Math 1080: Spring 2005

Homework #2

SOLUTIONS

Problem 1:

Let **A** be an $m \times n$ matrix ($m \geq n$), and let $\mathbf{A} = \hat{\mathbf{Q}}\hat{\mathbf{R}}$ be a reduced QR factorization. Show that **A** has full rank (i.e., rank *n*) if and only if all the diagonal entries of $\hat{\mathbf{R}}$ are nonzero.

SOLUTION:

If **A** has full rank then its columns $\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n$ are linearly independent vectors. Therefore, the subspaces $span\{\mathbf{a}_1\}$, $span\{\mathbf{a}_1, \mathbf{a}_2\}$, ... $span\{\mathbf{a}_1, ...\mathbf{a}_n\}$ are all distinct. Thus $\mathbf{a}_k = \hat{\mathbf{Q}}\mathbf{r}_k$ cannot be in the span of $span\{\mathbf{q}_1, ...\mathbf{q}_{k-1}\} = span\{\mathbf{a}_1, ...\mathbf{a}_{k-1}\}$ and hence $r_{kk} \neq 0$.

Conversely, if $r_{kk} = 0$ then $span\{\mathbf{a}_1, ...\mathbf{a}_{k-1}\} = span\{\mathbf{a}_1, ...\mathbf{a}_k\}$ and the matrix **A** does not have full rank.

Problem 2:

Using classical or modified Gram-Schmidt orthogonalization compute the QR factorization of the following matrix

$$\mathbf{A} = \begin{bmatrix} 12 & -20 & 41 \\ 9 & -15 & -63 \\ 20 & 50 & 35 \end{bmatrix}$$

SOLUTION:

Using the classical Gram-Schmidt orthogonalization we compute:

$$r_{11} = \|\mathbf{a}_1\| = 25 , \qquad \mathbf{q}_1 = \frac{\mathbf{a}_1}{r_{11}} = \frac{1}{25}\begin{bmatrix} 12 \\ 9 \\ 20 \end{bmatrix}$$

$$r_{12} = \mathbf{q}_1^{\mathrm{T}}\mathbf{a}_2 = 25 , \quad \mathbf{v}_2 = \mathbf{a}_2 - r_{12}\mathbf{q}_1 = \begin{bmatrix} -32 \\ -24 \\ 30 \end{bmatrix} , \quad r_{22} = \|\mathbf{v}_2\| = 50 , \quad \mathbf{q}_2 = \frac{\mathbf{v}_2}{r_{22}} = \frac{1}{25}\begin{bmatrix} -16 \\ -12 \\ 15 \end{bmatrix}$$

$$r_{13} = \mathbf{q}_1^{\mathrm{T}}\mathbf{a}_3 = 25, \qquad r_{23} = \mathbf{q}_2^{\mathrm{T}}\mathbf{a}_3 = 25, \qquad \mathbf{v}_3 = \mathbf{a}_3 - r_{13}\mathbf{q}_1 - r_{23}\mathbf{q}_2 = \begin{bmatrix} 45 \\ -60 \\ 0 \end{bmatrix},$$

$$r_{33} = \|\mathbf{v}_3\| = 75 \ , \qquad \mathbf{q}_3 = \frac{\mathbf{v}_3}{r_{33}} = \frac{1}{5}\begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$$

Thus, the resulting matrices **Q** and **R** are

$$\mathbf{Q} = \frac{1}{25}\begin{bmatrix} 12 & -16 & 15 \\ 9 & -12 & -20 \\ 20 & 15 & 0 \end{bmatrix} \qquad \mathbf{R} = 25\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

Modified Gram-Schmidt orthogonalization gives the same result.

Computer Assignment 1:

(a) Write a function **`[Q,R]=gs(A)`** that computes a reduced QR factorization $\mathbf{A} = \hat{\mathbf{Q}}\hat{\mathbf{R}}$ of an $m \times n$ matrix **A** with $m \geq n$ using the Gram-Schmidt orthogonalization. The output variables are the $m \times n$ matrix **Q** and $n \times n$ upper triangular matrix **R**.

(b) Use the function to calculate the QR factorization of the following matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1+\varepsilon & 1+2\varepsilon \\ 1 & 1 & 1+\varepsilon \\ 1 & 1 & 1 \end{bmatrix}$$

with $\varepsilon = 10^{-10}, 10^{-12}, 10^{-14}$.

(c) Test the loss of orthogonality by calculating $\mathbf{E} = \mathbf{Q}^{\mathrm{T}}\mathbf{Q} - \mathbf{I}$.

Hand in the printout of your program and the resulting **Q** and **E** matrices for each ε.

FUNCTION:

```
function [Q,R] = gs(A)
[m,n] = size(A);
R = zeros(n);
Q = zeros(m,n);
for j=1:n
    v  = A(:,j);
    for i=1:j-1
        R(i,j) = Q(:,i)'*A(:,j);
        v = v - R(i,j)*Q(:,i);
    end
    R(j,j)=sqrt(v'*v);
    D = R(j,j);
    while abs(D) == 0
        v = rand(m,1);
        for i=1:j-1
            p = Q(:,i)'*v;
            v = v - p*Q(:,i);
        end
        D = sqrt(v'*v);
    end
    Q(:,j)=v/D;
end
```

OUTPUT:

```
>> eps = 1e-10;
>> A = [1 1+eps 1+2*eps; 1 1 1+eps; 1 1 1];
>> [Q,R] = gs(A)

Q =
```

```
    0.5774    0.8165    0.8165
    0.5774   -0.4083   -0.4082
    0.5774   -0.4083   -0.4083


R =

    1.7321    1.7321    1.7321
         0    0.0000   -0.0000
         0         0    0.0000

>> E = Q'*Q-eye(3)

E =

    0.0000   -0.0000   -0.0000
   -0.0000         0    1.0000
   -0.0000    1.0000    0.0000


>> eps = 1e-12;
>> A = [1 1+eps 1+2*eps; 1 1 1+eps; 1 1 1];
>> [Q,R] = gs(A)

Q =

    0.5774    0.8160    0.8160
    0.5774   -0.4087   -0.4087
    0.5774   -0.4087   -0.4087


R =

    1.7321    1.7321    1.7321
         0    0.0000   -0.0014
         0         0    0.0014

>> E = Q'*Q-eye(3)

E =

    0.0000   -0.0008   -0.0008
   -0.0008         0    1.0000
   -0.0008    1.0000   -0.0000

>> eps = 1e-14;
>> A = [1 1+eps 1+2*eps; 1 1 1+eps; 1 1 1];
>> [Q,R] = gs(A)

Q =

    0.5774    0.7587    0.7587
    0.5774   -0.4606   -0.4606
    0.5774   -0.4606   -0.4606
```

```
R =

    1.7321    1.7321    1.7321
         0    0.0000   -0.1626
         0         0    0.1626

>> E = Q'*Q-eye(3)

E =

    0.0000   -0.0939   -0.0939
   -0.0939         0    1.0000
   -0.0939    1.0000   -0.0000



>> eps = 0.1;
>> A = [1 1+eps 1+2*eps; 1 1 1+eps; 1 1 1];
>> [Q,R] = gs(A)

Q =

    0.5774    0.8165    0.0000
    0.5774   -0.4082    0.7071
    0.5774   -0.4082   -0.7071


R =

    1.7321    1.7898    1.9053
         0    0.0816    0.1225
         0         0    0.0707

>> E = Q'*Q-eye(3)

E =

  1.0e-013 *

    0.0022   -0.0155   -0.0455
   -0.0155         0    0.4197
   -0.0455    0.4197   -0.0022
```