|\mathbf{v}_2\| = \sqrt{134}/3, \qquad \mathbf{q}_2 = \frac{1}{3\sqrt{134}}\begin{bmatrix} 20 \\ 25 \\ 9 \\ 10 \end{bmatrix},$$

$$Q = \begin{bmatrix} 2/3 & 20/(3\sqrt{134}) \\ -2/3 & 25/(3\sqrt{134}) \\ 0 & 9/(3\sqrt{134}) \\ 1/3 & 10/(3\sqrt{134}) \end{bmatrix}$$

$$R = \begin{bmatrix} 3 & -1/3 \\ 0 & \sqrt{134}/3 \end{bmatrix}$$

Computer Assignment 2:

a) Write a MATLAB function **`[Q,R]=house(A)`** that computes full QR factorization of an $m \times n$ matrix **A** with $m \geq n$ using Householder triangularization. The output variables are the orthogonal $m \times m$ matrix **Q**, and the upper triangular $m \times n$ matrix **R**.

b) For the following matrix

$$\mathbf{Z} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 7 \\ 4 & 2 & 3 \\ 4 & 2 & 2 \end{bmatrix}$$

compute three QR factorizations using (1) the Gram-Schmidt algorithm **`gs`**, (2) the Householder subroutine **`house`**, and (3) the built-in command **`[Q,R]=qr(A)`**.

Compare the results and comment on any differences you find.

PROGRAMS:

```
function [Q,R] = house(A)
[m,n] = size(A);
Q = eye(m);
for j = 1:n
    B = A(j:end,j:end);
    b = B(:,1);
    e1 = zeros(m-j+1,1);
    e1(1)=sign(b(1))*sqrt(b'*b);
    v = -e1-b;
    if v'*v == 0
        F = eye(m-j+1);
    else
        F = eye(m-j+1)-2/(v'*v)*v*v';
    end
    A(j:end,j:end) = F*B;
    Q(:,j:end) = Q(:,j:end)*F;
end
R = A;
```

OUTPUT:

```
>> A = [1 2 3; 4 5 6; 7 8 7; 4 2 3; 4 2 2];
>> [Q,R] = gs(A)

Q =

    0.1010    0.3162    0.5420
    0.4041    0.3534    0.5162
    0.7071    0.3906   -0.5248
    0.4041   -0.5580    0.3871
    0.4041   -0.5580   -0.1204


R =

    9.8995    9.4954    9.6975
         0    3.2919    3.0129
         0         0    1.9701

>> Q'*Q-eye(3)

ans =

  1.0e-014 *

         0    0.0167    0.0160
    0.0167    0.0444   -0.1277
    0.0160   -0.1277    0.0222

>> [Q,R] = house(A)

Q =

   -0.1010   -0.3162    0.5420   -0.6842   -0.3577
   -0.4041   -0.3534    0.5162    0.3280    0.5812
   -0.7071   -0.3906   -0.5248    0.0094   -0.2683
   -0.4041    0.5580    0.3871    0.3656   -0.4918
   -0.4041    0.5580   -0.1204   -0.5390    0.4695


R =

   -9.8995   -9.4954   -9.6975
         0   -3.2919   -3.0129
    0.0000         0    1.9701
    0.0000    0.0000    0.0000
         0    0.0000         0

>> Q'*Q-eye(5)

ans =
```

```
  1.0e-015 *

   -0.2220         0    0.0763   -0.0833         0
         0   -0.1110    0.0278   -0.2220   -0.0555
    0.0763    0.0278         0   -0.0278    0.0694
   -0.0833   -0.2220   -0.0278   -0.3331         0
         0   -0.0555    0.0694         0         0


>> [Q,R] = qr(A)

Q =

    0.1010    0.3162    0.5420    0.3408   -0.6928
    0.4041    0.3534    0.5162   -0.5730    0.3422
    0.7071    0.3906   -0.5248    0.2684    0.0028
    0.4041   -0.5580    0.3871    0.5006    0.3534
    0.4041   -0.5580   -0.1204   -0.4825   -0.5273


R =

    9.8995    9.4954    9.6975
         0    3.2919    3.0129
         0         0    1.9701
         0         0         0
         0         0         0

>> Q'*Q-eye(5)

ans =

  1.0e-015 *

         0    0.1665   -0.0694   -0.0278   -0.1110
    0.1665         0    0.0971   -0.1665    0.1110
   -0.0694    0.0971   -0.3331    0.1596   -0.0833
   -0.0278   -0.1665    0.1596   -0.2220    0.0555
   -0.1110    0.1110   -0.0833    0.0555   -0.6661
```

Conclusion: As can be seen from the check for error in the factorization, $Q^T Q - I$, the built in subroutine and the Householder algorithm yield the most accurate results. All methods are very accurate with absolute error less than 10^{-14}. We are safely away from the region of instability of the Gram-Schmidt algorithm.