Math 1080: Spring 2011

Homework #1

SOLUTIONS

***Problem 1*:**

Consider the matrix

$$\mathbf{Q} = \begin{bmatrix} 1/2 & 0 & \sqrt{3}/2 \\ 0 & 1 & 0 \\ -\sqrt{3}/2 & 0 & 1/2 \end{bmatrix}$$

Show that **Q** is orthogonal. What transformation of $\mathbb{R}^3$ does it correspond to ?

(Hint: Find the vector **a** that is invariant under **Q**. Pick a vector **b** orthogonal to **a**. Find the angle α between **b** and **Qb**. If this angle is independent of the choice of **b**, then **Q** corresponds to a rotation about **a** by the angle α. Think about other possibilities.)

SOLUTION:

$$\mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \begin{bmatrix} 1/2 & 0 & -\sqrt{3}/2 \\ 0 & 1 & 0 \\ \sqrt{3}/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & \sqrt{3}/2 \\ 0 & 1 & 0 \\ -\sqrt{3}/2 & 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

Hence **Q** is orthogonal.

Since **Q** is orthogonal, the mapping $\mathbf{x} \to \mathbf{Qx}$ corresponds to either rotation or reflection of $\mathbb{R}^3$. Let us find the vectors **a** that are invariant under **Q:**

$$\mathbf{Qa} = \mathbf{a}$$

$$(\mathbf{Q} - \mathbf{I})\mathbf{a} = \mathbf{0}$$

$$\begin{bmatrix} -1/2 & 0 & \sqrt{3}/2 \\ 0 & 0 & 0 \\ -\sqrt{3}/2 & 0 & -1/2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution is $\mathbf{a} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^{\mathrm{T}}$ or its scalar multiples. Therefore, **Q** corresponds to either a reflection through the *y*-axis or a rotation about that axis.

To confirm or rule out the rotation we pick a vector **b** orthogonal to **a** and find the angle α between **b** and **Qb**. A vector **b** is orthogonal to **a** if and only if

$$\mathbf{b}^{\mathrm{T}}\mathbf{a} = [b_1 \quad b_2 \quad b_3]\begin{bmatrix}0\\1\\0\end{bmatrix} = b_2 = 0 \quad .$$

Thus, any vector $\mathbf{b} = [b_1 \quad 0 \quad b_3]$ is orthogonal to **a**. The angle α between **b** and **Qb** is given by the formula

$$\cos\alpha = \frac{\mathbf{b}^{\mathrm{T}}(\mathbf{Qb})}{\|\mathbf{b}\|\|\mathbf{Qb}\|} \quad .$$

Because $\|\mathbf{b}\| = \sqrt{b_1^2 + b_3^2}$ and, by orthogonality of **Q**, $\|\mathbf{Qb}\| = \|\mathbf{b}\|$, we have

$$\cos\alpha = \frac{1}{b_1^2 + b_3^2}[b_1 \quad 0 \quad b_3]\begin{bmatrix}1/2 & 0 & \sqrt{3}/2\\ 0 & 1 & 0\\ -\sqrt{3}/2 & 0 & 1/2\end{bmatrix}\begin{bmatrix}b_1\\0\\b_3\end{bmatrix} = \frac{1}{2} \quad .$$

Thus, the angle between **b** and **Qb** is 60º regardless of the choice of **b** (provided **b** is orthogonal to **a**) and **Q** corresponds to the rotation about the *y*-axis by 60º.

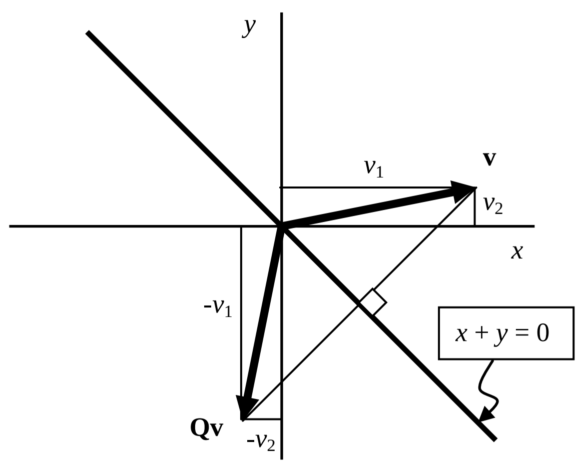
***Problem 2*:**

Find the 3×3 orthogonal matrix that corresponds to the reflection of $\mathbb{R}^3$ through the plane $x + y = 0$

SOLUTION:

A reflection of $\mathbb{R}^3$ through the plane $x + y = 0$ takes the vector $[v_1 \quad v_2 \quad v_3]^{\mathrm{T}}$ into the vector $[-v_2 \quad -v_1 \quad v_3]^{\mathrm{T}}$ (see the Figure). Thus the problem calls for finding the orthogonal matrix **Q** such that

$$\mathbf{Q}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}-v_2\\-v_1\\v_3\end{bmatrix} \quad .$$



Thus

$$\mathbf{Q}\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\-1\\0\end{bmatrix},\qquad \mathbf{Q}\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}-1\\0\\0\end{bmatrix},\qquad \mathbf{Q}\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

and hence

$$\mathbf{Q}=\mathbf{QI}=\mathbf{Q}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&1\end{bmatrix}.$$

***Problem 3*:**

Find the orthogonal projector **P** onto range(**A**) where

$$\mathbf{A}=\begin{bmatrix}1&0\\1&0\\0&1\end{bmatrix}$$

What is null(**P**)? What is the image under **P** of the vector $\begin{bmatrix}1 & 2 & 4\end{bmatrix}^{\mathrm{T}}$?

SOLUTION:

A formula given in the lecture tells us that $\mathbf{P}=\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}$.

We have

$$\mathbf{A}^T\mathbf{A}=\begin{bmatrix}1&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0\\1&0\\0&1\end{bmatrix}=\begin{bmatrix}2&0\\0&1\end{bmatrix}$$

$$(\mathbf{A}^T\mathbf{A})^{-1}=\begin{bmatrix}2&0\\0&1\end{bmatrix}^{-1}=\begin{bmatrix}1/2&0\\0&1\end{bmatrix}$$

$$\mathbf{P}=\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}=\begin{bmatrix}1&0\\1&0\\0&1\end{bmatrix}\begin{bmatrix}1/2&0\\0&1\end{bmatrix}\begin{bmatrix}1&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1/2&1/2&0\\1/2&1/2&0\\0&0&1\end{bmatrix}$$

Since **P** is a projector,

$$\operatorname{null}(\mathbf{P})=\operatorname{range}(\mathbf{I}-\mathbf{P})=\operatorname{span}\left(\begin{bmatrix}1/2\\-1/2\\0\end{bmatrix},\begin{bmatrix}-1/2\\1/2\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\end{bmatrix}\right)=\operatorname{span}\left(\begin{bmatrix}1\\-1\\0\end{bmatrix}\right).$$

The image under **P** of the vector $[1 \;\; 2 \;\; 4]^{\mathrm{T}}$ is

$$P\begin{bmatrix}1\\2\\4\end{bmatrix}=\begin{bmatrix}1/2 & 1/2 & 0\\1/2 & 1/2 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\2\\4\end{bmatrix}=\begin{bmatrix}3/2\\3/2\\4\end{bmatrix}.$$